published).
[9]V. Barger (unpublished).
[10]S. Mandelstam, Nuovo Cimento 30, 1150 (1963).
[11]J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).
[12]R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968); R. Aviv and D. Horn, *ibid.* 186, 1510 (1969).
[13]R. K. Logan, Phys. Rev. Letters 16, 414 (1965); R. J. N. Phillips and W. Rarita, Phys. Rev. 139, B1336 (1965); F. Arbab and C. B. Chiu, *ibid.* 167, 1045 (1966).
[14]N. F. Bali *et al.*, Phys. Rev. 150, 1358 (1966).
[15]R. J. N. Phillips, Rutherford Laboratory report, 1971 (unpublished).
[16]H. Högassen and W. Fischer, Phys. Letters 22, 516 (1966).
[17]R. J. N. Phillips, Nuovo Cimento 45, 265 (1966).
[18]Vincent M. de Lany *et al.*, Phys. Rev. Letters 18, 149 (1967).
[19]V. Singh, Phys. Rev. 129, 1889 (1963).
[20]C. B. Chiu, R. J. N. Phillips, and W. Rarita, Phys. Rev. 153, 1185 (1967).
[21]V. Barger and R. J. N. Phillips, Phys. Rev. Letters 20, 564 (1968).
[22]F. Henyey *et al.*, Phys. Rev. 182, 1579 (1969).
[23]G. Cohen-Tannoudji *et al.*, Nuovo Cimento 48A, 1075 (1967).
[24]R. C. Arnold and M. L. Blackmon, Phys. Rev. 176, 2082 (1968).
[25]R. J. Rivers and L. M. Saunders, Nuovo Cimento 58A, 385 (1968).
[26]J. N. J. White, Nucl. Phys. B13, 139 (1969).

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Regge-Pole Model of Pion-Nucleon Scattering with Explicit Quarks

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A U(6)-symmetric Regge-pole model with explicit quark spin is applied to meson and baryon exchange in the πN system. Attention is focused on the general form of the polynomial residues which result from including the required projection operators. Detailed calculations are exhibited for forward charge-exchange scattering within the context of a dual model with fixed cuts. For the case of baryon exchange a Regge residue appropriate to the symmetric-quark-model spectrum is presented and studied.

Recent work by several authors, of whom we can refer to only a few,[1-3] has stimulated a renewed interest in considering a U(6)-symmetric quark picture of hadrons. For our purposes the essential feature of this scheme is the utilization of explicit quark-spin structure to generate the desired particle spectrum. This leads to polynomial Regge residues which exhibit many desirable features. Several of these features are independent of detailed assumptions about the particle spectrum, such as those made in Ref. 3, and we shall present the results of the quark-spin calculation for meson-baryon scattering in a general form so that these properties can be exhibited without further assumptions. Then we shall discuss the results of assuming the detailed structure appropriate to the dual model with fixed cuts presented in Ref. 1.[4] In particular we shall be concerned with ρ exchange in forward πN charge-exchange scattering and Δ exchange in backward $\pi^- p$ scattering.

To define the desired particle spectrum we shall assume that mesons are composed of a quark-antiquark pair and belong to a mass-degenerate $(6, \bar{6}; L)$ representation of the group $\mathrm{U}(6)\times\mathrm{U}(6)\times\mathrm{O}(3)$, i.e., the usual $\underline{36}$ multiplet. Similarly, baryons are taken to be composites of three quarks and to appear in the $(56, \bar{1}; L)$ and $(70, \bar{1}; L)$ representations for the case of meson-baryon scattering. We shall also assume that all couplings occur via $\mathrm{U}(6)_W\times\mathrm{O}(2)_{L_z}$-invariant vertices.

Once the quarks have been explicitly introduced via the usual external U(6) wave functions, which are given along with other details in the Appendix, the desired quark-model spectrum can be obtained by utilizing projection operators for the individual quarks.[1] The structure introduced by these projection operators is the essential feature to be studied in the present work. These operators serve to prevent the negative-parity components (MacDowell twins) of the spin-$\frac{1}{2}$ quarks from contributing to the resonances. In the case of the mesons the $q\bar{q}$ propagator must include a factor $(1+\not{k}/M)(1-\not{k}/M)$ near the pole, $k^2\cong M^2$, where M is the resonance mass and the appropriate indices are understood to be present. This will ensure that only resonances which belong to the $(6, \bar{6})$ representation will appear and that there will be no contributions from $(\bar{6}, 6)$, $(35, 1)$, $(1, 35)$, and $(1, 1)$ which would otherwise appear.[5] Assuming that the internal excitations of the mesons are described by the usual Veneziano amplitude,[6] we find that the Reggeized meson "propagator" for the leading trajectory, in Sommerfeld-Watson-transformation notation, has the form

$$D^{(a)(c)}_{(b)(d)} = \frac{1}{2\pi i}\int_{\supset} \frac{d\lambda\,\Gamma(-\lambda)}{\lambda - l(t)}\, s^{\lambda}[X(\lambda, t)\delta^a_b\delta^c_d + (\not{k}^a_b\delta^c_d - \delta^a_b\not{k}^c_d)Y(\lambda, t) - (\not{k}^a_b\not{k}^c_d)Z(\lambda, t)]\,, \tag{1}$$

where $l(t) = l_0 + l't$ is the linear trajectory describing the internal excitations. The integration contour is defined to include the contribution from the explicit pole at $\lambda = l(t)$ (in a positive sense) and any implicit singularities in the functions X, Y, and Z, but not the poles in $\Gamma(-\lambda)$. The complex λ plane is essentially the complex angular momentum plane for the meson channel. To ensure the correct projection properties at the pole $t = M^2$ these functions must have the property that when $\lambda = l(t) = l_0 + l'M^2$,

$$X/Y = Y/Z = \sqrt{t} = M\,. \tag{2}$$

Although further properties of X, Y, and Z will depend on the specific model used, we shall see that just the assumption that they are nonsingular at $t = 0$ will already lead to some interesting results. The important feature present in Eq. (1) is that it will yield polynomials in the Regge residue. That these polynomials appear implies some very definite assumptions about couplings and about how to continue away from the poles. For example, the couplings for vector mesons in this picture contain both γ_μ and $\not{k}\gamma_\mu$ terms which make quite different contributions away from the pole.[7]

For the baryon propagator we need in general a factor like

$$\left(1 + \frac{\not{k}}{M}\right)\left(1 + \frac{\not{k}}{M}\right)\left(1 + \frac{\not{k}}{M}\right)$$

at the pole $k^2 = M^2$ with the indices appropriately defined. However, the Reggeized propagator is expected to be more complicated than in the meson case because of the presence of two types of multiplets, both the $(56, 1)$ and $(70, 1)$, with different symmetry properties. The specific form one arrives at depends on the structure of the baryon spectrum one assumes, e.g., whether one wants $\underline{56}$ even L and $\underline{70}$ odd L or the more degenerate

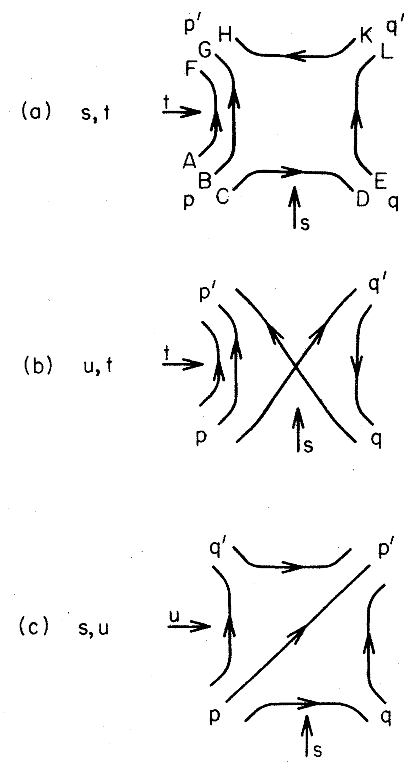

FIG. 1. Quark diagrams for meson-baryon scattering. A, B, C, etc. are indices as they appear in Eq. (A3).

spectrum of the symmetric quark model. We shall return to this question later.

MESON EXCHANGE

We proceed to calculate the contribution of meson exchange to forward meson-baryon scattering by calculating the contribution of the appropriate quark graphs exhibited in Fig. 1. The graphs serve to tell us how to attach the indices of the external wave functions to those of the propagator defined in Eq. (1). For meson exchange we have contributions from the s, t and u, t diagrams, the sum of which exhibits the usual signature factor. The definitions of the external wave functions and the actual expressions to be evaluated are given in the Appendix. We note that the calculations yield the usual $U(6)_W$ F/D values for the t-channel amplitudes B and A', i.e., $F/D|_B = \frac{2}{3}$ and $F/D|_{A'} = \infty$. For the case of πN charge-exchange scattering we find the following structure for the usual t-channel helicity-nonflip and -flip amplitudes, where trivial over-all numerical constants have been absorbed into the coupling constant g^2 and we have set $l' = 1$:

$$A'^{(-)} \simeq \frac{6g^2}{m^2\mu^2}\frac{s}{2(2\pi i)}\int_{\supset}\frac{d\lambda\,\Gamma(-\lambda)s^\lambda}{\lambda - l(t)}(1+e^{-i\pi\lambda})\{[X(\lambda,t)+tZ(\lambda,t)](4m\mu+t)+4tY(\lambda,t)(m+\mu)\}\,, \tag{3a}$$

$$B^{(-)} \simeq \frac{20g^2}{m^2\mu^2}\frac{(4m^2-t)}{\sqrt{2}(2\pi i)}\int_{\supset}\frac{d\lambda\,\Gamma(-\lambda)s^\lambda}{\lambda - l(t)}(1+e^{-i\pi\lambda})\{(m+\mu)[X(\lambda,t)+tZ(\lambda,t)]+Y(\lambda,t)(4m\mu+t)\}\,. \tag{3b}$$

As mentioned above, the implied contour integration encloses both the Regge pole and any singularities implicit in X, Y, and Z.

An essential feature of Eq. (3a) is that in the helicity-nonflip amplitude both Y and Z are multiplied by a factor t. The reason for this coefficient is easily understood from the form of the propagator in Eq. (1). Both Y and Z appear multiplied by $\not{k}_t$ and so the quark-spin calculation must lead to a coefficient which vanishes at $k_t \to 0$, i.e., the coefficient must be a power of t. This is not a constraint for the helicity-flip amplitude B since it appears in $d\sigma/dt$ multiplied by t for kinematic reasons. One important result of the presence of these t factors is that the continuation of the A' amplitude from the ρ pole $(t=\mu^2)$ to $t=0$, in order to find $d\sigma/dt|_{t=0}$ in terms of the ρ coupling constant, is independent of the values of Y and Z at $t=0$ as long as they are not infinite. In particular if we set $X\equiv 1$ and assume Y and Z have the values $1/\mu$ and $1/\mu^2$ at $t=\mu^2$, as required by Eq. (2), and are regular at $t=0$, as required by the usual analyticity constraints, we find

$$\begin{aligned}\left.\frac{d\sigma}{dt}\right|_{t=0} &= \frac{|A'|^2}{16\pi s^2}\\ &\simeq \frac{36g^4}{2(16\pi)}\frac{|\Gamma(\frac{1}{2})|^2}{m^2\mu^2}s^{2l_0}|1+e^{-i\pi l_0}|^2(16m^2\mu^2)\\ &= \frac{1}{36s}\left(\frac{g_\rho}{1+\mu/2m}\right)^4\\ &\approx \frac{g_\rho{}^4}{164s}\ \text{mb/GeV}^2\,,\end{aligned} \tag{4}$$

where we have used $\mu=\frac{2}{3}m$, $l_0=-0.5$, and the usual universality assumptions. This gives quite good agreement for $g_\rho{}^2\approx 28$ and $s=25$ GeV^2 when the measured value of the differential cross section is approximately 0.19 mb/GeV^2. The result is the same for any model which fulfills the two constraints mentioned above, including the models of Refs. 1 and 3. It should be noted, however, that the continuation to larger positive t will be quite model-dependent. A model with Y and Z constant, as in Ref. 3, will have a residue which increases much faster than one where Y and Z behave as $1/t$ and $1/\sqrt{t}$ near the resonances as in Ref. 1. The data seem to favor the slower increase.[8]

The other very interesting feature of Eq. (3) is the zero structure of the polynomials in t which have appeared and which are peculiar to this quark propagation picture. The same t factors mentioned above cause the helicity-nonflip amplitude to vanish at small negative t whereas in the helicity-flip amplitude, where X and Y are now adding, there are no small-t zeros. The actual location of these zeros in the real and imaginary parts is, of course, dependent on the specific forms of X, Y, and Z. For the case $X=1$, $Y=1/\mu$, $Z=1/\mu^2$, the A' amplitude, both real and imaginary parts, vanishes at $t\simeq -0.2$. In the dual-cut model discussed below, the zero is at somewhat more negative t but still in a reasonable location considering that absorption has not yet been explicitly included. Although the discussion of the t factors given earlier does not constitute a proof, the general feature of having a small-t zero in the nonflip amplitude seems to be rather basic to this quark propagator picture, in qualitative agreement with nature. Similar structure for the flip and nonflip amplitudes appears also for backward scattering.

The expressions in Eq. (3) have been evaluated as functions of t to find $d\sigma/dt$ utilizing the forms of X, Y, and Z suggested by Ref. 1. We have taken

$$\begin{aligned} X&=1\,,\\ Y&=\frac{F(l(t)-\lambda)}{[(\lambda-l_0)/l']^{1/2}}\,,\\ Z&=\frac{F(l(t)-\lambda)}{(\lambda-l_0)/l'}\,.\end{aligned} \tag{5}$$

Note that Y and Z contain fixed singularities in the λ plane.[9] These singularities are necessary in order for Y and Z to behave appropriately at all resonances and still be regular at the origin. The function $F(z)$ is the result of including the required neutralizer in the original dual-amplitude integral.[10] Although it is not uniquely defined in the dual model, it must have the general properties that $F(0)=1$ and $F(z)$ vanishes faster than any inverse power of $|z|$ as $|z|\to\infty$, with z constrained to be outside of some as yet unspecified region about the positive real axis, e.g., $|\arg z|>\epsilon$. It must, of course, be rather badly behaved along the positive real axis in order to satisfy the usual theorems about analytic functions.[11] Following the suggestion of the usual Veneziano model in which the wedge about the positive real axis, where the amplitude has poor asymptotic behavior, is treated as a cut, we take the attitude that the function F also represents a cut. In the calculations discussed here we have used $F(z)=e^{-k\sqrt{-z}}$,

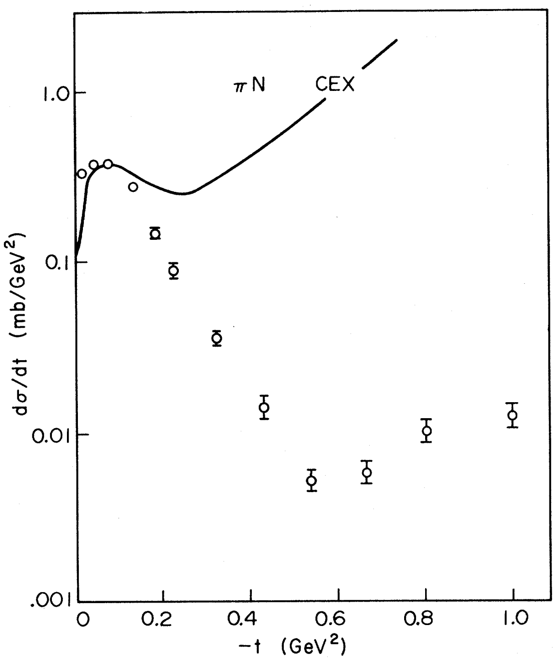

FIG. 2. Calculated differential cross section with no neutralizer. P_L is 5.9 GeV/c. Some data points are shown for comparison.

which shows the cut explicitly.[12]

The result of calculating $d\sigma/dt$ for $k=0$, no neutralizer, is illustrated in Fig. 2. This is clearly a catastrophe. The rapid growth results from the fixed-cut and -pole terms which behave like s^{l_0} times polynomials in t. The results for $k=2.8$ and 3.6 are illustrated in Figs. 3(a), 3(b). In all these calculations the values $\mu^2=0.6$ GeV^2 and $m^2=1.0$ GeV^2 have been used. We see that the general structure of the individual amplitudes is reasonable, at least in terms of the zeros present. However, the values for $d\sigma/dt$ shown in Fig. 3(b) can, at best, only be considered as being in qualitative agreement with the data. Another problem is polarization which turns out to be negative in the present model in clear disagreement with the data. This is primarily due to the fixed-cut structure plus the fact that the neutralizer form used here has little effect on the phase of Y and Z. One could, in principle, try to find a member of the general class of neutralizer functions which would give a better description of the data. However, without a more specific model in mind, this does not seem very instructive.

BARYON EXCHANGE

Now let us briefly survey the situation for baryon exchange[13] as calculated in the present picture. The major feature of the data which the polynomial Regge residues appearing in such

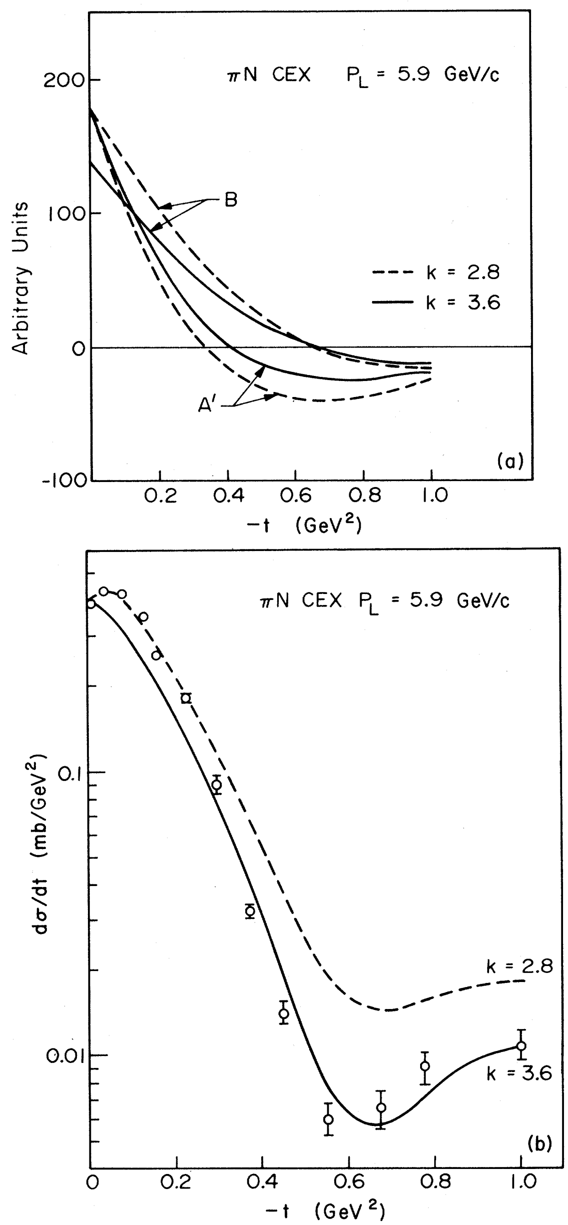

FIG. 3. (a) Calculated imaginary parts of the individual amplitudes with the neutralizer parameter k taking the values shown and $P_L=5.9$ GeV/c. (b) Differential cross section for the k values shown and $P_L=5.9$ GeV/c.

quark models hold some hope of explaining is the rapidly varying residue of the Δ exchange. Such variation does not appear in simple Regge or Veneziano models.

Before proceeding it is useful to review again why these polynomial residues arise. They result from making very specific assumptions about how the amplitudes are defined in terms of external U(6) wave functions, about which representations of U(6)×U(6)×O(3) should appear as resonances, and about how to continue away from the poles. In the present work we shall go again to a quark picture in order to decide which resonances appear. In particular we shall assume that the spectrum of the symmetric quark–harmonic-oscillator model of baryons[2] is a reasonable approximation of nature. As suggested by Mandelstam[14] the resonances on the leading baryon trajectory will obey such a spectrum if we calculate the contribution of the u, t quark diagram [see Fig. 1(b)] using the usual Veneziano amplitude for the internal excitations but include an extra factor $(\frac{1}{2})^n$, where n is the degree of excitation, in the s, u contribution [Fig. 1(c)]. This extra factor appears in a straightforward fashion in the harmonic-oscillator formalism for the Veneziano amplitude[15] and can be interpreted as accounting for the extra degree of freedom for the baryons (two internal sets of harmonic oscillators) as compared with the mesons (one set of harmonic oscillators). Now we need only define quark projection operators for each participating quark line as we did for the meson case, i.e., a $(1+\not{k}/M)$ factor for each quark. This yields a form analogous to Eq. (1) except for the absence of minus signs and the presence of a $W(\lambda, t)$ term which behaves like $1/M^3$. The results of this procedure will be given explicitly below.

First it is important to note that, although the symmetric-quark-model spectrum has pure $\underline{56}$ at $L=0$ and pure $\underline{70}$ at $L=1$, both $\underline{56}$ and $\underline{70}$ representations are present at all higher L. This does not correspond to the simple pair of exchange-degenerate $\underline{36}$ trajectories which appear in the meson case, for this baryon-spectrum signature will no longer appear in the usual way.

We present here the amplitudes calculated as described above[16] for the leading trajectories in the $I_u=\frac{3}{2}$ and $I_u=\frac{1}{2}$ channels. This is the quark spin-$\frac{3}{2}$ contribution which corresponds to the Δ_δ-N_β exchange in the usual Regge theory. The two amplitudes given correspond to the usual s-channel helicity amplitudes at large s ($\sigma=m+\mu$, $l'=1$):

$$(A+mB)^{I_u=3/2} \simeq \frac{2g^2}{m^2\mu^2}\frac{s}{2\pi i}\int_{\supset}\frac{d\lambda\,\Gamma(-\lambda)}{\lambda-l(u)}\{(s)^\lambda[(3X+5uZ)(u+\sigma^2)+2\sigma u(7Y+uW)] + (-\tfrac{1}{2}s)^\lambda[(5X+11uZ)(u+\sigma^2)+2\sigma u(13Y+3Wu)]\}\,, \tag{6a}$$

$$(B)^{I_u=3/2} \simeq \frac{-2g^2}{m^2\mu^2}\frac{s}{2\pi i}\int_{\supset}\frac{d\lambda\,\Gamma(-\lambda)}{\lambda-l(u)}\{(s)^\lambda[(3X+13uZ)2\sigma+(u+\sigma^2)(5Y+3uW)] + (-\tfrac{1}{2}s)^\lambda[(3X+13uZ)2\sigma+(u+\sigma^2)(11Y+5Wu)]\}\,, \tag{6b}$$

$$(A+mB)^{I_u=1/2} \simeq \frac{g^2}{m^2\mu^2}\frac{s}{2\pi i}\int\frac{d\lambda\,\Gamma(-\lambda)}{\lambda-l(u)}\{(s)^\lambda[(15X-11uZ)(u+\sigma^2)+2\sigma u(17Y-13uW)] + (-\tfrac{1}{2}s)^\lambda[(10X-14uZ)(u+\sigma^2)+2\sigma u(8Y-12uW)]\}\,, \tag{7a}$$

$$(B)^{I_u=1/2} \simeq \frac{g^2}{m^2\mu^2}\frac{s}{2\pi i}\int\frac{d\lambda\,\Gamma(-\lambda)}{\lambda-l(u)}\{(s)^\lambda[(13X-17uZ)2\sigma+(u+\sigma^2)(11Y-15uW)] + (-\tfrac{1}{2}s)^\lambda[(12X-8uZ)2\sigma+(u+\sigma^2)(14Y-10uW)]\}\,. \tag{7b}$$

Note that the usual signature factor is definitely absent but that the new structrue does have zeros in the appropriate places. Specifically the $I=\frac{1}{2}$ amplitude vanishes for $\lambda=0$, a ground-state $\underline{56}$, and the $I=\frac{3}{2}$ amplitude vanishes for $\lambda=1$, the pure $\underline{70}$ $L=1$.

Looking at the $I=\frac{3}{2}$ term we may determine g^2 in terms of the $\Delta(1236)$ coupling constant and then calculate backward π^-p scattering. We find that

$$\left.\frac{d\sigma}{du}\right|_{u=0} = \frac{|A+mB|^2}{64\pi s M_N^2} \approx \frac{2.5}{M_N^2}(l's)^{-1.8}\left[\frac{g_\Delta(m+\mu)}{10(2m+\mu)}\right]^4 \approx 0.62 s^{-1.8}\ \text{mb/GeV}^2. \tag{8}$$

This is in reasonable agreement with the data, which show

$$d\sigma/du|_{u=0} \sim 1.9\times10^{-3}\ \text{mb/GeV}^2 \text{ at } s=20\ \text{GeV}^2.$$

We note that again the continuation from the first resonance ($u=M_\Delta^2$ in this case) to $u=0$ does not depend on the specific form of Y, Z, and W as long as they have the appropriate values at the resonance and are regular at $u=0$. However, there is some dependence on the signature structure of the amplitude, i.e., the presence or absence of ordinary signature and the choice of which representations are present as discussed above. So to some extent, the agreement of Eq. (8) with the data is a confirmation of the symmetric quark model, at least as represented here.

A more instructive test is the continuation from the $\Delta(1236)$ resonance to the $F_{37}(1950)$ resonance. In order to make comparisons with previous work as outlined in Ref. 13 we study the baryon reduced Regge residue defined as

$$\gamma_R = \gamma_{A+mB} - \sqrt{u}\,\gamma_B\,, \tag{9}$$

where the γ's are the residues at the pole in the λ plane of Eq. (6) times the factors

$$s^{-1-l(u)}[\Gamma(-l(u))]^{-1}\,\frac{2[1+l(u)]}{1+e^{-i\pi l(u)}}\,,$$

where the last factor accounts for the absence of signature in the present model. If we assume X, Y, Z, W to have the values 1, $1/\sqrt{u}$, $1/u$, $1/u^{3/2}$, at the poles we find that γ_R changes by a factor of approximately $\frac{5}{2}$ in going from the Δ to the F_{37}, in quite good agreement with the observed values. If one assumed constant values for X, Y, Z, and W and the usual signature factor, the ratio of the residues at the two resonances is of order 10 in serious disagreement with the data.

CONCLUSION

We have seen that by using a model with explicit quark spin to construct Regge amplitudes for πN scattering, with projection operators included to ensure the appropriate resonance structure on the Regge trajectory, we are led to polynomial Regge residues. Independent of specific assumptions about the structure of the resonance spectrum beyond the first resonance on the Regge trajectory, we can already notice some encouraging results. Continuations of the Regge residue from the first pole down to $t=0$ or $u=0$ yield cross sections which agree quite well with nature. The polynomials also exhibit zero structure which is very suggestive of what is observed. Calculations utilizing assumptions and detailed structure appropriate to a specific dual-quark model yield results which are interesting but not conclusive due to the ambiguity of the neutralizer function, an essential feature of the model. A more specific picture is required in order to proceed.[17]

In general the continuation of the Regge residues away from the region between the first resonance and $t=0$ or $u=0$ is quite model-dependent and deserves further study. The results discussed above suggest that the form present in the dual model, where the resonance structure of the symmetric quark model appears at all levels of excitation on the leading trajectory, agrees quite well with the data. More detailed research including the study of the nonleading terms, e.g., the pion and nucleon trajectories, should serve to illuminate the usefulness of the quark picture more fully.

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APPENDIX

The external pseudoscalar-meson states are described by[18]

$$M(q)^{(b\,B)}_{(a\,A)} = \frac{1}{\sqrt{2}}\,[(1+\not{q}/\mu)\gamma_5]^{\,b}_{\,a}P^B_A\,, \tag{A1}$$

where μ is the $\underline{36}$ multiplet mass and P is the usual pseudoscalar $\mathrm{U}(3)$ matrix. The external nucleon is given by[19]

$$\psi(p)^{ABC}_{a\,b\,c} = \frac{1}{\sqrt{6}}\,\{[(1+\not{p}/m)\gamma_5 C]_{a\,b}\,\epsilon_{ABD}u_c(p)B^D_C$$

$$+\text{cyclic permutations}\}\,, \tag{A2}$$

where m is the $\underline{56}$ multiplet mass, $u_c(p)$ is a Dirac spinor, and B is the baryon U(3) matrix. The matrix C has the properties that

$$C^T = C^+ = C^{-1} = -C\,, \quad C^{-1}\gamma_\mu C = -(\gamma_\mu)^T\,,$$

$$C^{-1}\gamma_5 C = \gamma_5^T\,, \quad \text{and} \quad \overline{C} = C\,.$$

To within some over-all unknown coupling constant the contribution to the scattering amplitude of the s, t quark diagram [Fig. 1(a)] is found by evaluating the expression

$$T_{s,t} \propto \overline{\psi}(p')\begin{Bmatrix} FGH \\ fgh \end{Bmatrix} M(q')^{(l\,L)}_{(k\,K)} M(q)^{(d\,D)}_{(e\,E)}\psi(p)\begin{Bmatrix} ABC \\ abc \end{Bmatrix}$$

$$\times (1)^e_l\,\delta^E_L(1)^a_f\,\delta^A_F(1)^b_g\,\delta^B_G\,\delta^K_H\,\delta^C_D\,D^{(k)(c)}_{(h)(d)}\,. \tag{A3}$$

The calculation of the results given in the text is straightforward but tedious. To obtain the given normalization the constant factors in the external wave functions have been absorbed into the over-all coupling constant. The calculation of the u, t contribution to meson exchange is the same as the s, t contribution with the exchanges $s \to -s$, $\overline{P} \leftrightarrow P$, and $q \leftrightarrow -q'$. Baryon exchange is calculated in an analogous fashion for the appropriate quark diagrams using the baryon projection operator.

[1]R. Carlitz, S. Ellis, P. G. O. Freund, and Satoshi Matsuda, Caltech Report No. CALT-68-260, 1970 (unpublished).

[2]R. P. Feynman, M. Kislinger, and F. Ravndal, Phys. Rev. D 3, 2706 (1971).

[3]R. Delbourgo and P. Rotelli, Phys. Letters 35B, 65 (1971).

[4]For more details of this dual model study see S. D. Ellis, Caltech Ph.D. thesis, 1971 (unpublished).

[5]S. Mandelstam, Phys. Rev. 183, 1374 (1969); K. Bardakci and M. B. Halpern, *ibid.* 183, 1456 (1969).

[6]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[7]This explains why the present model gives different results from that of R. Carlitz and M. Kislinger, Phys. Rev. D 2, 336 (1970).

[8]E. L. Berger and G. C. Fox, Phys. Rev. 188, 2120 (1969).

[9]R. Carlitz and M. Kislinger, Phys. Rev. Letters 24, 186 (1970), and Phys. Rev. D 2, 336 (1970).

[10]The neutralizer function is an essential feature of the model of Ref. 1. It is required in order to prevent the $k\!\!\!/_t/m$ term in the quark propagator from contributing to poles in any other channel except the t channel. If allowed to contribute, this term would not only break the $U(6)_W$ symmetry but also lead to new poles displaced by $\frac{1}{2}$ in the complex angular momentum plane.

[11]See, for example, E. C. Titchmarsh, *Theory of Functions*, 2nd ed. (Oxford Univ. Press, London, 1939), p. 177.

[12]In the language of Ref. 1 this corresponds to

$$\Phi(z)=\int_0^z \frac{dx}{[\ln(1/x)]^2}\,\frac{\exp\{-k^2/4[\ln(1/x)]^2\}}{x}.$$

This function lends itself more easily to numerical calculations than the example given in Ref. 1.

[13]A recent review of data and current theories is given by E. Berger and G. Fox, Nucl. Phys. B26, 1 (1971).

[14]S. Mandelstam, Phys. Rev. D 1, 1720 (1970); 1, 1734 (1970).

[15]G. Frye, C. W. Lee, and L. Susskind, Nuovo Cimento 69A, 497 (1970).

[16]Included in the constants is the fact that there are two independent ways to construct the s, u diagram both of which give the same contribution.

[17]A program of study similar to the one discussed here, but utilizing other dual models with spin, is currently under way. For a review of the current situation in dual models with spin and further references, see M. A. Virasoro, talk at the International Conference on Duality and Symmetry in Hadron Physics, Tel-Aviv, Israel, 1971 (unpublished).

[18]A. Salam, R. Delbourgo, and I. Strathdee, Proc. Roy. Soc. (London) A284, 146 (1965); B. Sakita and K. C. Wali, Phys. Rev. 139, B1355 (1965).

[19]R. Delbourgo and M. A. Rashid, Proc. Roy. Soc. (London) A286, 412 (1965).

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Low-Energy Theorem for $\gamma \to 3\pi$

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A low-energy theorem relating $\gamma\to 3\pi$ to $\pi^0\to 2\gamma$ is derived by using anomalous Ward identities and by using Schwinger's proper-time technique. We discuss the theoretical significance of this low-energy theorem. In particular, the theorem is meaningful only if the partially conserved axial-vector current anomaly is in fact responsible for the decay $\pi^0\to 2\gamma$.

I. INTRODUCTION

Recently Adler *et al.*[1] discovered a low-energy theorem relating $\gamma\to 3\pi$ to $\pi\to 2\gamma$. The theorem states that

$$eF^{3\pi}=F^{\pi}f_{\pi}^{-2}, \tag{1.1}$$

where $F^{3\pi}$ and F^{π} are "coupling constants," to be defined in Sec. II, describing $\gamma\to 3\pi$ and $\pi\to 2\gamma$, respectively. The theorem rests upon the following assumptions:

(a) gauge invariance;

(b) Gell-Mann's current algebra and the hypothesis of the partial conservation of the axial-vector current (PCAC); and

(c) that the electromagnetic *current* commutes with the neutral axial *charge* at equal times.

We should emphasize that Eq. (1.1) is independent of the nature of chiral-symmetry breaking. It was