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Lower Bound for the Hadronic Contribution to the Muon Magnetic Moment*

G. Auberson and Ling-Fong Li
The Rockefeller University, New York, New York 10021
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A new lower bound is derived for the hadronic contribution to the anomalous magnetic moment of the muon. In spite of a careful treatment of the ρ-dominance region, the numerical improvement over a previous bound due to Langacker and Suzuki is not very significant.

I. INTRODUCTION

Recently, various interesting inequalities imposing restrictions on some vertex functions involving electromagnetic[1] or weak[2] hadronic currents have been derived by assuming cut-plane analyticity. Technically, the methods used in these derivations essentially rely on the Schwarz inequality or upon a quite different maximum principle originally introduced by Meiman[3] and recently improved by Okubo and the present authors.[4]

More specifically, Langacker and Suzuki,[5] and Palmer[6] have obtained in this way lower bounds for the hadronic contribution to the anomalous magnetic moment of the muon, $\Delta a_{(\mu)}$, in terms of the charge radius of the pion r_π. The bound derived in Ref. 6 by using Okubo's method is the best possible one with the total charge and charge radius of the pion given, and is free of any phenomenological input. On the other hand, the method of Ref. 5 assumes a once-subtracted dispersion relation for the form factor and requires in addition some knowledge of the p-wave $\pi\pi$ cross section, resulting in a substantial numerical improvement. Although the lower bound thus derived is a direct consequence of the Schwarz inequality, it can be shown that it is an optimal one if the pion charge radius and the p-wave cross section are the *only* input. However, the pion form factor which saturates this lower bound clearly violates Watson's theorem,[7] another condition which arises as soon as one decides to include the rather reliable experimental information on the low-energy p-wave $\pi\pi$ scattering, dominated by the ρ-meson contribution. Hence, one can expect a further improvement if the method can be modified in such a way that the phase of the pion form factor is enforced to have the correct behavior in the ρ region. The present paper is devoted to the solution of this problem. It turns out that the practical gain over the Langacker-Suzuki bound is rather poor. Yet we shall describe the mathematical method in some detail as it may be of more general interest. Though not obvious from the mathematical point of view, the fact that we get only a minor improvement is physically not too surprising, since the Langacker-Suzuki bound is almost saturated when the pure vector-meson dominance values of both $\Delta a_{(\mu)}$ and r_π are inserted. It turns out that, given any reasonable value of r_π, the minimum value of $\Delta a_{(\mu)}$ is not very sensitive to the phase of the pion form factor on the cut.

In Sec. II, after listing our definitions and assumptions and formulating the problem more precisely, we write down its solution in the form of an algorithm which allows us to compute the lower bound. The derivation of the relevant formulas is given in Sec. III, with some details deferred to an Appendix. Section IV contains the numerical results and a short discussion.

II. FORMULATION OF THE PROBLEM AND RESULTS

The pion electromagnetic form factor $F(t)$ is defined by

$$(p-p')_\mu F(t)=\langle 0|J_\mu(0)|\pi^+(\vec{p})\pi^-(\vec{p}\,'),\text{in}\rangle\,, \qquad (2.1)$$

where $t=(p+p')^2$ and $J_\mu(x)$ is the hadronic electromagnetic current. The form factor $F(t)$ is normalized in such a way that

$$F(0)=1\,. \qquad (2.2)$$

The charge radius of the pion is then related to the derivative of $F(t)$ by

$$F'(0)=\tfrac{1}{6}\,r_\pi^{\,2}\,. \qquad (2.3)$$

For convenience, we rewrite Eq. (2.1) in the form

$$F(t)=\frac{2}{t-4M_\pi^{\,2}}\langle 0|\vec{p}\cdot\vec{J}(0)|\pi^+(\vec{p})\pi^-(-\vec{p}),\text{in}\rangle\,, \qquad (2.4)$$

where $\vec{p}$ is the momentum of π^+ in the c.m. frame of the two pions. We now assume that $F(t)$ is holomorphic in the t plane deprived of the cut $[4M_\pi^{\,2},\infty]$ and at most one subtraction is needed for the dispersion relation. Therefore we have

$$\tfrac{1}{6}r_\pi^{\,2}=F'(0)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{\text{Im}F(t)\,dt}{t^2}\,, \qquad (2.5)$$

with

$$t_0=4M_\pi^{\,2}\,.$$

By applying the reduction technique to Eq. (2.4), one can write

$$\text{Im}F(t)=\sum_n a_n(t)b_n(t) \qquad (t\geq t_0)\,, \qquad (2.6)$$

where

$$|a_n(t)|^2=\frac{(2\pi)^4}{t-t_0}\,\delta^4(p+p'-p_n)|\langle 0|\vec{p}\cdot\vec{J}(0)|n\rangle|^2\,, \qquad (2.7)$$

$$|b_n(t)|^2=\frac{(2\pi)^4}{t-t_0}\,\delta^4(p+p'-p_n)|\langle n|P_1 j^{\pi^-}(0)|\pi^+(\vec{p})\rangle|^2\,. \qquad (2.8)$$

Here, $j^{\pi^-}(x)$ is the source of the pion field; $\{|n\rangle\}$, any complete set of in- (or out-) states; $\sum_n$ the integral over the usual invariant measure, and P_1 the projection onto the $J=1$ subspace. The $a_n(t)$ and $b_n(t)$ are related to the experimentally measurable quantities, by

$$\sum_n |a_n(t)|^2=\frac{t^2}{32\pi^2\alpha^2}\,\sigma_{e^+e^-}(t)\,, \qquad (2.9)$$

$$\sum_n |b_n(t)|^2=2\left(\frac{t}{t-t_0}\right)^{1/2}\sigma_T^1(t)\equiv 2S(t)\,, \qquad (2.10)$$

where $\sigma_{e^+e^-}(t)$ is the total cross section for e^+e^- annihilation, and $\sigma_T^1(t)$ is the $\pi\pi$ total cross section in the p wave at energy $\sqrt{t}$. Moreover, the hadronic contribution $\Delta a_{(\mu)}$ to the muon anomalous magnetic moment $a_{(\mu)}=\frac{1}{2}(g_\mu-2)$ due to vacuum polarization effects is related to $\sigma_{e^+e^-}(t)$ by[8]

$$\Delta a_{(\mu)}=\frac{1}{4\pi^3}\int_{t_0}^{\infty}dt\,\sigma_{e^+e^-}(t)G(t)t^2$$

or

$$\Delta a_{(\mu)}=\frac{8\alpha^2}{\pi}\int_{t_0}^{\infty}dt\,G(t)\sum_n|a_n(t)|^2\,, \qquad (2.11)$$

with

$$G(t)=\frac{1}{t^2}\int_0^1 dz\,\frac{z^2(1-z)}{z^2+(t/m_\mu^{\,2})(1-z)}\,.$$

Hence if we know $\Delta a_{(\mu)}$, Eq. (2.11) serves as a constraint on $a_n(t)$. If we also have information on the cross section $\sigma_T^1(t)$, we get a constraint on $b_n(t)$. (The p-wave $\pi\pi$ cross section is known for the low-energy region and the high-energy part can be bounded by unitarity.) To get a bound on r_π, we just have to maximize $\text{Im}\,F(t)=\sum_n a_n(t)b_n(t)$ subjected to the constraints Eqs. (2.10) and (2.11). This maximum gives an upper bound for r_π in terms of $\Delta a_{(\mu)}$. Or turning it around, we get a lower bound of $\Delta a_{(\mu)}$ in terms of r_π. The maximization procedure can be carried out by using the functional variation technique with Lagrange multipliers. However, it turns out that the bound derived in this way is the same as that of Langacker and Suzuki. This means that the bound derived by using the Schwarz inequality is an optimal one with the information given above. Therefore to improve the bound, we have to put in more information.

We now assume that the $\pi\pi$ scattering is purely elastic up to some energy $\sqrt{t_1}$ (in the application, we shall choose $\sqrt{t_1}\simeq 1.1$ GeV, much larger than the inelastic threshold, taking the elastic approximation for granted in the whole ρ-resonance region). Then, by using elastic unitarity in conjunction with Eqs. (2.4) and (2.6)–(2.8), one obtains Watson's theorem, i.e.,

$$\text{phase of } F(t)=\delta_1(t)\ (\text{mod}\pi) \text{ for } t_0 \le t \le t_1, \tag{2.12}$$

where $\delta_1(t)$ is the $\pi\pi$ p-wave phase shift. The elastic approximation also implies, according to Eqs. (2.4) and (2.7),

$$\sum_n |a_n(t)|^2 = \frac{1}{96\pi}\, t \left(\frac{t-t_0}{t}\right)^{3/2} |F(t)|^2 \quad \text{for } t_0 \le t \le t_1 .$$

Thus Eq. (2.11) can be rewritten as

$$\Delta a_{(\mu)} = \frac{\alpha^2}{12\pi^2}\int_{t_0}^{t_1} dt\, H(t)|F(t)|^2 + \frac{8\alpha^2}{\pi}\int_{t_1}^{\infty} dt\, G(t) \sum_n |a_n(t)|^2 , \tag{2.13}$$

where

$$H(t)=\frac{(t-t_0)^{3/2}}{t^{1/2}}\, G(t). \tag{2.14}$$

Let us notice that the functions $S(t)$, $G(t)$, and $H(t)$ are all positive.

Now the problem to solve is the following: By using the Eqs. (2.13) and (2.6), find the minimum of the functional $\Delta a_{(\mu)}$ under the constraints (2.2), (2.5), (2.10), and (2.12).

The solution is worked out in the following section, where a (unique) extremum of the functional is found, $\Delta a_{(\mu)\text{extre}}$. Since we know from previous works that a lower bound must exist, $\Delta a_{(\mu)\text{extre}}$ is the minimum we are looking for. It is given by the following set of equations. Define

$$X(t)=\frac{1}{t-t_1}\exp\left(\frac{1}{\pi}\int_{t_0}^{t_1} dt'\, \frac{\delta_1(t')}{t'-t}\right), \tag{2.15}$$

$$\sigma_n(t')=\left(\frac{S(t')}{G(t')}\right)^{1/2} \frac{1}{X(0)X(t')t'^2}\int_{t_0}^{t_1} dt\, t^{n+1}\frac{H(t)|X(t)|^2}{t-t'} \quad (n=1,2), \tag{2.16}$$

$$K(t',t'')=\left(\frac{S(t')S(t'')}{G(t')G(t'')}\right)^{1/2}\frac{1}{X(t')X(t'')t'^2t''^2}\,\frac{1}{48\pi^3}\int_{t_0}^{t_1} dt\, t^4\, \frac{H(t)|X(t)|^2}{(t-t')(t-t'')}. \tag{2.17}$$

The function $X(t)$ is holomorphic in the t plane deprived of the cut $[t_0, t_1]$. The real functions $\sigma_n(t')$ and $K(t', t'')$ are defined on $[t_1,\infty]$ and $[t_1,\infty]\otimes[t_1,\infty]$, respectively.

Let $R(t',t'')$ be the resolvent of the (symmetric) kernel $K(t',t'')$. Introduce the "scalar products"

$$(\sigma_n|\sigma_m)=(\sigma_m|\sigma_n)=\frac{1}{48\pi^3}\int_{t_1}^{\infty} dt\, \sigma_n(t)[(\underline{1}-\underline{R})\sigma_m](t) \tag{2.18}$$

and the "moments"

$$I_n=\frac{1}{X^2(0)}\int_{t_0}^{t_1} dt\, t^n H(t)|X(t)|^2 \quad (n=0,1,2). \tag{2.19}$$

Then the lower bound $\Delta a_{(\mu)\min}$ is given by

$$\Delta a_{(\mu)\min}=\frac{\alpha^2}{12\pi^2}\left[[I_2-(\sigma_2|\sigma_2)]\left(\frac{r_\pi^{\,2}}{6}-\frac{X'(0)}{X(0)}\right)^2+2[I_1-(\sigma_1|\sigma_2)]\left(\frac{r_\pi^{\,2}}{6}-\frac{X'(0)}{X(0)}\right)+[I_0-(\sigma_1|\sigma_1)]\right]. \tag{2.20}$$

Remark. From Eqs. (2.14) and (2.17), it is easy to show that in general the kernel $K(t', t'')$ is *not* of the Hilbert-Schmidt type, because of a singularity at $t=t_1$. As we shall see later, there is no practical difficulty in coping with this peculiarity.

III. DERIVATION

We first give a formal proof of Eq. (2.20), and next go into the properties of the kernel $K(t', t'')$.

(i) Consider the analytic function $f(t)\equiv F(t)/X(t)$. From Eq. (2.15), one can see that $X(t)$ is constructed in such a way that it has the same phase as $F(t)$ (up to $n\pi$) in the region $t_0 \le t \le t_1$. This gives

$$\text{phase of } f(t)=0\ (\text{mod}\pi) \text{ for } t_0 \le t \le t_1 .$$

Hence $f(t)$ is holomorphic in the t plane deprived of the cut $[t_1,\infty]$, since $X(t)$ has no zeros in the region

$t_0 \leq t \leq t_1$. Moreover, $X(t) \sim t^{-1}$ as $|t| \to \infty$, so that a once-subtracted dispersion relation for $F(t)$ implies a twice-subtracted dispersion relation for $f(t)$. Thus, taking the normalization (2.2) into account,

$$F(t) = X(t)\left[\frac{t^2}{\pi}\int_{t_1}^{\infty} dt' \frac{\operatorname{Im}F(t')}{t'^2(t'-t)X(t')} + \frac{1+bt}{X(0)}\right], \tag{3.1}$$

where, from Eq. (2.3), b is related to the charge radius by

$$b = \frac{r_\pi^{\,2}}{6} - \frac{X'(0)}{X(0)}\,. \tag{3.2}$$

[Let us remark that the representation (3.1) is nothing but a particular solution to the so-called nonhomogeneous Hilbert problem.[9]] In order to make $\operatorname{Im}F(t)$ an explicitly real function in Eq. (2.6), we shall write

$$a_n(t) = \alpha_n(t)e^{i\psi_n(t)},$$

$$b_n(t) = \beta_n(t)e^{-i\phi_n(t)},$$

with $0 \leq \psi_n(t),\ \phi_n(t) < \pi$ and $\alpha_n(t),\ \beta_n(t)$ real. Then

$$\operatorname{Im}F(t) = \sum_n \alpha_n(t)\beta_n(t)\cos\theta_n(t) \quad (\theta_n = \psi_n - \phi_n)\,. \tag{3.3}$$

We now want to express everything in terms of $\alpha_n(t)$, $\beta_n(t)$, and $\theta_n(t)$. In Eq. (2.13), the first term contains $|F(t)|^2$, for which we substitute the representation (3.1) and integrate over t to get

$$\begin{aligned}\frac{12\pi^2\Delta a_{(\mu)}}{\alpha^2} &\equiv \Phi[\alpha_n, \beta_n, \theta_n]\\ &= \int_{t_1}^{\infty} dt' \int_{t_1}^{\infty} dt'' k(t',t'')\left[\sum_n \alpha_n(t')\beta_n(t')\cos\theta_n(t')\right]\left[\sum_m \alpha_m(t'')\beta_m(t'')\cos\theta_m(t'')\right]\\ &\quad + 2\int_{t_1}^{\infty} dt'[\Sigma_1(t') + b\Sigma_2(t')]\left[\sum_n \alpha_n(t')\beta_n(t')\cos\theta_n(t')\right] + 96\pi\int_{t_1}^{\infty} dt' G(t')\sum_n \alpha_n^{\,2}(t') + I_0 + 2bI_1 + I_2\,,\end{aligned} \tag{3.4}$$

where

$$k(t',t'') = \frac{1}{X(t')X(t'')t'^2t''^2}\frac{1}{\pi^2}\int_{t_0}^{t_1} dt\, t^4 \frac{H(t)|X(t)|^2}{(t-t')(t-t'')}, \tag{3.5}$$

$$\Sigma_n(t') = \frac{1}{X(0)X(t')t'^2}\frac{1}{\pi}\int_{t_0}^{t_1} dt\, t^{n+1}\frac{H(t)|X(t)|^2}{t'-t} \qquad (n=1,2)\,. \tag{3.6}$$

Now, our initial problem is reduced to an extrema problem for the functional (3.4), under the constraint (2.10), which can be handled by the usual Lagrange multiplier method.

Introducing

$$\Psi[\alpha_n, \beta_n, \theta_n] = \Phi[\alpha_n, \beta_n, \theta_n] + \int_{t_1}^{\infty} dt'\nu(t')\left[\sum_n \beta_n^{\,2}(t') - 2S(t')\right],$$

where $\nu(t')$ is the Lagrange multiplier function, we get from $\delta\Psi/\delta\theta_n = 0$

$$-2\sin\theta_n(t')\alpha_n(t')\beta_n(t')\left(\int_{t_1}^{\infty} dt'' k(t',t'')\sum_m \alpha_m(t'')\beta_m(t'')\cos\theta_m(t'') + 1\right) = 0 \quad \text{for all } n\,.$$

It is easy to show that the integral equation for $g(t'')$,

$$\int_{t_1}^{\infty} dt'' k(t',t'')g(t'') + 1 = 0,$$

has no solution, so that we must have $\sin\theta_n(t)\alpha_n(t)\beta_n(t) = 0$. According to Eq. (3.3), this just means that we can choose

$$\theta_n(t) = 0 \quad \text{for all } n\,. \tag{3.7}$$

Next, $\delta\Psi/\delta\alpha_n = 0$ and $\delta\Psi/\delta\beta_n = 0$ imply for all n

$$2\beta_n(t')\left(\int_{t_1}^{\infty} dt''\, k(t',t'')\sum_m \alpha_m(t'')\beta_m(t'') + \Sigma_1(t') + b\Sigma_2(t')\right) + 192\pi G(t')\alpha_n(t') = 0, \tag{3.8a}$$

$$2\alpha_n(t')\left(\int_{t_1}^{\infty} dt''k(t', t'')\sum_m \alpha_m(t'')\beta_m(t'') + \Sigma_1(t') + b\Sigma_2(t')\right) + 2\nu(t')\beta_n(t') = 0\,. \tag{3.8b}$$

By requiring these two simultaneous homogeneous equations to have a nontrivial solution (at least for one n), the Lagrange multiplier is fixed to be

$$\nu(t') = \frac{1}{96\pi G(t')}\left(\int_{t_1}^{\infty} dt''k(t', t'')\sum_m \alpha_m(t'')\beta_m(t'') + \Sigma_1(t') + b\Sigma_2(t')\right)^2.$$

Then solving Eqs. (3.8) for α_n, β_n and using Eq. (2.10), we get

$$\sum_n \alpha_n(t')\beta_n(t') = -\frac{S(t')}{48\pi G(t')}\left(\int_{t_1}^{\infty} dt''k(t', t'')\sum_m \alpha_m(t'')\beta_m(t'') + \Sigma_1(t') + b\Sigma_2(t')\right).$$

It is convenient to symmetrize this equation by introducing

$$y(t') = \left[\frac{G(t')}{S(t')}\right]^{1/2}\sum_n \alpha_n(t')\beta_n(t') \tag{3.9}$$

and replacing the quantities (3.5) and (3.6) by (2.16) and (2.17), respectively. In this way, we are led to the following integral equation for $y(t')$:

$$y(t') + \int_{t_1}^{\infty} dt''K(t', t'')y(t'') = \frac{1}{48\pi^2}[\sigma_1(t') + b\sigma_2(t')]. \tag{3.10}$$

It is shown below that the kernel $K(t', t'')$ has a well-defined (symmetric) resolvent $R(t', t'')$. Thus, in the operator form

$$\underline{K}\underline{R} = \underline{R}\underline{K} = \underline{K} - \underline{R} \tag{3.11}$$

and the (unique) solution of the integral equation can be written as

$$y = \frac{1}{48\pi^2}(\underline{1} - \underline{R})(\sigma_1 + b\sigma_2)\,. \tag{3.12}$$

Also, it follows from Eqs. (3.8) and (3.12) that

$$\sum_n \alpha_n^{\,2}(t) = \frac{1}{(48\pi^2)^2}\,\frac{1}{2G}[(\underline{1} - \underline{R})(\sigma_1 + b\sigma_2)]\,. \tag{3.13}$$

Then, inserting Eqs. (3.7), (3.12), and (3.13) into the expression for Φ, we get

$$\begin{aligned}\Phi_{\text{extre}} = {} & \frac{1}{48\pi^3}((\underline{1} - \underline{R})(\sigma_1 + b\sigma_2), \underline{K}(\underline{1} - \underline{R})(\sigma_1 + b\sigma_2)) \\ & - \frac{2}{48\pi^3}((\sigma_1 + b\sigma_2), (\underline{1} - \underline{R})(\sigma_1 + b\sigma_2)) \\ & + \frac{1}{48\pi^3}((\underline{1} - \underline{R})(\sigma_1 + b\sigma_2), (\underline{1} - \underline{R})(\sigma_1 + b\sigma_2)) \\ & + I_0 + 2bI_1 + I_2\,.\end{aligned}$$

Finally, a simple reduction using Eq. (3.13), the symmetry of $\underline{R}$, and Eq. (3.2) leads to the result (2.20).

(ii) In order to investigate the properties of the kernel $K(t', t'')$, we have to know the behavior of the various functions involved, at infinity and in the neighborhood of the singular point $t = t_1$:

$$S(t) \underset{t\to\infty}{\leq} 48\pi/t\,, \tag{3.14}$$

$$G(t) \underset{t\to\infty}{\sim} m_\mu^{\,2}/3t^3\,, \tag{3.15}$$

$$X(t) \underset{t\to\infty}{\sim} 1/t\,, \tag{3.16}$$

$$X(t) \underset{t\to t_1}{\sim} C(t - t_1)^{\delta_1(t_1)/\pi - 1}\,. \tag{3.17}$$

Equation (3.14) follows from the unitarity bound $\sigma_T^1(t) \leq 48\pi/(t - t_0)$, and Eqs. (3.15)–(3.17) follow directly from the definitions.

Let us notice that the behavior (3.17) eventually allows the form factor to diverge at $t = t_1$, when $\delta_1(t_1) < \pi$ [see Eq. (3.1)]. In our case, however, $\delta_1(t_1)$ is very close to π; though $F(t_1) = \infty$, it is a rather "weak" and perfectly admissible singularity [in fact, the purpose of introducing the factor $(t - t_1)^{-1}$ in Eq. (2.15) is to prevent an unnecessary zero of $F(t)$ at $t = t_1$]. In any case, we have to assume that $\delta(t_1) > \frac{1}{2}\pi$ in order to ensure the convergence of the integrals in Eqs. (2.16) and (2.17). On the other hand, the threshold behavior of the p-wave phase shift $\delta_1(t) \sim (t - t_0)^{3/2}$ as $t \to t_0$ implies that $X(t_0)$ is finite.

By using Eqs. (3.14)–(3.17), we obtain from Eqs. (2.15) and (2.16) the following behaviors:

$$\begin{aligned}&\sigma_n(t') \underset{t'\to\infty}{\leq} \frac{\text{const}}{t'}\,, \\ &\sigma_n(t') \underset{t'\to t_1}{\sim} \text{const}(t' - t_1)^{\delta_1(t_1)/\pi - 1} \quad (n = 1, 2)\,,\end{aligned} \tag{3.18}$$

$$\begin{aligned}&K(t', t'') \underset{t'\to\infty}{\leq} \frac{\text{const}}{t't''}\,, \\ &K(t_1 + \alpha\rho, t_1 + \beta\rho) \underset{\rho\to 0}{\sim} \frac{\text{const}}{\rho}\,.\end{aligned} \tag{3.19}$$

Equation (3.18) shows that $\sigma_{1,2}(t') \in L^2(t_1, \infty)$. However, we see from Eq. (3.19) that the integral of $|K(t', t'')|^2$, though converging at infinity, diverges logarithmically at $t' = t'' = t_1$. Thus $K(t', t'')$ is not a Hilbert-Schmidt kernel. In order to isolate its singular part, we replace $X(t)$ by the expression (3.17) in Eq. (2.17), and all the other functions regular at $t = t_1$ by their value at this point. We get

$$K_S(t', t'') = S(t_1)\frac{(t_1-t_0)^{3/2}}{\sqrt{t_1}}(t'-t_1)^{\eta}(t''-t_1)^{\eta} \times \frac{1}{48\pi^3}\int_{t_0}^{t_1}\frac{dt}{(t_1-t)^{2\eta}(t-t')(t-t'')},$$

where

$$\eta = 1-\delta(t_1)/\pi \quad (0<\eta<\tfrac{1}{2}). \tag{3.20}$$

Although $K_S(t', t'')$ is not a Hilbert-Schmidt kernel, it is proved in the Appendix that $\underline{K}_S$ is a *bounded* operator, when acting on the Hilbert space $L^2(t_1, \infty)$. More precisely, its norm is shown to be bounded by

$$\|\underline{K}_S\| \leq \tan^2\delta_1(t_1). \tag{3.21}$$

On the other hand, it is easy to see from Eqs. (2.17) and (3.20) that the regular part $\underline{K}_r$ defined by

$$K(t', t'') = K_r(t', t'') + K_S(t', t'')$$

is of the Hilbert-Schmidt type, i.e.,

$$\|\underline{K}_r\|_{\text{HS}} = \int_{t_1}^{\infty} dt'dt''|K(t', t'')|^2 < \infty. \tag{3.22}$$

Hence $\underline{K}$ itself is a bounded operator acting on $L^2(t_1, \infty)$. In our case, we can evaluate an upper bound for its norm from Eqs. (3.21) and (3.22):

$$\|\underline{K}\| < \|\underline{K}_S\| + \|\underline{K}_r\|_{\text{HS}} < 0.066 + 0.27 \approx 0.34.$$

Then, since $\|\underline{K}\|<1$, the operator $(\underline{1}+\underline{K})$ has a well-defined bounded inverse $(\underline{1}-\underline{R})$. This proves the existence of the bounded (self-adjoint) resolvent $\underline{R}$ and the uniqueness of the solution of the integral equation (3.10) in the space $L^2(t_1, \infty)$.

Furthermore, from the (strongly convergent) Neuman expansion $\underline{R} = \underline{K} - \underline{K}^2 + \underline{K}^3 - \cdots$, we get

$$\|\underline{R}\| < \|\underline{K}\|/(1-\|\underline{K}\|) \lesssim 0.51.$$

This means that the term containing $\underline{R}$ in Eq. (2.18) must appear as a rather small correction, which is confirmed by the detailed numerical analysis. Also the fact that $\|\underline{K}_S\| << \|\underline{K}\|$ means that the singularity of the kernel at $t=t_1$ is harmless in the actual computation of the resolvent (which may be carried out consistently with the method of Ref. 10, for instance). Finally, we have to make sure that $L^2(t_1, \infty)$ is the suitable space in which a solution of Eq. (3.9) has to be found. But this is an immediate consequence of the definition (3.9) and Eqs. (2.10) and (2.11). Indeed, we have

$$|y(t')|^2 \leq \frac{G(t')}{S(t')}\left[\sum_n \alpha_n^2(t')\right]\left[\sum_n \beta_n^2(t')\right] = 2G(t')\left[\sum_n \alpha_n^2(t')\right],$$

which implies

$$\|y\|^2 \leq \frac{\pi}{4\alpha^2}\Delta a_{(\mu)} < \infty.$$

With these ingredients, it is easy to convert the formal derivation given in (i) into a rigorous argument.

IV. NUMERICAL RESULTS AND DISCUSSION

Our main result is stated in Eq. (2.20). The numerical evaluation is straightforward but rather tedious. From the formula for the lower bound, we see that we have to calculate the "moments" I_n and the scalar products $(\sigma_n|\sigma_m)$. For the evaluation of these quantities, we use a method similar to that of Langacker and Suzuki for the treatment of $\pi\pi$ scattering data in the p wave, in order to see the improvement over the simple Schwarz inequality argument. This method consists of the following steps.

(i) Use the experimental p-wave $\pi\pi$ phase shift[11] from the elastic threshold t_0 up to $\sqrt{t_1} \simeq 1.1$ GeV.

(ii) Extrapolate $\sigma_T^1(t)$ from $\sqrt{t_1} \simeq 1.1$ GeV with a horizontal straight line up to $\sqrt{t} \simeq 20$ GeV at which

$$\sigma_T^1(t_1) = \frac{48\pi}{t_1 - t_0}\sin^2\delta_1(t_1)$$

reaches the unitarity limit $48\pi/(t-t_0)$.

(iii) Replace $\sigma_T^1(t)$ by its unitarity limit above $\sqrt{t} = 20$ GeV.

Clearly, step (ii) is not completely secure, because the real $\pi\pi$ cross section could go above the straight line extrapolation. We will come back to this point below.

We first evaluate the functions $S(t)$ and $X(t)$ by using the experimental data, and then apply the algorithm given at the end of Sec. II. As discussed in the previous section, the singular part of the kernel $K(t', t'')$ plays no significant role, so that we can compute the resolvent $R(t', t'')$ by the matrix inversion method. The final result is

$$\Delta a_{(\mu)} \geq 4.5\times 10^{-7}[(\tfrac{1}{6}r_\pi^{\ 2} - 0.0697)^2 \times 21.72 + 2(\tfrac{1}{6}r_\pi^{\ 2} - 0.0697)\times 1.45 + 0.11], \tag{4.1}$$

where r_π is the charge radius of the pion in fermis. This has to be compared with the Langacker-Suzuki bound[12]

$$\Delta a_{(\mu)} \geq 2.61\times 10^{-7} r_\pi^{\ 4}. \tag{4.2}$$

Since the "experimental" value for the charge radius of the pion is still somewhat uncertain,[13] we have plotted both bounds (4.1) and (4.2) as functions of r_π in Fig. 1. For reasonable values of r_π, the improvement of our bound over Eq. (4.2) is not particularly impressive. If $r_\pi = 0.86$ F, for instance,[13] we obtain a gain of ~5%, which is not very significant in view of the uncertainty on the experimental value of r_π and the rather critical dependence of the bound on this variable.

Now we come back to step (ii), namely the extrapolation to the unitarity limit. In the bound derived directly from the Schwarz inequality, the integral between $\sqrt{t_1} \approx 1.1$ GeV to $\sqrt{t} \approx 20$ GeV contributes only 15% of the total integral. This means for instance that if we increase this linear extrapolation by a factor of two, the lower bound decreases by 15%. Roughly the same thing happens with our improved bound. Thus we see that our result must not be very sensitive to the detailed form of the $\pi\pi$ cross section between 1.1 and 20 GeV.

We want to stress here again that even though the numerical improvement over the Schwarz inequality is rather deceptive in this case, the method we have described is flexible enough to make a full use of various pieces of information, and could be used in dealing with other similar problems.

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We are indebted to Professor G. Tiktopoulos for drawing our attention to a useful boundedness criterion.

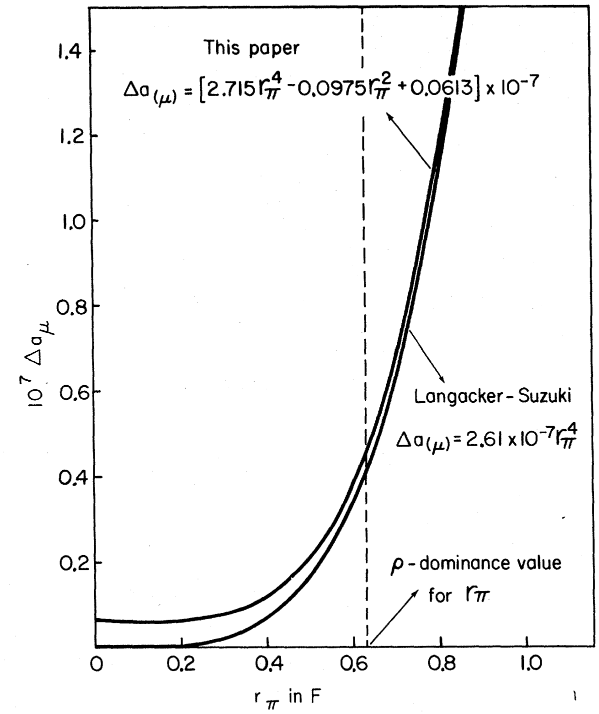

FIG. 1. The lower bound stated in Eq. (4.1) as compared to the bound of Langacker and Suzuki.

APPENDIX

We want to prove that the bound (3.21) holds for the norm of the "singular" kernel $K_S(t', t'')$. This is easily done by using a very simple formula due to Tiktopoulos.[14] It can be written in our case as

$$\|\underline{K}_S\| \leq \operatorname*{Sup}_{t' \geq t_1} \left(\frac{1}{p(t')} \int_{t_1}^{\infty} dt'' |K_S(t', t'')| p(t'') \right), \tag{A1}$$

where $p(t')$ is any positive (measurable) function such that the integral is convergent [$p(t')$ is not required to belong to $L^2(t_1, \infty)$]. Using

$$S(t_1) = 48\sqrt{t_1}\, \sin^2\delta_1(t_1)/(t_1 - t_0)^{3/2}$$

and inserting Eq. (3.20) into Eq. (A1), we obtain, after a simple change of variables,

$$\|\underline{K}_S\| \leq \frac{1}{\pi^2} \sin^2\delta_1(t_1) \operatorname*{Sup}_{x' \geq 0} \left(\frac{x'^{\eta}}{\bar{p}(x')} \int_0^{\infty} dx'' \bar{p}(x'') x''^{\eta} \int_0^{t_1 - t_0} \frac{dx}{x^{2\eta}(x + x')(x + x'')} \right). \tag{A2}$$

If one restricts oneself to a simple class of functions of the form $\bar{p}(x') = x'^{\alpha}$, it turns out that the right-hand side of Eq. (A2) is maximized for $\alpha = -\frac{1}{2}$. With this particular choice, we get

$$\|\underline{K}_S\| \leq \frac{1}{\pi^2} \sin^2\delta_1(t_1) \operatorname*{Sup}_{x' \geq 0} \left(x'^{1/2+\eta} \int_0^{t_1 - t_0} \frac{dx}{x^{2\eta}(x + x')} \int_0^{\infty} \frac{dx''}{x''^{1/2-\eta}(x + x'')} \right)$$

$$= \frac{1}{\pi^2} \sin^2\delta_1(t_1) \operatorname*{Sup}_{x' \geq 0} \left(\int_0^{(t_1 - t_0)/x'} \frac{dy}{y^{1/2+\eta}(1 + y)} \right) \int_0^{\infty} \frac{dy''}{y''^{1/2-\eta}(1 + y'')}$$

$$= \frac{\sin^2\delta_1(t_1)}{\sin[(\frac{1}{2} - \eta)\pi] \sin[(\frac{1}{2} + \eta)\pi]} = \tan^2\delta_1(t_1). \quad \text{Q.E.D.}$$

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Symmetry Breaking and the Pionic Decays of K Mesons

D. N. Goswami*
National Research Council of Canada, Sussex Drive, Ottawa 3, Canada

and

J. Schechter and Y. Ueda†
Physics Department, Syracuse University, Syracuse, New York 13210

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Symmetry-breaking corrections to the current-algebra formulas for $K\to 2\pi$ and $K\to 3\pi$ decays are calculated in the framework of a general form of the linear SU(3) σ model.

I. INTRODUCTION

The first generation of workers on "current algebras" produced some interesting results[1] on the $K\to 2\pi$ and $K\to 3\pi$ decays. Nevertheless these are still among the most mysterious processes in all of particle physics. One of the associated problems is to find the symmetry-breaking corrections to the current-algebra formulas. This has been attacked by many authors with different kinds of results. The basic difficulty is that the situation is rather complicated so that a fairly large number of assumptions must be brought into the picture. In the present paper we shall calculate the symmetry-breaking corrections in the framework of an SU(3) σ model of spin-0 mesons. The advantage of this model[2] is that, while it is realistic enough to give all the current-algebra formulas in the appropriate limits, it is simple enough so that we can perform the calculations in a self-consistent way without introducing extra assumptions. Specifically, we will consider corrections to the $K\to 3\pi$ amplitudes resulting from the SU(3) noninvariance of the "vacuum," and also corrections to the $K\to 2\pi$ amplitudes resulting both from the SU(3) noninvariance and the SU(2) (electromagnetic) noninvariance of the "vacuum." These effects are similar to the so-called "tadpole" effects but *not* to the strangeness-changing tadpoles. In our work we shall assume that the weak nonleptonic interaction is of current-current form and can be effectively represented by a pure octet in SU(3) space.

The main results in this model are as follows:

(1) For $K^+\to\pi^+\pi^0$ the "tadpole" contribution is most likely much too small to explain the entire decay rate.

(2) For $K\to 3\pi$ the effect of symmetry breaking is possibly in the right direction to improve the agreement with experiment.

(3) A comparison of $K\to 3\pi$ and $\eta\to 3\pi$, which has been discussed in the present framework elsewhere,[3] shows that the predicted spectrum shape is the same even though $K\to 3\pi$ arises from a current-current interaction and $\eta\to 3\pi$ arises from a tadpole-type interaction. Thus the apparent experimental similarity of these two spectra need not indicate that both arise from effective inter-