[23]M. Gell-Mann, Phys. Rev. Letters 12, 155 (1964); H. Cabibbo, *ibid.* 12, 62 (1964).

[24]C. Michael and R. Odorico, Phys. Letters 34B, 422 (1971).

[25]Similar results were obtained by Schechter (Ref. 20) with effectively the approach of Brown and Sommerfield (Ref. 3) for $p$ waves.

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# Asymptotic Bounds on the Absorptive Parts of the Elastic Scattering Amplitudes*

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We establish exact bounds on the absorptive parts $A(s,t)$ of an elastic scattering amplitude (spinless case) and evaluate them for positive $t$ values lying within the Lehmann-Martin ellipse [the major axis $=2(1+t_0/2k^2)$]. These bounds are used to derive a number of asymptotic results; e.g., (i) the "diffraction-peak width" $W$ is larger than $W_{\min}\sim 4t_0/(1+\lambda)^2 \times(1-\frac{1}{2}\sigma)(\ln s)^2$ (for $s\to\infty$); (ii) the leading Regge trajectory for $t_0>t>0$ lies below $[1+(t/t_0)^{1/2}]-\lambda[1-(t/t_0)^{1/2}]$; (iii) there are no complex zeros of $A(s,t)$ for $|t|<4t_0/(1+\lambda)^2e^2(\ln s)^2$ (for $s\to\infty$) and no real zeros for $t_0>t>-W_{\min}$, where $\lambda=\lim_{s\to\infty}\ln\sigma_{\text{tot}}(s)/\ln s$ and $\sigma=\lim_{s\to\infty}t_0\sigma_{\text{tot}}(s)/4\pi(\ln s)^2$.

## I. INTRODUCTION

The investigation of bounds on scattering amplitudes, and in particular on absorptive parts, following from the general principles of analyticity, unitarity, crossing, etc., has proved to be quite fruitful in the study of the strong interactions.[1] The restrictions on the absorptive part, for a given total cross section $\sigma_{\text{tot}}(s)$ ($\sqrt{s}$ = c.m. energy), were first studied by Martin.[2] The exact solution to this problem, as well as to the one with both the total and elastic cross sections given, was given by Singh and Roy, who succeeded in constructing the correct Fresnel-plate solution.[3] A comparison of the unitarity upper bound, involving the total and elastic cross section, with the experimental data in the diffraction-peak region showed that the bound was almost achieved.[3,4] The purpose of the present paper is to show how one can establish bounds on the absorptive parts which, apart from unitarity constraints, also take into account their polynomial boundedness. These bounds are evaluated in the positive-momentum-transfer region, lying within the Lehmann-Martin ellipse and have important consequences for the "diffraction-peak-width," Regge behavior, and zeros of the amplitude.

In order to derive these new bounds we make use of only (i) unitarity, (ii) analyticity within the Lehmann-Martin ellipse [the major axis $=2(1+t_0/2k^2)$, $k$ = c.m. momentum], and (iii) the Jin-Martin upper bound [5]

$$A(s,t_0)\underset{s\to\infty}{<}(s/s_0)^2, \tag{1.1}$$

where $A(s,t)$ is the absorptive part of the elastic scattering amplitude (we restrict ourselves to the spinless problem for simplicity) and $s$ and $t$ are, respectively, the squared c.m. energy and momentum-transfer variables. No assumption shall be made about the high-energy behavior of the total cross section $\sigma_{\text{tot}}(s)$.

## II. BASIC THEOREMS

The absorptive part $A(s,t)$ of the elastic scattering amplitude has the following partial-wave expansion, valid within the Lehmann-Martin ellipse:

$$A(s,t)=\frac{\sqrt{s}}{k}\sum_{l=0}^{\infty}(2l+1)\operatorname{Im}a_l(s)P_l(z), \tag{2.1}$$

where $t=-2k^2(1-z)$. We also have the following unitarity restrictions on partial-wave amplitudes:

$$\frac{k}{\sqrt{s}}A(s,0)=\frac{k^2}{4\pi}\sigma_{\text{tot}}(s)$$

$$=\sum_{l=0}^{\infty}(2l+1)\operatorname{Im}a_l(s) \tag{2.2}$$

and

$$\text{(positivity): } \operatorname{Im} a_l(s) \geq 0, \tag{2.3a}$$

$$\text{(boundedness): } 1 \geq \operatorname{Im} a_l(s). \tag{2.3b}$$

We further know that

$$A(s,t_0) = \frac{\sqrt{s}}{k} \sum_{l=0}^{\infty} (2l+1) \operatorname{Im} a_l(s) P_l(x), \tag{2.4}$$

where $t_0 = 2k^2(x-1)$, satisfies the Jin-Martin upper bound (1.1).

In order to take into account the unitarity restrictions as well as the polynomial boundedness of $A(s,t)$, we formulate the following mathematical problem.[6,7] Find extrema of $A(s,t)$ such that $A(s,0)$ and $A(s,t_0)$ have given fixed values and the partial-wave amplitudes $\operatorname{Im} a_l$ satisfy both the positivity and the boundedness constraints (2.3).

It is convenient to define the sets of $l$ values $U(M,N;z,x)$ and $W(M,N;z,x)$ as follows:

$$\begin{aligned} U(M,N;\, z,x) &\equiv \{l \,|\, P(M,N,l;\, z,x) < 0\}, \\ W(M,N;\, z,x) &\equiv \{l \,|\, P(M,N,l;\, z,x) \geq 0, \quad l \neq M,N\}, \end{aligned} \tag{2.5}$$

where $x \geq z$, $M > N$ and

$$\begin{aligned} P(M,N,l;\, z,x) \equiv\; & P_M(z)[P_l(x) - P_N(x)] \\ & + P_l(z)[P_N(x) - P_M(x)] \\ & + P_N(z)[P_M(x) - P_l(x)]. \end{aligned} \tag{2.6}$$

Using the definitions (2.5) and (2.6), we can state the solution to the extrema problems as follows:

*Theorem 1.* The upper bound on $A(s,t)$ is given by

$$A_{\max}(s,t) \geq A(s,t)$$

for $t$ real and within the Lehmann-Martin ellipse, where

$$(k/\sqrt{s})A_{\max}(s,t) = \sum_U (2l+1)P_l(z) + (2m+1)\epsilon_m P_m(z) + (2n+1)\epsilon_n P_n(z) \tag{2.7}$$

and $\sum_U$ stands for the summation over $l$ values contained in the set $U(m,n;\, z,x)$. The non-negative integers $m,n$ $(m>n)$ and the numbers $\epsilon_m, \epsilon_n$ $(1 > \epsilon_m, \epsilon_m \geq 0)$ are to be determined by

$$\begin{aligned} & \sum_U (2l+1) + (2m+1)\epsilon_m + (2n+1)\epsilon_n = \frac{k}{\sqrt{s}} A(s,0) \\ & \text{and} \\ & \sum_U (2l+1)P_l(x) + (2m+1)\epsilon_m P_m(x) + (2n+1)\epsilon_n P_n(x) = \frac{k}{\sqrt{s}} A(s,t_0). \end{aligned} \tag{2.8}$$

*Proof.* Using Eqs. (2.2), (2.4), and (2.8) we have

$$\begin{aligned} & \sum_U (2l+1)(1-c_l) + (2m+1)(\epsilon_m - c_m) + (2n+1)(\epsilon_n - c_n) = \sum_W (2l+1)c_l \\ & \text{and} \\ & \sum_U (2l+1)(1-c_l)P_l(x) + (2m+1)(\epsilon_m - c_m)P_m(x) + (2n+1)(\epsilon_n - c_n)P_n(x) = \sum_W (2l+1)c_l P_l(x), \\ & \text{where} \\ & c_l \equiv \operatorname{Im} a_l(s) \end{aligned} \tag{2.9}$$

and $\sum_W$ stands for the summation over the $l$ values contained in the set $W(m,n;\, z,x)$.

We now consider

$$\begin{aligned} D &\equiv (k/\sqrt{s})[A_{\max}(s,t) - A(s,t)] \\ &= \sum_U (2l+1)(1-c_l)P_l(z) + (2m+1)(\epsilon_m - c_m)P_m(z) + (2n+1)(\epsilon_n - c_n)P_n(z) - \sum_W (2l+1)c_l P_l(z). \end{aligned}$$

Eliminating $(\epsilon_m - c_m)$ and $(\epsilon_n - c_n)$ from $D$, by using Eq. (2.9) we obtain

$$[P_m(x) - P_n(x)]D = \sum_U (2l+1)(1 - \operatorname{Im} a_l)[-P(m,n,l;\, z,x)] + \sum_W (2l+1)\operatorname{Im} a_l P(m,n,l;\, z,x).$$

If we now use Eq. (2.3), the definition (2.5), and

$$P_m(x)>P_n(x) \quad \text{for } x>1,\ m>n,$$

we obtain

$$D \geq 0,$$

i.e.,

$$A_{\max}(s,t) \geq A(s,t). \qquad \text{Q.E.D.}$$

*Theorem 2.* The lower bound on $A(s,t)$ is given by

$$A(s,t) \geq A_{\min}(s,t)$$

for $t$ real and within the Lehmann-Martin ellipse, where

$$(k/\sqrt{s})A_{\min}(s,t)=\sum_{W'}(2l+1)P_l(z)+(2m'+1)\epsilon_{m'}P_{m'}(z)+(2n'+1)\epsilon_{n'}P_{n'}(z) \tag{2.10}$$

and $\sum_{W'}$ stands for the summation over the $l$ values contained in the set $W(m',n';z,x)$. The non-negative integers $m'$, $n'$ $(m'>n')$ and the numbers $\epsilon_{m'}$, $\epsilon_{n'}$ $(1>\epsilon_{m'},\ \epsilon_{n'}>0)$ are to be determined by

$$\begin{aligned}&\sum_{W'}(2l+1)+(2m'+1)\epsilon_{m'}+(2n+1)\epsilon_{n'}=\frac{k}{\sqrt{s}}A(s,0),\\&\sum_{W'}(2l+1)P_l(x)+(2m'+1)\epsilon_{m'}P_{m'}(x)+(2n'+1)\epsilon_{n'}P_{n'}(x)=\frac{k}{\sqrt{s}}A(s,t_0).\end{aligned} \tag{2.11}$$

A proof of Theorem 2 can be given along the same lines as that of Theorem 1 and so it shall be omitted.

*Remarks.* (i) Theorems 1 and 2 are true for all energies above threshold and not just for high energies.

(ii) The upper bound given by Theorem 1 is saturated for the following set of the partial-wave amplitudes:

$$\begin{aligned}\operatorname{Im} a_l &= 1, && l\in U(m,n;z,x)\\ &=\epsilon_m, && l=m\\ &=\epsilon_n, && l=n\\ &=0, && l\in W(m,n;z,x).\end{aligned}$$

Since the upper bound is achieved, and the theorem has been proved by the "direct subtraction" method, it follows that the upper bound is the best possible one under the stated assumptions. Similar remarks apply to Theorem 2.

(iii) Even for a problem with spin there is always an amplitude which is formally similar to the one we considered in the spinless case.[8] For example, the amplitude $\sqrt{s}(f_1+zf_2)$ for $\pi N$ scattering is such an amplitude, where $f_1$ and $f_2$ are respectively the standard $\pi N$ spin-nonflip and -flip amplitudes. All our considerations equally apply to such an amplitude.

(iv) *Dependence of the extrema of $A(s,t)$ on $A(s,t_0)$.* The bounds given by Theorems 1 and 2 are obviously functions of $A(s,0)$, i.e., of $\sigma_{\text{tot}}(s)$ and $A(s,t_0)$. We however do not in general know the value of $A(s,t_0)$. Asymptotically $A(s,t_0)$ satisfies, of course, the upper bound (1.1). We would thus obtain interesting asymptotic results from Theorems 1 and 2 by combining it with the Jin-Martin bound if, and only if, the upper (lower) bound on $A(s,t)$ is an increasing (decreasing) function of $A(s,t_0)$ for a fixed $A(s,0)$, i.e., fixed total cross section. We, therefore, note that:

*Lemma 1.*

$$\delta(A_{\max}(s,t))=\frac{P_m(z)-P_n(z)}{P_m(x)-P_n(x)}\,\delta(A(s,t)),$$

provided

$$\delta(A(s,0))=0.$$

*Proof.* Let us introduce small changes in $A(s,t_0)$ by introducing small changes $\delta\epsilon_m$ in $\epsilon_m$ and $\delta\epsilon_n$ in $\epsilon_n$ but keeping $m$ and $n$ unchanged. We obtain

$$\delta\left(\frac{k}{\sqrt{s}}A(s,t_0)\right)=(2m+1)P_m(x)\delta\epsilon_m+(2n+1)P_n(x)\delta\epsilon_n.$$

The small changes $\delta\epsilon_m$ and $\delta\epsilon_n$ are, however, subject to

$$(2m+1)\delta\epsilon_m+(2n+1)\delta\epsilon_n=\delta\left(\frac{k}{\sqrt{s}}A(s,0)\right)=0.$$

The changes $\delta\epsilon_m$, $\delta\epsilon_n$ introduce a change in $A_{\max}(s,t)$ given by

$$\delta\left(\frac{k}{\sqrt{s}}A_{\max}(s,t)\right)=(2m+1)P_m(z)\delta\epsilon_m+(2n+1)P_n(z)\delta\epsilon_n.$$

Eliminating $\delta\epsilon_m$ and $\delta\epsilon_n$ between the expressions for $\delta(A(s,t_0))$, $\delta(A(s,0))$, and $\delta(A_{\max}(s,t))$ we obtain

Lemma 1. Similarly, we have:

*Lemma 2.*

$$\delta(A_{\min}(s,t)) = \frac{P_{m'}(z) - P_{n'}(z)}{P_{m'}(x) - P_{n'}(x)}\,\delta(A(s,t_0)),$$

provided

$$\delta(A(s,0)) = 0.$$

These lemmas also follow from the general analysis given by Einhorn and Blankenbecler.[6]

(v) A variant of the above problem with total elastic cross section, instead of the total cross section, given has been considered earlier by Einhorn and Blankenbecler.[6]

## III. EVALUATION OF THE UPPER BOUND ON $A(s,t)$ IN THE REGION $t_0 > t > 0$.

It is obvious from Lemma 1 that if we wish to use the Jin-Martin upper bound (1.1), in conjunction with the extrema theorems, we can do so only for the upper bound, Theorem 1, in the region $t_0 \geq t \geq 0$ (i.e., $x \geq z \geq 1$). For an explicit evaluation of $A_{\max}(s,t)$ it is useful to know the following result.

*Lemma 3.*

$$U(m,n;z,x) = \{l \mid m > l > n\}$$

and

$$W(m,n;z,x) = \{l \mid n > l \geq 0\} + \{l \mid l > m\}$$

for

$$x \geq z \geq 1,\quad m > n. \tag{3.1}$$

*Proof.* Consider the function $f(y)$ defined by

$$f(y) = -[(1-\eta)P_n(z) + \eta P_{n+1}(z)]$$

for

$$y = (1-\eta)P_n(x) + \eta P_{n+1}(x),$$

$$n = 0, 1, 2, \ldots,\quad 1 > \eta \geq 0,\quad x \geq z \geq 1.$$

By varying $n$ in integral steps and $\eta$ continuously between 0 and 1, the variable $y$ takes all possible values from one to infinity; the numbers $x$ and $z$ are fixed numbers. It has been shown by Singh that the function $f(y)$ is a convex function of $y$ for $y > 1$.[7] We therefore have

$$(y_2 - y)f(y_1) + (y_1 - y_2)f(y) + (y - y_1)f(y_2) \geq 0$$

for all

$$y_2 \geq y \geq y_1 \geq 1.$$

Choosing

$$(y_2, y, y_1) = (P_m(x), P_l(x), P_n(x))$$

we obtain

$$(-)P(m,n,l;z,x) \geq 0$$

for

$$m > l > n,\quad x \geq z \geq 1.$$

Similarly, by choosing $(y_2, y, y_1) = (P_l(x), P_m(x), P_n(x))$ and $(P_m(x), P_n(x), P_l(x))$ we find that

$$P(m,n,l;z,x) > 0$$

for

$$x \geq z \geq 1,\quad m > n > l,\quad \text{and}\quad l > m > n.$$

This proves Lemma 3.

If we combine Theorem 1 with Lemmas 1 and 3 we obtain the following theorem.

*Theorem 3.*

$$A_{\max}(s,t) \geq A(s,t),$$

where

$$\frac{k}{\sqrt{s}}A_{\max}(s,t) = [P_{m-1}'(z) + P_m'(z) + (2m+1)\epsilon_m P_m(z)] - [P_{n-1}'(z) + P_n'(z) + (2n+1)(1-\epsilon_n)P_n(z)]$$

for

$$x \geq z \geq 1. \tag{3.2}$$

The non-negative integers $m$ and $n$ ($m > n$) and the numbers $\epsilon_m, \epsilon_n$ ($1 > \epsilon_m, \epsilon_n > 0$) are to be determined by

$$\frac{k^2}{4\pi}\sigma_{\text{tot}}(s) = [m^2 + (2m+1)\epsilon_m] - [n^2 + (2n+1)(1-\epsilon_n)]$$

and (3.3)

$$\frac{k}{\sqrt{s}}\hat{A}(s,t_0) = [P_{m-1}'(x) + P_m'(x) + (2m+1)\epsilon_m P_m(x)] - [P_{n-1}'(x) + P_n'(x) + (2m+1)(1-\epsilon_n)P_n(x)],$$

where $\hat{A}(s,t_0)$ is any number larger than $A(s,t_0)$.

We now proceed to evaluate the asymptotic upper bound on $A(s,t)$ given by Theorem 3. We use

$$\hat{A}(s,t_0) = (s/s_0)^2. \tag{3.4}$$

Solving Eqs. (3.3) we obtain

$$m \underset{s\to\infty}{\sim} \frac{k}{\sqrt{t_0}}(1+\lambda)\ln s + \cdots$$

and (3.5)

$$n \underset{s\to\infty}{\sim} \frac{k}{\sqrt{t_0}}(1+\lambda)(1-\sigma)^{1/2}\ln s + \cdots,$$

where

$$\lambda = -\lim_{s\to\infty}\frac{\ln\sigma_{\text{tot}}(s)}{\ln s}, \tag{3.6}$$

and

$$\sigma = \lim_{s\to\infty}\frac{t_0\sigma_{\text{tot}}(s)}{4\pi(\ln s)^2}. \tag{3.7}$$

Note that $\lambda$ is nonzero only if $\sigma_{tot}(s)$ has a power decrease with $s$.

We thus obtain, using the solution (3.5), the asymptotic bound:

*Theorem 3(a).*

$$\frac{A(s,t)}{A(s,0)} \underset{s\to\infty}{\leq} [\tilde{I}_1((1+\lambda)(t/t_0)^{1/2}\ln s) - (1-\sigma)\tilde{I}_1((1+\lambda)(1-\sigma)^{1/2}(t/t_0)^{1/2}\ln s)]/\sigma, \tag{3.8}$$

where $\tilde{I}_1(y) \equiv 2I_1(y)/y$ and $I_1(y)$ is the modified Bessel function of the first order.

*Remark.* If $\sigma=0$, then Theorem 3(a) leads to the result

$$\frac{A(s,t)}{A(s,0)} \underset{s\to\infty}{\leq} I_0((1+\lambda)(t/t_0)^{1/2}\ln s). \tag{3.9}$$

## IV. SOME CONSEQUENCES OF THE UPPER-BOUND THEOREMS 3 and 3(a)

We shall now discuss some of the important consequences which can be deduced as corollaries of Theorems 3 and 3(a).

*(a) The "diffraction-peak width" W.* We define

$$W^{-1} = \left[\frac{d\ln A(s,t)}{dt}\right]_{t=0}. \tag{4.1}$$

It follows from Theorem 3 that

$$\frac{A_{\max}(s,t)-A(s,0)}{tA(s,0)} \geq \frac{A(s,t)-A(s,0)}{tA(s,0)}$$

for

$$t_0 \geq t \geq t.$$

Taking the limit $t\to 0+$ we obtain

$$\lim_{t\to 0+}\frac{A_{\max}(s,t)-A(s,0)}{tA(s,0)} \geq W^{-1}. \tag{4.2}$$

We now use Eqs. (3.2) and (3.3) to obtain:

*Theorem 4.*

$$W_{\min}^{-1} \geq W^{-1},$$

where

$$[m^2+(2m+1)\epsilon_m - n^2-(2n+1)(1-\epsilon_n)][8k^2W_{\min}^{-1}] \equiv m^2(m^2-1)+2m(m+1)(2m+1)\epsilon_m - n^2(n^2-1)-2n(n+1)(2n+1)(1-\epsilon_n) \tag{4.3}$$

and $m, n, \epsilon_m, \epsilon_n$ are given by Eqs. (3.3).

Similarly, using Theorem 3(a), we obtain:

*Theorem 4(a).*

$$W_{\min}^{-1} \underset{s\to\infty}{\sim} \frac{(1+\lambda)^2(1-\sigma/2)(\ln s)^2}{4t_0} \underset{s\to\infty}{\geq} W^{-1}. \tag{4.4}$$

Theorem 4(a) removes the major arbitrariness in the upper bound on $W^{-1}$ given by Kinoshita,[9] i.e.,

$$\text{const}(\ln s)^2 > W^{-1} \quad \text{for } s\to\infty$$

and improves an earlier result of Singh.[7] If we combine Theorem 4(a) with the unitarity lower bound on $W^{-1}$ given by Martin, i.e.,

$$W^{-1} \underset{s\to\infty}{\geq} \frac{\sigma_{tot}}{32\pi},$$

we obtain the Lukaszuk-Martin upper bound[10] on the total cross section.

*(b) Asymptotic power behavior.* If the amplitude $A(s,t)$ has a power behavior like $s^{\alpha(t)}$ then using Theorem 3(a) one obtains the following upper bound:

*Theorem 5.*

$$\alpha(t) \leq [1+(t/t_0)^{1/2}] - \lambda[1-(t/t_0)^{1/2}] \quad \text{for } 0\leq t\leq t_0. \tag{4.5}$$

This improves a result of Martin[11] given by $\alpha(t)\leq 1+(t/t_0)^{1/2}$ for $t_0\geq t>0$.

*Remark.* If there is a $J$-plane singularity at $J=\alpha(t)$ then it is well known that unitarity implies the presence of further $J$-plane singularities having positions $J=\alpha_n(t)\equiv n\alpha(t/n^2)-1$, $n=2,3,4,\ldots$. It follows therefore that if $J=\alpha(t)$ is the leading singularity for $t_0\geq t\geq 0$, then it must satisfy

$$\alpha(t)\geq n\alpha(t/n^2)-1,\ n=2,3,\ldots\ (t_0\geq t\geq 0).$$

It is nice to note that our bound on $\alpha(t)$ does have this desirable property both for $\lambda\neq 0$ and $\lambda=0$.

*(c) Zeros of $A(s,t)$ in the complex t plane.* The zeros of the scattering amplitude near the forward direction have been investigated in connection with the possible violation of the Pomeranchuk theorem.[12] Let $n(y)$ be the number of zeros of the function $A(s,t)$ for $|t|<y<t_0$. Then from Jensen's theorem we have $(t_0>r>0)$

$$\int_0^r \frac{dy\,n(y)}{y} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \ln\left|\frac{A(s,t=re^{i\phi})}{A(s,0)}\right|.$$

Using

$$|A(s,t=re^{i\phi})| < A(s,r)$$

we obtain

$$\int_0^r dy\frac{n(y)}{y} < \ln\left(\frac{A(s,r)}{A(s,0)}\right).$$

If we now combine this result with Theorem 3(a) we get:

*Theorem 6(a).*

$$\int_0^r dy \frac{n(y)}{y} < (1+\lambda)\left(\frac{r}{t_0}\right)^{1/2} \quad \text{for } t_0 > r > 0, \ s \to \infty. \tag{4.6}$$

We can also extract a bound on $n(y)$ itself from Theorem 6(a) by noting (for $t_0 > r > r_1 > 0$) that

$$\int_0^r \frac{n(y)}{y} dy \geq n(r_1) \int_{r_1}^r \frac{dy}{y} = n(r_1) \ln\left(\frac{r}{r_1}\right), \tag{4.7}$$

since $n(y)$ is an increasing positive definite function of $y$. Optimal results are obtained by choosing $(r/r_1) = e^2$ for $t_0/e^2 > r_1 > 0$ and $(r/r_1) = t_0/r_1$ for $r_1 > t_0/e^2$. Therefore we have the result:

*Theorem 6(b).*

$$n(r) < \frac{e(1+\lambda)}{2}\left(\frac{t}{t_0}\right)^{1/2} \ln s \quad \text{for} \ \frac{t_0}{e^2} > r > 0, \ s \to \infty$$

and

$$n(r) < \frac{(1+\lambda)\ln s}{\ln(t_0/r)} \quad \text{for } t_0 > r > \frac{t_0}{e^2}, \ s \to \infty.$$

*Corollary.* The absorptive part can not have a zero in the complex $t$ plane within the circle

$$|t| < \frac{4t_0}{(1+\lambda)^2 e^2} \frac{1}{(\ln s)^2} \quad \text{for } s \to \infty.$$

These results make some of the earlier results of Eden *et al.*[12] more precise by determining the unknown constants in their results.

On the real $t$ axis we can obtain a stronger result as follows. Using the inequality

$$P_l(z) \geq [1 - \tfrac{1}{2} l(l+1)(1-z)] \quad \text{for } 1 > z > -1$$

it is easy to show that

$$A(s,t) \geq A(s,0)(1 + t/W) \quad \text{for } 0 > t > -4k^2.$$

It follows that:

*Theorem 7.* For $0 \geq t \geq -4k^2$,

$$A(s,t) \geq A(s,0)(1 + t/W_{\min}). \tag{4.8}$$

*Corollary.* The absorptive part $A(s,t)$ can not have a zero on the real $t$ axis for $0 > t > -W_{\min}$. The $A(s,t)$ does not, of course, have a zero for $t_0 > t > 0$.


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[1] A. Martin and F. Cheung, *Analyticity Properties and Bounds of the Scattering Amplitudes* (Gordon and Breach, New York, 1970); Virendra Singh, Fields and Quanta 1, 151 (1971); R. J. Eden, Rev. Mod. Phys. 43, 15 (1971).

[2] A. Martin, Phys. Rev. 129, 1432 (1963).

[3] Virendra Singh and S. M. Roy, Phys. Rev. Letters 24, 28 (1970), and Phys. Rev. D 1, 2638 (1970).

[4] M. A. Jacobs, A. Joseph, S. Nussinov, and A. A. Rangwala, Phys. Rev. D 2, 1970 (1970); D. G. Ravenhall and W. J. Pardee, *ibid.* 2, 589 (1970). M. B. Einhorn, R. Blankenbecler, and R. Savit, SLAC report, 1971 (unpublished).

[5] Y. S. Jin and A. Martin, Phys. Rev. 135, B1375 (1964).

[6] For general mathematical background of this type of extremum problem consult, e.g., S. I. Gass, *Linear Programming* (McGraw Hill, New York, 1958); M. R. Hestenes, *Calculus of Variations and Optimal Control Theory* (Wiley, New York, 1966); D. G. Luenberger, *Optimization by Vector Space Methods* (Wiley, New York, 1969); M. B. Einhorn and R. Blankenbecler, Ann. Phys. (N.Y.) 67, 480 (1971). We especially recommend to the physicist reader the last reference.

[7] For the solution in the absence of the boundedness condition, see Virendra Singh, Phys. Rev. Letters 26, 530 (1971).

[8] See, e.g., G. Mahoux and A. Martin, Phys. Rev. 174, 2140 (1968); J. S. Bell, Nuovo Cimento 61A, 541 (1969).

[9] T. Kinoshita, in *Lectures in Theoretical Physics*, edited by W. E. Brittin, A. O. Barut, and M. Guenin (Gordon and Breach, New York, 1966), Vol. IXB.

[10] L. Lukaszuk and A. Martin, Nuovo Cimento 52A, 122 (1967).

[11] A. Martin, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, Edinburgh, Scotland, 1963).

[12] J. Finkelstein, Phys. Rev. Letters 24, 172 (1970); 24, 432(E) (1970); R. C. Casella, *ibid.* 24, 1463 (1970); R. J. Eden and L. Lukaszuk, Nuovo Cimento 47, 817 (1967); R. J. Eden and G. D. Kaiser, Phys. Rev. D 3, 2286 (1971); G. Auberson, T. Kinoshita, and A. Martin, *ibid.* 3, 3185 (1971).