Comments and Addenda

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Equivalence Between the S Matrix and Potential Formalism of K_S -K_L Decay*

Sudhir Kumar
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556
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It is shown that an S matrix for two overlapping resonances can always be written exactly in a form derivable from a potential theory of scattering. The overlap between resonant states and the matrix elements of a Hermitian potential are explicitly constructed in terms of the parameters of the given S matrix.

Some time ago McGlinn and Polis[1] attempted to derive the Bell-Steinberger unitary sum rule[2] directly from a phenomenological S matrix. The difficulty[3] with such an approach lies in describing the two resonances (K_S and K_L) in terms of the single-particle, strong-interaction states, say K^0 and $\bar{K}^0$. The projection operators occurring in the K-matrix formalism were used to define the overlap of these strong-interaction states, which has no direct and trivial relation with the overlap defined in the Bell-Steinberger relation, and as a result a sum rule[1] different from the Bell-Steinberger sum rule was derived. This paper has been widely and reasonably criticized in the literature.[4] It has been shown,[5] however, that the McGlinn-Polis S matrix can be cast exactly into a more commonly used (and fairly general) form given by Durand and McVoy[4]:

$$S(E)=1-i\frac{\Gamma_S g_S \tilde{h}_S}{E-M_S}-i\frac{\Gamma_L g_L \tilde{h}_L}{E-M_L}\,, \tag{1}$$

where $M_i=\mathrm{Re}M_i - i\frac{1}{2}\Gamma_i$, $i=S, L$, and g_i and $\tilde{h}_i$ are column and row vectors representing decay and production amplitudes of these above-mentioned overlapping resonances. We have omitted here the background scattering term for simplicity and, of course, without any loss of generality in the main ideas presented.

It has been shown, interestingly enough, by Stodolsky[6] and by Gien[7] independently that given a Hermitian potential V which connects the states K^0 and $\bar{K}^0$ to the continuum states, a potential theory of scattering (without any reference to perturbation theory) dominated by two resonances, namely, the ones represented by $|S\rangle$ and $|L\rangle$, gives an S matrix of the form

$$S_{\alpha\beta}(E)=\delta_{\alpha\beta}-i\left\langle\alpha\middle| V\left(\frac{1}{E-\underline{\mathfrak{M}}}\right)V\middle|\beta\right\rangle, \tag{2}$$

where $\underline{\mathfrak{M}}$ is a 2×2 complex mass matrix having the right and left eigenvectors $|i\rangle$ and $\langle i'|$, respectively, with the same eigenvalue M_i $(i=S, L)$, $|\alpha\rangle$ $(\alpha=1,\ldots,N)$ refers to a channel state into which the resonance $|i\rangle$ can decay, and the unitarity condition for such an S matrix is nothing but the Bell-Steinberger sum rule[2] itself,

$$i(\underline{\mathfrak{M}}-\underline{\mathfrak{M}}^\dagger)_{ij}=\sum_\alpha\langle i|V|\alpha\rangle\langle\alpha|V|j\rangle \quad (i,j=S, L). \tag{3}$$

It is actually not very difficult to show that an S matrix of the above form, i.e., Eq. (2), can be written in the form of Eq. (1) if one uses the following standard normalization for states[6,7] $|i\rangle$ and $|i'\rangle$:

$$\langle i|i\rangle=\langle i|i'\rangle=1, \qquad \text{for all } i,j=S, L,$$
$$\langle i'|j\rangle=0 \quad (i\neq j),$$

implying

$$|S'\rangle=N(|S\rangle-\langle L|S\rangle|L\rangle), \tag{4a}$$

$$|L'\rangle=N(|L\rangle-\langle S|L\rangle|S\rangle), \tag{4b}$$

where

$$N = (1 - |\langle S|L\rangle|^2)^{-1}, \tag{4c}$$

and the following identification[8]:

$$g_i^\alpha \equiv \Gamma_i^{-1/2}\langle \alpha| V|i\rangle, \tag{5a}$$

$$h_i^\alpha \equiv \Gamma_i^{-1/2}\langle i'| V|\alpha\rangle. \tag{5b}$$

This means that a phenomenological S matrix such as Eq. (1) is derivable from a potential theory of scattering.

In this note we show that even the converse of the above is true, i.e., given an arbitrary S matrix dominated by two resonances, e.g., Eq. (1), the overlap between the two resonant states can be defined such that the Bell-Steinberger sum rule results, and the matrix elements of a Hermitian potential V can be completely determined[9] in terms of the parameters of the given S matrix such that they have standard relations[10] with the particle decay widths.

One can write down the unitarity conditions for Eq. (1) exactly as

$$\tilde{h}_S = N[g_S^\dagger - (\Gamma_L/\Gamma_S)^{1/2}\alpha g_L^\dagger], \tag{6a}$$

$$\tilde{h}_L = N[g_L^\dagger - (\Gamma_S/\Gamma_L)^{1/2}\alpha^* g_S^\dagger], \tag{6b}$$

where

$$\alpha = \frac{i(\Gamma_S\Gamma_L)^{1/2}(g_S^\dagger g_L)}{M_S^* - M_L} \tag{7}$$

and

$$N = (1 - |\alpha|^2)^{-1}, \tag{8}$$

with the normalization for decay amplitudes

$$g_S^\dagger g_S = g_L^\dagger g_L = 1. \tag{9}$$

Now the identification Eq. (5a) alone is enough to show Eq. (5b) through the use of Eqs. (4) and (6) and an identification

$$\alpha \equiv \langle S|L\rangle. \tag{10}$$

Equation (10) is the Bell-Steinberger sum rule itself, and, by Eq. (7), the overlap $\langle S|L\rangle$ is completely defined in terms of the given parameters. It is not difficult to show now that by putting Eqs. (5) into Eq. (1) and using the fact that $1 = \sum_i |i\rangle\langle i'|$, Eq. (2) follows. In the inverse problem nothing has been said so far about the existence or nature of V except that the given parameters g_i^α are identified as the matrix elements $\langle\alpha| V|i\rangle$ through Eq. (5a).

To analyze the problem clearly, it is perhaps best to go to an orthonormal basis from the $|S\rangle, |L\rangle$ basis in which the matrix $\underline{\mathfrak{M}}$ of Eq. (2) is diagonal. A following change of basis will completely determine the transformation matrix in terms of the parameters of the given S matrix:

$$|S\rangle = [1/(1+r^2)^{1/2}](|1\rangle + r|2\rangle), \tag{11a}$$

$$|L\rangle = [1/(1+s^2)^{1/2}](|2\rangle + is|1\rangle), \tag{11b}$$

where r and s are real numbers and can be uniquely determined in terms of $\text{Re}\,\alpha$ and $\text{Im}\,\alpha$, which in turn are known through Eq. (7).

$$\alpha = \frac{r + is}{[(1+r^2)(1+s^2)]^{1/2}}\,. \tag{12}$$

In this new orthonormal basis $S(E)$ can be written for example as

$$S_{\alpha\beta}(E) = \delta_{\alpha\beta} - i\langle\alpha| V(|1\rangle \quad |2\rangle)\left(\frac{1}{E - \underline{\mu}}\right)\begin{pmatrix}\langle 1|\\ \langle 2|\end{pmatrix} V|\beta\rangle, \tag{13a}$$

where

$$\underline{\mu} = \frac{1}{1 - irs}\begin{pmatrix} M_S - irsM_L & is(M_L - M_S) \\ r(M_S - M_L) & M_L - irsM_S \end{pmatrix}. \tag{13b}$$

Through Eqs. (11) and (5a) it is possible to write the matrix elements $\langle\alpha| V|1\rangle$ and $\langle\alpha| V|2\rangle$ in terms of g_S^α and g_L^α as

$$\langle\alpha| V|1\rangle \equiv \frac{1}{1 - irs}[(1+r^2)^{1/2}\Gamma_S^{1/2}g_S^\alpha - r(1+s^2)^{1/2}\Gamma_L^{1/2}g_L^\alpha], \tag{14a}$$

$$\langle\alpha| V|2\rangle \equiv \frac{1}{1 - irs}[(1+s^2)^{1/2}\Gamma_L^{1/2}g_L^\alpha - is(1+r^2)^{1/2}\Gamma_S^{1/2}g_S^\alpha]. \tag{14b}$$

Similarly, using Eqs. (11) in Eqs. (4) and inverting them to give states $\langle 1|$ and $\langle 2|$ in terms of $\langle S'|$ and $\langle L'|$ and then using Eqs. (5b) and the unitarity condition Eqs. (6), one obtains, upon comparison with Eqs. (14),

$$\langle\alpha| V|1\rangle = \langle 1| V|\alpha\rangle^*, \tag{15a}$$

$$\langle\alpha| V|2\rangle = \langle 2| V|\alpha\rangle^*. \tag{15b}$$

Equations (15) clearly mean that the operator V whose matrix elements are the given parameters g_S^α and g_L^α is Hermitian and in Eq. (13a) the matrix elements of the left-hand V are complex conjugates of those of the right-hand V. For consistency it can be checked through Eqs. (14) and (15) that Eq. (13b) satisfies the Bell-Steinberger relation in the new basis:

$$i(\underline{\mu} - \underline{\mu}^\dagger)_{ij} = \sum_\alpha \langle i| V|\alpha\rangle\langle\alpha| V|j\rangle, \quad i,j = 1, 2. \tag{16}$$

This completes the inverse problem in the sense that it is possible to determine the matrix elements of a Hermitian "potential" in terms of the given parameters of a phenomenological S matrix.

It is worthwhile to note in passing that all of the above results are valid even for an S matrix dominated by three resonances, although the algebra is much more complicated. In fact, the results are perhaps true even for N resonances. Finally, inclusion of the background term or final-state interactions does not change the essential part of our arguments in any way.

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[8]Our identification of g_i^α and h_i^α differs from that of Wick or Gien by a factor of $(2\pi)^{1/2}$ but has, however, the same essential content.

[9]These matrix elements are, however, not uniquely determined, mainly due to the fact that in general, the total number of channels available for decay is much greater than the total number of resonances dominating the S matrix.

[10]It will be noted that the $g_i^\dagger g_i = 1$ normalization implies, through the unitarity conditions (6), $h_i^\dagger h_i = N \geq 1$. In fact, h_i's are completely determined by Eqs. (6) in terms of g_i's, and hence an experimental knowledge of the decay amplitudes g_i gives the production amplitudes h_i as well.

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Light-Cone Commutator and Callan-Gross Sum Rule*

S. Y. Lee

Department of Physics, University of California, San Diego, La Jolla, California 92037

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It is shown that the Callan-Gross sum rule can be rederived by the light-cone analysis of the current commutator.

The Callan-Gross sum rule[1] for deep-inelastic electron-proton scattering was first derived by using a dispersion relation and the Bjorken-Johnson-Low theorem. It is very interesting to see that it can also be derived from an entirely different approach, namely, the analysis of the current commutator near the light cone. Let us first consider the structure tensor $W_{\mu\nu}$ of the deep-inelastic e-p scattering defined by

$$W_{\mu\nu} = \frac{1}{2\pi}\int e^{iq\cdot x} d^4x \langle P|[J_\mu(x), J_\nu(0)]|P\rangle$$
$$= \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1$$
$$+ \left(P_\mu - \frac{P\cdot q}{q^2} q_\mu\right)\left(P_\nu - \frac{P\cdot q}{q^2} q_\nu\right)\frac{W_2}{m^2}, \qquad (1)$$

where J_μ is the electromagnetic current and P and q are the momenta of the proton and virtual photon, respectively, and an average over the proton spins is understood.

In the Bjorken scaling limit, we have

$$mW_1 \xrightarrow[\nu\to\infty;\,\omega \text{ fixed}]{} F_1(\omega), \qquad (2)$$

$$\nu W_2 \xrightarrow[\nu\to\infty;\,\omega \text{ fixed}]{} F_2(\omega), \qquad (3)$$

where $\nu = P\cdot q/m$, $\omega = -q^2/2P\cdot q$, $0 \leqslant \omega \leqslant 1$, and m is the mass of the proton. It has been pointed out by several authors[2,3] that in the Bjorken scaling limit one is probing the structure of the current commutator near the light cone, $x^2 \approx 0$. The matrix element of the current commutator near the light cone has been shown to have the following singular structure[2,4]: