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Nonplanar Helicity-Pole Couplings: Duality and the Feynman Graph. I

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We study, in the spirit of Gribov's Reggeon calculus, a particular nonplanar elastic six-point amplitude which contributes to the helicity-pole limit ($s, M^2 \to \infty$, $s/M^2 \to \infty$, and t fixed) of the single-particle distribution. We find "third double-spectral function" effects analogous to those which appear in 2-2 amplitudes. In particular we find (1) nonsense–triple-Regge–wrong-signature fixed poles, and (2) the triple-Pomeranchukon vertex to be finite at $t=0$ if the slope of the trajectory is nonzero and its intercept unity. In addition, we conjecture an asymptotic link between the high-energy Regge limits of ϕ^3 theory amplitudes and the high-energy Regge behavior of dual-tree and dual-loop amplitudes.

I. INTRODUCTION

It has been a tradition in particle physics, ranging over many years, to discuss both the analyticity and high-energy properties of a Feynman graph or some iterative sum of graphs in ϕ^3 theory. As various "new" developments appeared, the ϕ^3 theory was interrogated. One may simply cite, for example, the attempts to "prove" Mandelstam analyticity,[1] Regge behavior,[2] and more recently eikonalization.[3] Certainly the simplicity of the ϕ^3 theory, as compared with the more complex field theories such as quantum electrodynamics or the σ model, makes this ϕ^3 choice a bit irresistible.

In fact quite recently we[4] have again appealed[5] to ϕ^3 theory to investigate a beautiful new development due to Mueller,[6] which relates the single-particle distribution to a well-defined discontinuity of an elastic six-point function. In Ref. 4 we restricted ourselves to a particular sum of planar ϕ^3 Feynman graphs to study, in the strong-coupling regime, the helicity-pole limit[7,8] of the single-particle distribution. (See Fig. 1.) Here we shall investigate the same limit, yet now looking at a *nonplanar* set of graphs.[9] (See Fig. 2.) We obtain new results such as the *non*vanishing of the triple-Pomeranchukon vertex at $t=0$, $\alpha_P(0)=1$, and $\alpha_P' \neq 0$. In addition we see some "old" puzzles analogous to "third double-spectral function"[1] effects appearing in two-to-two amplitudes and which relate to the spurious singularities discussed in Ref. 4.

We have subtitled our paper "Duality and the Feynman Graph," in spite of the fact that none of the graphs considered here are dual, i.e., they do not have Regge behavior for any channel one can reach via crossing, nor is the amplitude identically equal to the sum over the resonances of the amplitude in a given crossed channel. As will be more fully discussed in a subsequent paper, "Pionization Limit for the Single-Particle Distribution: Duality and the Feynman Graph. II," in an asymptotic sense there appears to be a fascinating connection between the ϕ^3 results and the dual model. We shall discuss this asymptotic link in Sec. V.

Before concluding our introductory remarks, we should remind the reader of a long-standing dilemma of the ϕ^3 model, especially since one of our central themes will be abstracting from the theory properties of the dual model. To wit: the theory has no vacuum state.[10] Perhaps in some deep sense the vexing tachyon dilemma of the conventional dual-resonance model is somehow a reflection of the sickness of the ϕ^3 theory. Yet certainly our efforts are ultimately directed toward nature – and thus the set of graphs considered here can *never* literally be taken as a "truth." Hopefully what one can learn by means of abstracting definite characteristics may not be too distant from that "truth."

In Sec. II we define the model, in Sec. III we take the helicity-pole limit, in Sec. IV we discuss the cancellation of spurious singularities, in Sec. V we exhibit some puzzles, and in the Appendix we review briefly the Veneziano transform,[11,12] which we find quite helpful in obtaining our asymptotic results.

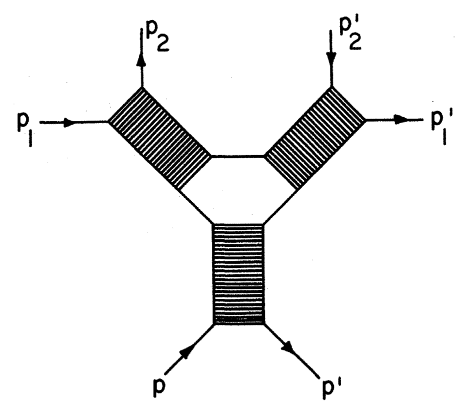

FIG. 1. The planar three-Reggeon graph.

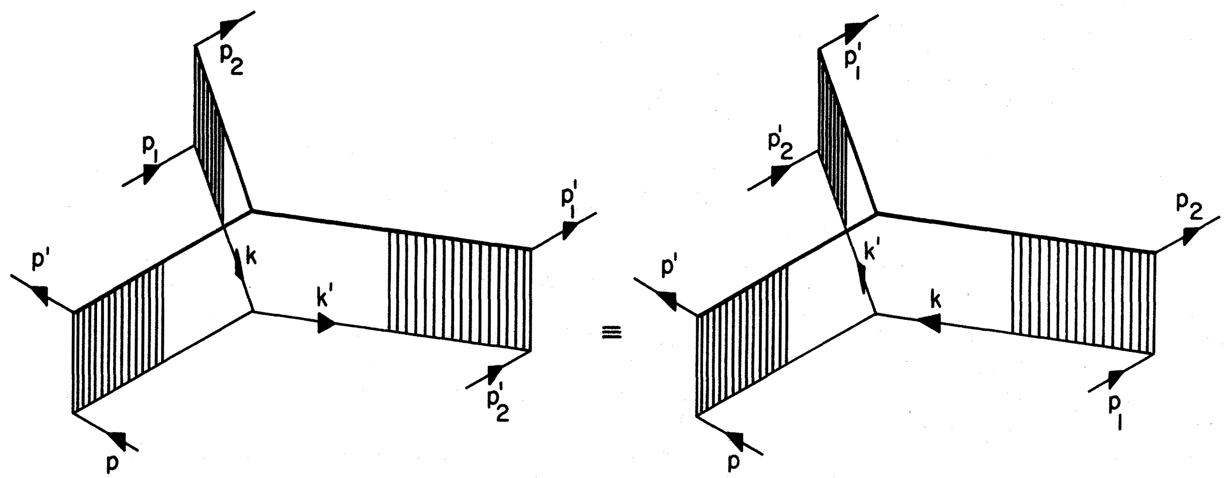

FIG. 2. The nonplanar three-Reggeon graph, for the model calculation considered here. (Note the obvious presence of the left- and right-hand M^2 singularities.)

II. THE MODEL

Our model is based on the diagram of Fig. 2, where for the moment we imagine the three black boxes to correspond to ladder graphs of ϕ^3 theory. Later in Sec. V we shall discuss a more general scheme. We adopt the Regge properties associated with these graphs, and thus have for the *unsignatured* "t"-channel trajectory amplitudes

$$R_t^{(1)} = \frac{-[(k-p_1)^2]^{\alpha_t}}{\sin\pi\alpha_t}\beta^{(1)}(k^2, (p_1-p_2-k)^2, t), \tag{2.1a}$$

$$R_{t'}^{(2)} = \frac{-[(k'-p_2')^2]^{\alpha_{t'}}}{\sin\pi\alpha_{t'}}\beta^{(2)}(k'^2, (p_2-p_1-k')^2, t'), \tag{2.1b}$$

and that amplitude associated with the zero momentum in the elastic limit,

$$R_v = \frac{-[(k-k'-p)^2]^{\alpha_v}}{\sin\pi\alpha_v}\beta((k-k')^2, (k-k'+p_1-p_1'-p_2+p_2')^2, \Delta^2), \tag{2.1c}$$

where

$$\alpha_t = \alpha_t(0)+f(t), \quad \alpha_{t'} = \alpha_t(0)+f(t'), \quad \alpha_v = \alpha_v(0)+\tilde{f}(\Delta^2), \text{ and } \Delta^2 = -(p-p')^2,$$

and at the appropriate time we shall go to the elastic limit in which case all primes on the external momenta may be dropped and $\tilde{f}(\Delta^2)\to 0$.

As in Ref. 4 we shall regard α_v, α_t, and $\alpha_{t'}$ as adjustable parameters. Since we are also anticipating the helicity-pole limit, $M^2\to\infty$, $s/M^2\to\infty$, t fixed, Eqs. (2.1a)–(2.1c) are appropriate approximations.

The 3-3 amplitude, $\mathfrak{M}$, is given by the equation,

$$\mathfrak{M} \sim \int\int d^4k\, d^4k' R_t^{(1)} R_{t'}^{(2)} R_v \frac{1}{k^2+\bar{\mu}_{x_1}{}^2}\frac{1}{k'^2+\bar{\mu}_{y_1}{}^2}\frac{1}{(p_1-p_2-k)^2+\bar{\mu}_{x_2}{}^2}\frac{1}{(p_2'-p_1'+k')^2+\bar{\mu}_{y_2}{}^2}$$
$$\times\frac{1}{(k-k')^2+\bar{\mu}_{z_2}{}^2}\frac{1}{(k-k'+p_1-p_2+p_2'-p_1')^2+\bar{\mu}_{z_1}{}^2}. \tag{2.2}$$

To facilitate the $d^4k\,d^4k'$ integrations we make use[4] of a spectral representation for the integrand based on the following identity:

$$\frac{-(-s)^\alpha}{\sin\pi\alpha} = \frac{1}{\pi}\int_0^\infty ds' \frac{s'^\alpha}{s'-s-i\epsilon} \quad (-1<\alpha<0), \tag{2.3}$$

for α_v, α_t, and $\alpha_{t'}$ in the *open* interval -1 to 0 (later we will analytically continue to the physically interesting region of positive α_v and α_t)[13] and obtain the resulting expression,

$$\mathfrak{M}\sim\int\int d^4k\,d^4k'\int\cdots\int\prod_{i=1}^{2}d\mu_{x_i}{}^2d\mu_{y_i}{}^2d\mu_{z_i}{}^2\;dm_{x_3}{}^2dm_{y_3}{}^2dm_{z_3}{}^2\rho(\mu_{x_1}{}^2,\mu_{x_2}{}^2,\ldots,\mu_{z_1}{}^2,\mu_{z_2}{}^2;t,t')$$
$$\times\frac{(m_{x_3}{}^2)^{\alpha_t}(m_{y_3}{}^2)^{\alpha_{t'}}(m_{z_3}{}^2)^{\alpha_v}}{[(p_1-k)^2+m_{x_3}{}^2][(k'+p_2')^2+m_{y_3}{}^2][(k-k'-p)^2+m_{z_3}{}^2]}$$
$$\times\frac{1}{(k^2+\mu_{x_1}{}^2)(k'^2+\mu_{y_1}{}^2)[(p_1-p_2-k)^2+\mu_{x_2}{}^2][(p_2'-p_1'+k')^2+\mu_{y_2}{}^2]}$$
$$\times\frac{1}{[(k-k')^2+\mu_{z_2}{}^2][(k-k'+p_1-p_2+p_2'-p_1')^2+\mu_{z_1}{}^2]}.\tag{2.4}$$

The sextuple spectral function ρ absorbs the free-particle propagation functions and provides for the off-mass-shell behavior of the Regge residues, $\beta^{(1)}$, $\beta^{(2)}$, and β. As was the case in Ref. 4, we shall see that the superconvergence properties of ρ play a central role in obtaining the familiar $(M^2)^{\alpha_v}(s/M^2)^{2\alpha_t}$ energy dependence of the helicity-pole limit ($s\to\infty$, $s/M^2\to\infty$, t fixed).

Our calculation is now reduced to computing an equivalent Feynman graph corresponding to Fig. 3. For simplicity we have assumed that all external particles are massless. We have checked the calculation for the massive case and found that the external-mass effects play no essential role for the asymptotic properties discussed here.[14] It is to our advantage to compute the graph by means of the Symanzik rules,[15] since they explicitly display the Mandelstam channels present in the graph. A straightforward yet tedious computation yields

$$\mathfrak{M}\sim I_s\int_0^\infty\int_0^\infty\int_0^\infty dm_{x_3}{}^2dm_{y_3}{}^2dm_{z_3}{}^2\int_0^1\cdots\int_0^1\prod_{i=1}^{3}dx_idy_idz_i\,\delta\Big(\sum_{i=1}^{3}(x_i+y_i+z_i)-1\Big)$$
$$\times\frac{C^3}{D^5(s,s';M_s^2,M_u^2;t,t')}(m_{x_3}{}^2)^{\alpha_t}(m_{y_3}{}^2)^{\alpha_{t'}}(m_{z_3}{}^2)^{\alpha_v},\tag{2.5}$$

where the kinematic variables are defined by the relations

$$s=-(p+p_1)^2,\quad s'=-(p'+p_1')^2,$$
$$t=-(p_1-p_2)^2,\quad t'=-(p_1'-p_2')^2,$$
$$M_s^2=-(p+p_1-p_2)^2,\quad M_u^2=-(p'-p_1+p_2)^2,$$

and where I_s is given by the following relation:

$$I_s=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{2}d\mu_{x_i}{}^2d\mu_{y_i}{}^2d\mu_{z_i}{}^2\rho(\mu_{x_1}{}^2,\mu_{x_2}{}^2,\ldots,\mu_{z_1}{}^2\mu_{z_2}{}^2;t,t'),\tag{2.6}$$

with ρ factorizing as

$$\rho=\rho^{(1)}(\mu_{x_1}{}^2,\mu_{x_2}{}^2,t)\rho^{(2)}(\mu_{y_1}{}^2,\mu_{y_2}{}^2,t')\rho^{(3)}(\mu_{z_1}{}^2,\mu_{z_2}{}^2).$$

We note an important symmetry of ρ which follows trivially from our graph (see Fig. 3.) in the *elastic* limit:

$$\rho^{(3)}(\mu_{z_1}{}^2,\mu_{z_2}{}^2)=\rho^{(3)}(\mu_{z_2}{}^2,\mu_{z_1}{}^2),\quad\rho^{(1)}(\mu_{x_1}{}^2,\mu_{x_2}{}^2,t)=\rho^{(2)}(\mu_{y_1}{}^2,\mu_{y_2}{}^2,t),$$

and thus

$$\rho\equiv\rho\text{ under the exchange }y_2\leftrightarrow x_2,\;x_1\leftrightarrow y_1,\text{ and }z_1\leftrightarrow z_2.\tag{2.6$'$}$$

We shall make use of this symmetry in Secs. III and IV. D is given[16] by the equation

$$D=sz_3x_3\Big(\sum_{i=1}^{3}y_i\Big)+s'z_3y_3\Big(\sum_{i=1}^{3}x_i\Big)+M_s^2z_3(x_2y_1-x_1y_3-y_2x_3-x_3y_3)+M_u^2z_3x_1y_2$$
$$+t\Big[x_1x_2\Big(y_3+\sum_{i=1}^{3}z_i+\sum_{i=1}^{2}y_i\Big)+x_1y_3z_3-\tfrac{1}{2}x_3y_3(z_1+z_2)\Big]+t'\Big[y_1y_2\Big(x_3+\sum_{i=1}^{3}z_i+\sum_{i=1}^{2}x_i\Big)+x_3y_2z_3-\tfrac{1}{2}x_3y_3(z_1+z_2)\Big]$$
$$-C\Big[m_{x_3}{}^2x_3+m_{y_3}{}^2y_3+m_{z_3}{}^2z_3+\sum_{i=1}^{2}(\mu_{x_i}{}^2x_i+\mu_{y_i}{}^2y_i+\mu_{z_i}{}^2z_i)\Big],\tag{2.7}$$

and C by the relation

$$C = x_3\left(\sum_{i=1}^{3} z_i + \sum_{i=1}^{3} y_i\right) + y_3\left(\sum_{i=1}^{3} z_i + \sum_{i=1}^{2} x_i\right) + z_3\left(\sum_{i=1}^{2} x_i + \sum_{i=1}^{2} y_i\right) + x_1\left(\sum_{i=1}^{2} y_i + \sum_{i=1}^{2} z_i\right) + x_2\left(\sum_{i=1}^{2} z_i + \sum_{i=1}^{2} y_i\right) + y_1 z_1 + y_2 z_2 . \qquad (2.8)$$

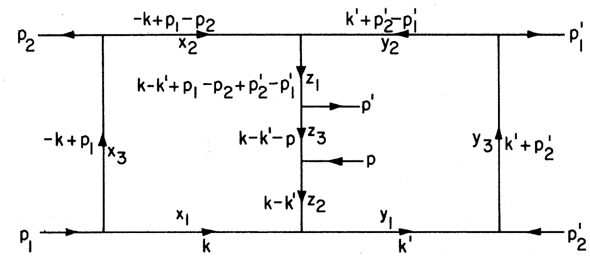

FIG. 3. The reduced equivalent nonplanar three-Reggeon graph.

It is understood that each internal squared mass has implicitly associated Feynman's $-i\epsilon$.

We have found it enormously useful, in both performing m^2 integrations and taking the high-energy limit by means of the Veneziano transform, to reexpress Eq. (2.5) in terms of the Nambu-Schwinger[17] representation. We thus obtain[18]

$$\mathfrak{M} \sim I_s \int_0^\infty \int_0^\infty \int_0^\infty dm_{x_3}^2 dm_{y_3}^2 dm_{z_3}^2 (m_{x_3}^2)^{\alpha_t} (m_{y_3}^2)^{\alpha_{t'}} (m_{z_3}^2)^{\alpha_v} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{3} dx_i dy_i dz_i C^{-2} \exp(D/C) . \qquad (2.9)$$

Using the relation

$$\int_0^\infty dx\, x^{a-1} e^{-xp} = p^{-a}\Gamma(a) , \qquad (2.10)$$

we easily perform the m^2 integrations which yields the equation

$$\mathfrak{M} \sim \Gamma(\alpha_t+1)\Gamma(\alpha_{t'}+1)\Gamma(\alpha_v+1) I_s \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{3} dx_i dy_i dz_i C^{-2} x_3^{-\alpha_t - 1} y_3^{-\alpha_{t'}-1} z_3^{-\alpha_v - 1} \exp\left(\frac{D(m_{x_3}^2 = m_{y_3}^2 = m_{z_3}^2 = 0)}{C}\right) . \qquad (2.11)$$

III. THE ASYMPTOTIC LIMIT

We go to the asymptotic limit in two steps: We first take the s, $s' \to \infty$ limit and then the $M_s^2 \to \infty$ limit. As can be seen upon inserting the kinematic relation

$$M_u^2 = -M_s^2 + t + t' - \Delta^2 \qquad (3.1)$$

into Eq. (2.7) the final limit will be the more subtle since clearly the coefficient of the M_s^2 term in Eq. (2.11) is not positive definite in the domain of integration. We proceed to the s, s' infinite limit.

Using the techniques developed in Refs. 11 and 12, and sketched in the Appendix, we multiple-transform $\mathfrak{M}$ in Eq. (2.11) on s and s' obtaining the expression

$$\tilde{\mathfrak{M}}(M_s^2, t, t'; \tau_s, \tau_{s'}) = \Gamma(\alpha_t+1)\Gamma(\alpha_{t'}+1)\Gamma(\alpha_v+1) I_s \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{3} dx_i dy_i dz_i x_3^{\tau_s - \alpha_t - 1} y_3^{\tau_{s'} - \alpha_{t'} - 1} z_3^{\tau_s + \tau_{s'} - \alpha_v - 1}$$
$$\times (x_1 + x_2 + x_3)^{\tau_{s'}} (y_1 + y_2 + y_3)^{\tau_s} C^{-2-\tau_s - \tau_{s'}} \mathfrak{F}_s^{\tau_s} \mathfrak{F}_{s'}^{\tau_{s'}}$$
$$\times \exp(\bar{D}/C) \exp\left[-z_3 C^{-1}\left(x_3 \sum_{i=1}^{3} y_i + y_3 \sum_{i=1}^{3} x_i\right)\right] , \qquad (3.2)$$

where $\bar{D}$, $\mathfrak{F}_s$, and $\mathfrak{F}_{s'}$ are given by

$$\bar{D} = D(s = s' = 0) ,$$

$$\mathfrak{F}_{s'} = F\left(z_3 y_3\left(\sum_{i=1}^{3} x_i\right) C^{-1}\right) ,$$

and

$$\mathfrak{F}_s = F\left(z_3 x_3\left(\sum_{i=1}^{3} y_i\right) C^{-1}\right) , \qquad (3.3)$$

with F defined by the equation

$$F=\left(\frac{1-e^{-x}}{x}\right). \tag{3.3'}$$

We remark that the $\mathfrak{F}$ functions are well behaved throughout the entire range of integration. Moreover, as can be easily seen, when their arguments approach infinity, the τ_s and $\tau_{s'}$ dependence vanishes from the integrand in Eq. (3.2). Thus all singularities in τ_s and $\tau_{s'}$ can only appear through the first six terms of the integrand of Eq. (3.2).

Transforming $\mathfrak{M}$ in Eq. (3.2) back and taking s and s' to minus infinity we find the following approximate asymptotic expression, viz.,

$$\mathfrak{M}(M_s^2,t,t')\sim\left(\frac{1}{2\pi i}\right)^2\int\int d\tau_s d\tau_{s'}\Gamma(-\tau_s)\Gamma(-\tau_{s'})(-s)^{\tau_s}(-s')^{\tau_{s'}}\Gamma(\alpha_t+1)\,\Gamma(\alpha_{t'}+1)\Gamma(\alpha_v+1)I_s\mathscr{I}, \tag{3.4}$$

where $\mathscr{I}$ is given by,

$$\mathscr{I}=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{3}dx_idy_idz_ix_3{}^{\tau_s-\alpha_t-1}y_3{}^{\tau_{s'}-\alpha_{t'}-1}z_3{}^{\tau_s+\tau_{s'}-\alpha_v-1}$$
$$\times\left(\sum_{i=1}^{3}x_i\right)^{\tau_{s'}}\left(\sum_{i=1}^{3}y_i\right)^{\tau_s}\mathfrak{F}_s{}^{\tau_s}\mathfrak{F}_{s'}{}^{\tau_{s'}}C^{-2-\tau_s-\tau_{s'}}\exp(\overline{D}/C)\exp\left[-z_3C^{-1}\left(x_3\sum_{i=1}^{3}y_i+y_3\sum_{i=1}^{3}x_i\right)\right]. \tag{3.5}$$

We remark that the apparent singularities at $\tau_s,\tau_{s'}=-1,-2,\ldots$ originating from the terms $(y_1+y_2+y_3)^{\tau_s}$ and $(x_1+x_2+x_3)^{\tau_{s'}}$ will be precisely canceled due to the superconvergence properties of the spectral function ρ. The integrand $\mathscr{I}$ in each case, respectively, will be either independent of the mass pairs $(\mu_{y_1}{}^2,\mu_{x_1}{}^2)$, $(\mu_{x_1}{}^2,\mu_{x_2}{}^2)$ or produce powers of μ^2 at the singular points. As was discussed in Ref. 4, at least for ϕ^3 theory, the required superconvergence property indeed keeps pace with singularities generated from the two terms $(y_1+y_2+y_3)^{\tau_s}$ and $(x_1+x_2+x_3)^{\tau_{s'}}$ in Eq. (3.5). A similar argument applies to potential singularities arising when $C\to 0$.

We are thus left with the singularities originating in τ_s and $\tau_{s'}$ appearing when x_3, y_3, and z_3 approach zero. We pick up the residues of the leading ones[19] when x_3 and $y_3\to 0$, to wit:

$$\tau_s=\alpha_t\quad\text{when}\quad x_3\to 0,$$
$$\tau_{s'}=\alpha_{t'}\quad\text{when}\quad y_3\to 0, \tag{3.6}$$

and obtain (we now set $\alpha_t=\alpha_{t'}$) the following relation,

$$\mathfrak{M}(M_s^2,t)\sim\frac{(-s)^{\alpha_t}(-s')^{\alpha_t}}{\sin^2\pi\alpha_t}\Gamma(\alpha_v+1)I_s\mathcal{G}, \tag{3.7}$$

where $\mathcal{G}$ is defined by the integral

$$\mathcal{G}=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{2}dx_idy_i\prod_{i=1}^{3}dz_iz_3{}^{2\alpha_t-\alpha_v-1}[(x_1+x_2)(y_1+y_2)]^{\alpha_t}\hat{C}^{-2-2\alpha_t}\exp(\hat{D}/\hat{C}), \tag{3.8}$$

and where $\hat{C}$ and $\hat{D}$ are given by the equations

$$\hat{C}=z_3\left(\sum_{i=1}^{2}x_i+\sum_{i=1}^{2}y_i\right)+x_1\left(\sum_{i=1}^{2}y_i+\sum_{i=1}^{2}z_i\right)+x_2\left(\sum_{i=1}^{2}z_i+\sum_{i=1}^{2}y_i\right)+y_1z_1+y_2z_2 \tag{3.9}$$

and

$$\hat{D}=M_s^2z_3(x_2y_1)+M_u^2z_3(x_1y_2)+t\left[x_1x_2\left(\sum_{i=1}^{3}z_i+\sum_{i=1}^{2}y_i\right)+y_1y_2\left(\sum_{i=1}^{3}z_i+\sum_{i=1}^{2}x_i\right)\right]-\hat{C}\sum_{i=1}^{2}(\mu_{x_i}{}^2x_i+\mu_{y_i}{}^2y_i+\mu_{z_i}{}^2z_i). \tag{3.10}$$

We note that singularities which would appear when the factors $\hat{C}$, (x_1+x_2), and (y_1+y_2) approach zero in Eq. (3.8) are fake, thanks again to superconvergence.

In turning our attention to the rather subtle limit $M_s^2\to\infty$, a bit more care is required with respect to our transform technique. The two central issues involved are (a) its very existence, and (b) an amusing symmetry of our amplitude which in the *elastic* limit even-signaturizes the amplitude in the Δ^2 channel.

With regard to point (b) we observe that

$$\mathcal{G}(M_s^2)=\mathcal{G}(M_u^2). \tag{3.11}$$

This result can be seen by noting that sending M_s^2 into M_u^2 is equivalent to interchanging $y_2 \leftrightarrow x_2$ and $x_1 \leftrightarrow y_1$, yet apart from the $\hat{C}$ function and the coefficient of $\hat{C}$ in Eq. (3.10), the remaining integrand is invariant under this transformation. However, thanks to the symmetry of ρ, Eq. (2.6′), $\hat{C}$ and the coefficient of $\hat{C}$ in Eq. (3.10) remain unchanged if in addition we interchange $z_1 \leftrightarrow z_2$. Thus the combined transformation

$$\begin{aligned} &y_2 \leftrightarrow x_2\,, \\ &x_1 \leftrightarrow y_1\,, \\ &z_1 \leftrightarrow z_2\,, \end{aligned} \tag{3.12}$$

results in Eq. (3.11).[20]

With respect to point (a), we shall observe that the transform exists in the domain $M_s^2 > 0$, $x_2 y_1 > x_1 y_2$ and $M_s^2 < 0$, $x_1 y_2 > x_2 y_1$. We are thus invited to reexpress Eq. (3.8) as

$$\mathcal{G} = \mathcal{G}^{(+)} + \mathcal{G}^{(-)}\,, \tag{3.13}$$

where

$$\mathcal{G}^{(+)} = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{2} dx_i dy_i \prod_{i=1}^{3} dz_i\, \theta(\beta) R\,, \tag{3.14a}$$

$$\mathcal{G}^{(-)} = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{2} dx_i dy_i \prod_{i=1}^{3} dz_i\, \theta(-\beta) R\,, \tag{3.14b}$$

with β denoting

$$\beta = x_2 y_1 - x_1 y_2\,, \tag{3.15}$$

and R the integrand of $\mathcal{G}$. We note that due to the above θ functions Eq. (3.14a) will have the right-hand cut and Eq. (3.14b) the left-hand cut.[21]

We further define a symmetrical pair of asymptotic variables by the equations

$$\tilde{M}_s^2 = M_s^2 - t$$

and (3.16)

$$\tilde{M}_u^2 = M_u^2 - t$$

and transform separately $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, to wit:

$$\tilde{\mathcal{G}}^{(+)}(\tau, t) = \frac{1}{2\pi i} \int_{-\eta - i\infty}^{-\eta + i\infty} d\tilde{M}_s^2\, \mathcal{G}^{(+)} B(\tilde{M}_s^2 + 1, \tau + 1) \quad (0 < \eta < 1)\,, \tag{3.17}$$

$$\tilde{\mathcal{G}}^{(-)}(\tau', t) = \frac{1}{2\pi i} \int_{-\eta' - i\infty}^{-\eta' + i\infty} d\tilde{M}_u^2\, \mathcal{G}^{(-)} B(\tilde{M}_u^2 + 1, \tau' + 1) \quad (0 < \eta' < 1)\,. \tag{3.18}$$

Upon performing the $d\tilde{M}_s^2$ and $d\tilde{M}_u^2$ integrations we obtain the following transformed amplitudes:

$$\begin{aligned} \tilde{\mathfrak{M}}^{(+)}(\tau, t) = \mathcal{L} I_s \int \cdots \int \prod_{i=1}^{2} dx_i dy_i \prod_{i=1}^{3} dz_i\, \theta(\beta) |\beta|^\tau z_3^{\tau + 2\alpha_t - \alpha_v - 1} \\ \times [(x_1 + x_2)(y_1 + y_2)]^{\alpha_t} \mathcal{F}_{\tilde{M}_s^2}{}^\tau \hat{C}^{-2-2\alpha_t - \tau} \exp(\tilde{D}/\hat{C}) \exp(-z_3 \hat{C}^{-1} |\beta|)\,, \end{aligned} \tag{3.19a}$$

and

$$\begin{aligned} \tilde{\mathfrak{M}}^{(-)}(\tau', t) = \mathcal{L} I_s \int \cdots \int \prod_{i=1}^{2} dx_i dy_i \prod_{i=1}^{3} dz_i\, \theta(-\beta) |\beta|^{\tau'} z_3^{\tau' + 2\alpha_t - \alpha_v - 1} \\ \times [(x_1 + x_2)(y_1 + y_2)]^{\alpha_t} \mathcal{F}_{\tilde{M}_u^2}{}^{\tau'} \hat{C}^{-2-2\alpha_t - \tau'} \exp(\tilde{D}/\hat{C}) \exp(-z_3 \hat{C}^{-1} |\beta|)\,, \end{aligned} \tag{3.19b}$$

where $\tilde{D}$ is given by the expression

$$\tilde{D} = t\left[x_1 x_2 \left(\sum_{i=1}^{3} z_i + \sum_{i=1}^{2} y_i\right) + y_1 y_2 \left(\sum_{i=1}^{3} z_i + \sum_{i=1}^{2} x_i\right) + z_3 (x_2 y_1 + x_1 y_2)\right] - \hat{C} \sum_{i=1}^{2} (\mu_{x_i}{}^2 x_i + \mu_{y_i}{}^2 y_i + \mu_{z_i}{}^2 z_i) \tag{3.20}$$

and $\mathcal{L}$ is given by the relation

$$\mathcal{L}=\frac{(-s)^{\alpha_t}(-s')^{\alpha_t}}{\sin^2\pi\alpha_t}\Gamma(\alpha_v+1)\,. \tag{3.21}$$

Recalling Eq. (3.3′) $\mathfrak{F}_{\tilde{M}_s{}^2}$ and $\mathfrak{F}_{\tilde{M}_u{}^2}$ are both given by the expression

$$\mathfrak{F}_{\tilde{M}_s{}^2}=\mathfrak{F}_{\tilde{M}_u{}^2}=F(z_3\hat{C}^{-1}|\beta|)\,. \tag{3.22}$$

We observe that there are important poles in the τ variable in both $\tilde{\mathfrak{M}}^{(+)}$ and $\tilde{\mathfrak{M}}^{(-)}$ at

$$\left.\begin{matrix}\tau\\ \tau'\end{matrix}\right\}=\alpha_v-2\alpha_t-n\,,\quad n=0,1,2,\ldots \tag{3.23}$$

when $z_3\to 0$ and require no special discussion. However, we will return to them shortly to obtain the helicity-pole limit. The poles which seem to appear when $\beta\to 0$ at

$$\left.\begin{matrix}\tau\\ \tau'\end{matrix}\right\}=-n\,,\quad n=1,2,3,\ldots \tag{3.24}$$

require a more delicate discussion.

Note that at the tip of the cut in the Nambu-Schwinger parameter space ($x_1=x_2=y_1=y_2=0$) the singularities at $\tau=-n$ are canceled due to the superconvergence properties of the spectral function, yet of course the singularities will remain for finite x_1, x_2, y_1, and y_2.[22]

Before transforming $\tilde{\mathcal{G}}^{(+)}$ back in Eq. (3.17) and $\tilde{\mathcal{G}}^{(-)}$ in Eq. (3.18), it is useful to rewrite $|\beta|^\tau\theta(\beta)$ and $|\beta|^\tau\theta(-\beta)$ in terms of generalized functions[23]:

$$\beta_+\equiv|\beta|^\tau\theta(\beta)=\frac{(-)^{n-1}\delta^{(n-1)}(\beta)}{(n-1)!(\tau+n)}+F^+_{-n} \tag{3.25a}$$

and

$$\beta_-\equiv|\beta|^{\tau'}\theta(-\beta)=\frac{\delta^{(n-1)}(\beta)}{(n-1)!(\tau'+n)}+F^-_{-n}\,, \tag{3.25b}$$

where F^+_{-n} and F^-_{-n} are regular at τ, $\tau'=-n$.

The generalized functions permit a simple evaluation of the residues of the poles at τ, $\tau'=-n$ in Eqs. (3.19a) and (3.19b). Those at the even negative integers are zero since apart from the factor $\delta^{(n-1)}(\beta)$ the remaining function is even[24] in β, and hence when integrated over β gives zero.

We thus obtain, for *odd n*,

$$\mathrm{Res}\tilde{\mathfrak{M}}^+(\tau=-n,t)=\mathcal{L}I_s\int\cdots\int\prod_{i=1}^{2}dx_i\prod_{i=1}^{3}dz_i[(x_1+x_2)(y_1+y_2)]^{\alpha_t}z_3{}^{-n+2\alpha_t-\alpha_v-1}$$
$$\times\hat{C}^{-2-2\alpha_t-n}\mathfrak{F}_{M_s}{}^{2-n}\exp(\tilde{D})\left[\frac{(-)^{n-1}\delta^{(n-1)}(\beta)}{(n-1)!}\right], \tag{3.26a}$$

and

$$\mathrm{Res}\tilde{\mathfrak{M}}^-(\tau'=-n,t)=\mathcal{L}I_s\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{2}dx_i\prod_{i=1}^{3}dz_i[(x_1+x_2)(y_1+y_2)]^{\alpha_t}z_3{}^{-n+2\alpha_t-\alpha_v-1}$$
$$\times\hat{C}^{-2-2\alpha_t-n}\mathfrak{F}_{M_u}{}^{2-n}\exp(\tilde{D})\left[\frac{\delta^{(n-1)}(\beta)}{(n-1)!}\right]. \tag{3.26b}$$

Equations (3.26a) and (3.26b) lead to the following asymptotic results:

$$\mathfrak{M}_F^{(+)}\sim\mathcal{L}I_s\sum_{n(\mathrm{odd})=1}^{\infty}(-M_s{}^2)^{-n}\mathrm{Res}\mathfrak{M}^+(\tau=-n,t)\Gamma(n) \tag{3.27a}$$

and

$$\mathfrak{M}_F^{(-)}\sim\mathcal{L}I_s\sum_{n(\mathrm{odd})=1}^{\infty}(M_s{}^2)^{-n}\mathrm{Res}\mathfrak{M}^-(\tau'=-n,t)\Gamma(n)\,, \tag{3.27b}$$

which we recognize as a string of fixed poles at nonsense–wrong-signature points. Here is our first "third double-spectral-function effect" as was mentioned in the Introduction. Moreover, we note that because M_s^2 is associated with integral powers, $\mathfrak{M}_F^{(+)}$ and $\mathfrak{M}_F^{(-)}$ cannot contribute to the M_s^2 or M_u^2 discontinuity, and clearly not to the asymptotic behavior of the inclusive single-particle cross section. Furthermore, as may be easily seen, the sum, $\mathfrak{M}_F^{(+)}+\mathfrak{M}_F^{(-)}$, is identically zero. Thus the *full* amplitude has *no* fixed-power behavior. The fixed poles, nevertheless, make their presence known as was the case for the 2-2 amplitudes in modifying the structure of the Regge residue function from that which was obtained for the planar case. This will become evident as we now turn to the singularities which are generated when z_3 vanishes in Eqs. (3.19a) and (3.19b).

Upon evaluating the residue of the leading[25] singularity generated when $z_3\to 0$, in Eqs. (3.19a) and (3.19b), we find the following asymptotic behavior for their contribution to $\mathfrak{M}^{(+)}$ and $\mathfrak{M}^{(-)}$:

$$\mathfrak{M}_R^{(+)}\sim\frac{(-s)^{\alpha_t}(-s')^{\alpha_t}}{\sin^2\pi\alpha_t}(-\tilde{M}_s^2)^{\alpha_v-2\alpha_t}\Gamma(2\alpha_t-\alpha_v)\,I_sI_+ \tag{3.28a}$$

and

$$\mathfrak{M}_R^{(-)}\sim\frac{(-s)^{\alpha_t}(-s')^{\alpha_t}}{\sin^2\pi\alpha_t}(-\tilde{M}_u^2)^{\alpha_v-2\alpha_t}\Gamma(2\alpha_t-\alpha_v)\,I_sI_-\,, \tag{3.28b}$$

where

$$I_\pm=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{2}dx_idy_idz_i\beta_\pm^{\alpha_v-2\alpha_t}(\hat{C}_{|z_3=0})^{-2-\alpha_v}[(x_1+x_2)(y_1+y_2)]^{\alpha_t}$$
$$\times\exp\left(\left\{t\left[x_1x_2\left(\sum_{i=1}^2 z_i+\sum_{i=1}^2 y_i\right)+y_1y_2\left(\sum_{i=1}^2 z_i+\sum_{i=1}^2 x_i\right)\right]-\hat{C}_{|z_3=0}\sum_{i=1}^2(\mu_{x_i}^2x_i+\mu_{y_i}^2y_i+\mu_{z_i}^2z_i)\right\}(\hat{C}_{|z_3=0})^{-1}\right). \tag{3.28c}$$

We note that in Eq. (3.28a) $\mathfrak{M}_R^{(+)}$ will have the right-hand M^2 cut, and in Eq. (3.28b) $\mathfrak{M}_R^{(-)}$ will have the left-hand M^2 cut.

We can now trivially take, e.g., the M^2 discontinuity across the right-hand cut and continue s above the right-hand complex s-plane cut and s' below the complex s-plane cut and obtain

$$\mathfrak{A}^{(+)}\sim\left(\frac{s}{M_s^2}\right)^{2\alpha_t}\frac{(M_s^2)^{\alpha_v}}{\sin^2\pi\alpha_t}\frac{1}{\Gamma(1+\alpha_v-2\alpha_t)}I_s$$
$$\times\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^{2}dx_idy_idz_i\beta_+^{\alpha_v-2\alpha_t}[(x_1+x_2)(y_1+y_2)]^{\alpha_t}(\hat{C}_{|z_3=0})^{-2-\alpha_v}$$
$$\times\exp\left(\frac{1}{\hat{C}_{|z_3=0}}\left\{t\left[x_1x_2\left(\sum_{i=1}^2 z_i+\sum_{i=1}^2 y_i\right)+y_1y_2\left(\sum_{i=1}^2 z_i+\sum_{i=1}^2 x_i\right)\right]-\hat{C}_{|z_3=0}\sum_{i=1}^2(\mu_{x_i}^2x_i+\mu_{y_i}^2y_i+\mu_{z_i}^2z_i)\right\}\right), \tag{3.29}$$

which is, of course, directly proportional to the single-particle distribution, and is moreover the familiar helicity-pole limit.[7]

We observe that if we decree α_t and α_v to be the Pomeranchukon trajectories, the vanishing of the triple-Pomeranchukon vertex at $t=0$, i.e., the zero obtained previously in Refs. 4 and 26 will no longer hold due to the factor $\beta_+^{\alpha_v-2\alpha_t}$ which is singular when $\alpha_v-2\alpha_t=-1$. Furthermore, it is amusing to note that we will still have a *vanishing* result when the exponent $\alpha_v-2\alpha_t$ is a negative even integer for precisely the same reason that $\mathfrak{M}_F^{(+)}$ and $\mathfrak{M}_F^{(-)}$ had only wrong-signature fixed poles.

We remark that the potential singularities due to the terms $[(x_1+x_2)(y_1+y_2)]^{\alpha_t}$ and $\hat{C}^{-2-\alpha_v}$ will be canceled due to the superconvergence properties of I_s.

Finally, it is appropriate to mention at this stage the external masses which were taken to zero for reasons of simplicity of presentation. We have seen that the asymptotic limit of the absorptive parts of both $\mathfrak{M}_R^{(+)}$ and $\mathfrak{M}_R^{(-)}$ arose from singularities in the τ variables when x_3, y_3, and eventually z_3 were taken to zero. One can easily check that every external-mass factor will have either x_3, y_3, or z_3 as an over-all multiplicative factor, and thus for the helicity-pole limit the external-mass factors would eventually vanish from the final answer.

IV. THE CANCELLATION OF SPURIOUS POLES

As is quite evident from Eqs. (3.28a) and (3.28b), $\mathfrak{M}_R^{(+)}$ and $\mathfrak{M}_R^{(-)}$ in fact appear to have spurious singularities at

$$2\alpha_t - \alpha_v = 0, -1, -2, \ldots . \tag{4.1}$$

Using the relation

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

we rewrite $\mathfrak{M}_R^{(+)}$ and $\mathfrak{M}_R^{(-)}$ as

$$\mathfrak{M}_R^{(+)} \sim \frac{(-s)^{\alpha_t}(-s')^{\alpha_t}(-M_s^2)^{\alpha_v - 2\alpha_t}\pi}{\sin^2 \pi\alpha_t \sin\pi(2\alpha_t - \alpha_v)\Gamma(1+\alpha_v - 2\alpha_t)} I_s I_+ \tag{4.2a}$$

and

$$\mathfrak{M}_R^{(-)} \sim \frac{(-s)^{\alpha_t}(-s')^{\alpha_t}(M_s^2)^{\alpha_v - 2\alpha_t}\pi}{\sin^2 \pi\alpha_t \sin\pi(2\alpha_t - \alpha_v)\Gamma(1+\alpha_v - 2\alpha_t)} I_s I_- , \tag{4.2b}$$

and observe that again due to the symmetry of the amplitude under the transformation

$$\beta \to -\beta \text{ and } z_1 \leftrightarrow z_2, \tag{4.3}$$

there are, in fact, *no* spurious singularities at $2\alpha_t - \alpha_v = -1, -3, -5, \ldots,$ in the sum $\mathfrak{M}_R^{(+)} + \mathfrak{M}_R^{(-)}$.

The physical amplitude is, of course, the sum of Eqs. (4.2a) and (4.2b). However, thanks to the properties of β_+ and β_-[see Eq. (3.25a) and (3.25b)], the spurious singularities cancel upon summation.

Note added in proof. We thank J. H. Weis for pointing out that our conclusions reached in the preceding paragraph are wrong and furthermore might invalidate the concluding paragraph of *this* section. Equations (3.25a) and (3.25b) were misapplied in this context. At issue is the full singularity structure in the τ_s and $\tau_{s'}$ planes which may be exposed by examining in detail Eq. (3.2). We have examined [see Eq. (3.6)] the singularities at $\tau_s = \alpha_t$ and $\tau_{s'} = \alpha_{t'}$. Clearly there are others which will play a role in compensating the spurious singularities discussed here. We stress that these remarks do not alter any conclusions reached in the other sections of this paper.

This mechanism of cancellation is quite different from the planar case,[4] in which there was only a right-hand cut and the spurious singularities were canceled by means of a compensation mechanism. Here, however, both the amplitudes associated with the right- and left-hand cuts have spurious poles, and cancellation only occurs upon summation.

V. PUZZLES AND COMMENTS

As is well known,[27-29] consistency between an assumed Pomeranchukon-dominated constant total cross section $[\alpha_P(0)=1]$ and the nonvanishing of the triple-Pomeranchukon vertex at $t=0$ is an impossibility. Thus under the above assumptions, our final result, Eq. (3.29), is clearly incomplete. Certainly the simplest resolution (aside from decreeing that the unknown spectral function itself has an over-all zero at $t=0$) is to let the Pomeranchukon intercept lie below one by a small amount η. In this case η would be perhaps a quite fundamental, positive parameter as is the case in the schizophrenic Pomeranchukon model of Chew and Snyder.[30] Our graph might represent some small additional term to be considered within their framework.

At present, direct experimental evidence of a deviation from unity for the Pomeranchukon intercept is nonexistent. We feel a confrontation with this puzzle at this time is not at all an academic exercise. In fact, quite recently Abarbanel and Green[31] have addressed themselves to this issue by considering effects generated by inserting a single Regge cut in the vacuum channel (α_v) of the elastic six-point amplitude. We certainly share with them the attitude that Regge cuts must play a central role in the resolution of the puzzle (which for them involved "proving" that the residue of the nonsense–wrong-signature fixed pole vanishes at $t=0$), yet we find their argumentation incomplete. At issue is the frightening collision of singularities at $t=0$, viz., the fixed-pole, three-Pomeranchukon trajectories, and a three-fold infinity of Regge cuts. It appears to us that a systematic approach to the disentanglement of these singularities is required. Such a program has been carried out recently in a fascinating sequence of papers by Bronzan[32] and by Bronzan and Hui,[33] and earlier, using quite different techniques, by Gribov[34] and by Gribov and Migdal[35] for the elastic 2-2 amplitude. An investigation in the same spirit might now be appropriate for the elastic 3-3 amplitude.

In Sec. II we promised to suggest a somewhat more general model calculation than the one performed here. We have always had in mind the ϕ^3 theory, that is to say, the black boxes represented an iterative sum of ladder graphs, and the spectral function the solution to the ϕ^3 Bethe-Salpeter equation. Recently, Scherk[36] has discovered that a well-defined zero-slope limit of the dual-resonance perturbation expansion reduces to the Feynman-Dyson expansion of ϕ^3 theory.[37] Furthermore, he noted that the Pomeranchukon singularity vanishes in the limit. In the dual model the Pomeranchukon has an identifiable mathematical representation,[38] related to the experimentally sound[39] Freund-Harari[40] hypothesis. It appears that none of the ϕ^3 ladders can ever actually represent the Pomeranchukon in the sense suggested by Freund and Harari.

We are thus led to take the black boxes (in the spirit of Gribov) to actually represent the Pomeranchukon, and consequently are left at this stage with truly unknown spectral functions whose superconvergence properties must be assumed[41] in order to obtain the helicity-pole limit at $t=0$, $\alpha_P(0)=1$. Moreover, as will be discussed in a subsequent paper, "Pionization Limit for the Single-Particle Distribution: Duality and the Feynman Graph. II," our initial assumption of incorporating the free propagation functions for the scalar particles of the ϕ^3 graph into the spectral function appears to us too restrictive. We feel a somewhat more realistic model should include a far richer spectrum of particle states propagating along the lines labeled x_1, x_2; y_1, y_2; and z_1, z_2, in the graph of Fig. 3. This appears quite important if we are to compute other limits of the single-particle spectrum, such as for example, the asymptotic transverse-momentum distribution in the pionization limit.

We conclude with a conjecture (which in view of the recent work of Scherk[36] might possibly be not too difficult to prove), concerning an asymptotic link between ϕ^3 theory and the conventional dual model. We have observed that Regge limits (obtained from sums of ladders)[42] of *planar* ϕ^3 diagrams calculated in the strong-coupling regime have the form

$$a_{\text{planar}} \sim R \,⨍\, FX, \qquad (5.1)$$

where R symbolizes the Regge asymptotic power, F a known function which is *identical* to that obtained from the dual tree model, X the spectral function[43] (i.e., the solution of the Bethe-Salpeter equation), and $⨍$ indicates either a kind of convolution involving FX, or indeed for some limits there may be no convolution at all. For example, in Ref. 4 F was given by $\Gamma^{-1}(1+\alpha_v-2\alpha_t)\times\sin^{-2}(\pi\alpha_t)$.[44,45]

We further conjecture that a form similar to Eq. (5.1) holds for the Regge limits of nonplanar ϕ^3 configurations (such as discussed here), i.e.,

$$a_{\text{nonplanar}} \sim R' \,⨍\, F'X, \qquad (5.2)$$

where F' is essentially *identical* to the residue function obtained from the Regge limit of lowest-order dual loop or sum of loops[46,47] which asymptotically has the same Mandelstam channel as the reduced equivalent ϕ^3 graph (e.g., Fig. 3) and R' is the Regge power associated with the *non*-Pomeranchukon contribution to the limit. One of the possible lowest-order dual loop diagrams which is applicable here (there are certainly others) is given in Fig. 4. We note in passing that this graph has already been considered in Ref. 48 as an important contribution to the fragmentation limit of the single-particle distribution.

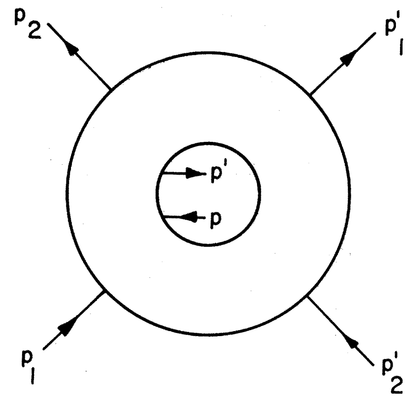

FIG. 4. A nonplanar dual amplitude (see Sec. V of text).

We began our paper with remarks concerning the interrogation of the ϕ^3 theory with regard to various "new" developments, which appeared from time to time, in our gradual attempts toward a fundamental understanding of that vexing yet beautiful aspect of nature – the world of hadrons. As we have seen, the link between the ϕ^3 theory and the dual-resonance model is indeed nontrivial. Moreover, the latter model appears to us to be a far more realistic representation of the experimental facts of the hadronic world. Hence, in the future we hope that the conventional dual-resonance model will be put, more frequently than is done at present, to that same important chore – the testing out of new ideas.

Note added: It has been pointed out to us that a similar model calculation of the same nonplanar graph (see Fig. 2) using quite different mathematical methods has been performed by Mueller and Trueman.[49] Their conclusions with regard to the nonvanishing (vanishing) of the helicity-pole vertex function at $\alpha_v-2\alpha_t=-1, (-2), -3, (-4), \ldots$ are identical to ours.

ACKNOWLEDGMENTS

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APPENDIX

In 1969 Veneziano proposed[11] the beta-function transform to study the J-plane analyticity structure of the four-point dual amplitude (at that time the dual n-point functions were in their infancy). More recently we[12] have generalized the transform in a rather straightforward manner into a multiple transform, and have found that the multiple transform of the n-point dual amplitude is again an n-point dual amplitude with well-defined shifts in the trajectory intercepts associated with the transform variables and consequently permitting a rather simple evaluation of Regge asymptotic limits.

It had not occurred to us that an object so closely indentified with the dual model should prove useful in other areas, such as taking the asymptotic limit of Feynman graphs. Yet, as we hope to have convinced the reader, it is indeed useful and in some respects perhaps one of the simplest transform devices to make use of in taking rather involved asymptotic limits.

Clearly the transform, which we shall define below, needs a far more thorough mathematical investigation than exists to date. Moreover, since after all its kernel is basically a kind of analytic continuation of the reciprocal of a binomial coefficient, we feel that the transform may find use in areas far afield from dual models, Feynman diagrams, inclusive amplitudes, etc. Below we shall define the transform, and inverse-transform, and as a trivial example of its applicability apply it to find the asymptotic limit ($s\to\infty$, t fixed) of the ϕ^3 box diagram. Finally, we shall define the multiple transform.

The transformed function $\tilde{V}$ is given by[11,50]

$$\tilde{V}(\tau, x)=\frac{i}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty} ds\, V(s,x)B(\tau+1, s+1) \quad (0<\eta<1), \tag{A1}$$

and we invert $\tilde{V}(t,x)$ by

$$V(s,x)=\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} d\tau\, \tilde{V}(\tau,x)B(-\tau,-s) \quad (0<\epsilon<1). \tag{A2}$$

$B(x,y)$ is the Euler beta function, s the asymptotic variable of interest, and x denotes collectively those variables which are kept fixed.

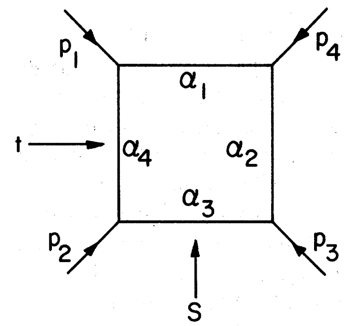

FIG. 5. The ϕ^3 box diagram (see Appendix).

We observe from Eq. (A2) that when $-s\to\infty$ we have the asymptotic result

$$\lim_{-s\to\infty} V(s,x)\sim\frac{1}{2\pi i}\int_{-\epsilon-i\infty}^{-\epsilon+i\infty} d\tau\, \tilde{V}(\tau,x)(-s)^{\tau}\Gamma(-\tau), \tag{A3}$$

and thus the burden of finding the asymptotic result rests on the singularity structure in the τ variable of $\tilde{V}$.

For the purposes of finding the asymptotic limit of a Feynman graph, it proves useful to make use of the Nambu-Schwinger representation of the graph, and the following integral representation of the beta function:

$$B(x,y)=\int_0^\infty dr\, e^{-xr}(1-e^{-r})^{y-1}. \tag{A4}$$

We consider the Feynman amplitude, M_B, for the box diagram and for simplicity set the external masses to zero (see Fig. 5):

$$M_B\sim\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^4 d\alpha_i C^{-2}\exp(D/C), \tag{A5}$$

where D and C are defined by the equations

$$D=\alpha_2\alpha_4 s+\alpha_1\alpha_3 t-\Big(\sum_{i=1}^4 \alpha_i m_i^2\Big)C, \tag{A6a}$$

$$C=\sum_{i=1}^4 \alpha_i. \tag{A6b}$$

Using Eq. (A1) we have

$$\tilde{M}_B(\tau,t)=\int_0^\infty\cdots\int_0^\infty \prod_{i=1}^4 d\alpha_i C^{-2}\exp\left\{\frac{\Big[\alpha_1\alpha_3 t - C\Big(\sum_{i=1}^4 \alpha_i m_i^2\Big)\Big]}{C}\right\} \times\frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty} ds\int_0^\infty dr\; e^{-s}\left(r-\frac{\alpha_2\alpha_4}{C}\right)e^{-r}(1-e^{-r})^{\tau}. \tag{A7}$$

We evaluate the integral

$$\frac{1}{2\pi i}\int_{-\eta-i\infty}^{-\eta+i\infty} ds\, e^{-s}\left(r-\frac{\alpha_2\alpha_4}{C}\right) \tag{A8}$$

by means of a Wick-like rotation, i.e., we define $s=+i|s|$, and obtain

$$\frac{1}{2\pi}\int_{i\eta-\infty}^{i\eta+\infty} d|s|\, e^{-i|s|\left(\frac{\alpha_2\alpha_4}{C}-r\right)}=\delta\left(r-\frac{\alpha_2\alpha_4}{C}\right). \tag{A9}$$

The integration in Eq. (A7) now becomes trivial and we have

$$\tilde{M}_B=\int_0^\infty\cdots\int_0^\infty\prod_{i=1}^4 d\alpha_i C^{-2-\tau}(\alpha_2\alpha_4)^\tau e^{-\alpha_2\alpha_4/C}\left(\frac{1-e^{-\alpha_2\alpha_4/C}}{\alpha_2\alpha_4/C}\right)^\tau \exp\left\{\frac{\left[\alpha_1\alpha_3 t-C\left(\sum_{i=1}^4\alpha_i m_i^2\right)\right]}{C}\right\} \tag{A10}$$

We see from Eq. (A10) that the leading τ singularity is a double pole at

$$\tau=-1 \tag{A11}$$

when α_2 and $\alpha_4\to 0$.

Inverting $\tilde{M}_B$ by means of Eq. (A2) and picking up the residue of the double pole, we obtain the leading asymptotic term

$$\lim_{-s\to\infty}\sim\frac{\ln s}{s}\int_0^\infty\int_0^\infty d\alpha_1 d\alpha_3\bar{C}^{-1}\exp\left[\frac{\alpha_1\alpha_3 t-\bar{C}(\alpha_1 m_1^2+\alpha_3 m_3^2)}{\bar{C}}\right], \tag{A12}$$

where $\bar{C}$ is given by

$$\bar{C}=\alpha_1+\alpha_3\,. \tag{A13}$$

This is, of course, a cumbersome method – to say the very least – to obtain the result depicted in Eq. (A12). We believe its utility is borne out when there are several asymptotic variables to be dealt with. Thus we define the multiple transform[12] and make use of it in Sec. III. To wit: For n asymptotic variables $s_1,\ldots,s_n$ the multiple transform is defined by

$$\tilde{V}(\tau_1,\tau_2,\ldots,\tau_n;X)=\frac{1}{2\pi i}\int_{-\eta_n-i\infty}^{-\eta_n+i\infty}\cdots\int_{-\eta_1-i\infty}^{-\eta_1+i\infty}\prod_{i=1}^n ds_i V(s_1,s_2,\ldots,s_n;X)\prod_{i=1}^n B(\tau_i+1,s_i+1)$$
$$(0<\eta_i<1;\ i=1,\ldots,n), \tag{A14}$$

and the inverse is defined by

$$V(s_1,s_2,\ldots,s_n;X)=\frac{1}{2\pi i}\int_{-\epsilon_n-i\infty}^{-\epsilon_n+i\infty}\cdots\int_{-\epsilon_1-i\infty}^{-\epsilon_1+i\infty}\prod_{i=1}^n d\tau_i\tilde{V}(\tau_1,\tau_2,\ldots,\tau_n;X)\prod_{i=1}^n B(-\tau_i,-s_i)$$
$$(0<\epsilon_i<1;\ i=1,\ldots,n), \tag{A15}$$

where again X denotes collectively the variables which are to be held fixed. Care must be taken when one uses (A15) where kinematic constraints require ratios of asymptotic variables to approach a limit (when the limit is unity, things can become quite tricky). If a discontinuity is to be taken, one *first* takes the discontinuity and then imposes the kinematic constraint. For the purpose of this paper, this word of caution really never arises, but it has been noted[51] that for the two-particle distribution such subtleties do indeed appear.

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[1]See, e.g., R. J. Eden, P. V. Landshoff, D. I. Olive, and J. C. Polkinghorne, *The Analytic S-Matrix* (Cambridge Univ. Press, Cambridge, England, 1966).

[2]R. J. Eden *et al.*, *The Analytic S-Matrix*, Ref. 1, and references cited therein.

[3]See, e.g., G. Tiktopoulos and S. B. Treiman, Phys. Rev. D 3, 1037 (1971); E. Eichten and R. Jackiw, *ibid.* 4, 439 (1971), and references cited therein.

[4]Shau-Jin Chang, David Gordon, F. E. Low, and S. B. Treiman, Phys. Rev. D 4, 3055 (1971); see also J. D. Dorren, Nucl. Phys. B36, 541 (1972), for the weak-coupling-limit calculation of the planar diagram.

[5]We believe our calculation need not be restricted to the ladder graphs of ϕ^3 theory, i.e., the essential requirements are that the black boxes have Regge behavior and the spectral functions satisfy a well-defined set of superconvergence conditions.

[6]A. H. Mueller, Phys. Rev. D 2, 2963 (1970).

[7]C. E. DeTar, C. E. Jones, F. E. Low, C.-I Tan, J. H. Weis, and J. E. Young, Phys. Rev. Letters 26, 675 (1971); C. E. Jones, F. E. Low, and J. E. Young, Phys. Rev. D 4, 2358 (1971).

[8]We use the term helicity-pole limit to mean $s\to\infty$, $M^2\to\infty$, $s/M^2\to\infty$, and t fixed. This is not to be con-

fused with the quite different triple-Regge limit in which a nonforward six-point amplitude has *six* channel variables taken to infinity. For an interesting discussion concerning the relation between these two limits see C. E. DeTar and J. H. Weis, Phys. Rev. D 4, 3141 (1971).

[9]We have chosen a particularly manageable subset of nonplanar graphs. There are certainly more involved nonplanar graphs such as those containing cuts in 25 Mandelstam channels (the maximal number) for the nonforward amplitude. We have no idea what new features might or might not emerge upon considering such complex graphs.

[10]W. Thirring, Helv. Phys. Acta 26, 33 (1953); we thank T. D. Lee and S. B. Treiman for pointing out this reference.

[11]G. Veneziano, in *Fundamental Interactions at High Energy I,* based on the proceedings of the 1969 Coral Gables Conference on Fundamental Interactions at High Energy, edited by T. Gudehus, G. Kaiser, and A. Perlmutter (Gordon and Breach, New York, 1969), p. 113.

[12]D. Gordon, Nuovo Cimento 6A, 107 (1971).

[13]As discussed in Ref. 4, continuation below −1 in α_v and α_t is prohibited since we would obtain contributions from the nonleading pieces (the Regge daughters) in the Regge black boxes.

[14]See the last paragraph of Sec. III. Strictly speaking, at $t=0$ we should retain some external masses for otherwise there would be no separation of the left- and right-hand M^2 cuts at $t=0$.

[15]K. Symanzik, Progr. Theoret. Phys. (Kyoto) 20, 690 (1958). We thank B. Hasslacher and D. K. Sinclair for discussions concerning the Symanzik rules.

[16]We have not included in the D function the two-particle $-(p-p')^2$ channel, nor any three-particle channel which yields a vanishing contribution to the elastic limit.

[17]Y. Nambu, Nuovo Cimento 25, 1292 (1962), and see, e.g., J. Schwinger, Phys. Rev. 82, 664 (1951).

[18]We thank S. D. Ellis and S. B. Treiman for discussions concerning the transition from the Feynman to the Nambu-Schwinger representation. See also R. J. Eden *et al.*, Ref. 1, p. 152. The actual representation we make use of is analogous to the Laplace representation as opposed to the Fourier version of the amplitude.

[19]An evaluation of the $z_3 \rightarrow 0$ contribution, which is a bit more subtle leads directly to an M^2-independent result, i.e., the amplitude, $\mathfrak{M}$, behaves like s^{α_v} in the elastic limit. A very similar phenomenon occurs for the planar graph considered in Ref. 4.

[20]These symmetries can be seen quite easily by inspecting the graph depicted in Fig. 3, with, however, the lines x_3 and y_3 contracted.

[21]We thank A. Sanda for several discussions concerning the separation of left- and right-hand M^2 cuts. We refer the reader to an interesting discussion by A. Sanda, Phys. Rev. D (to be published), on the analyticity properties of the six-point amplitude, which relates to the helicity-pole limit.

[22]It is amusing to note that the vanishing of the discontinuity across the tip of the cut in the Nambu-Schwinger parameter space is somewhat analogous in many respects to the well-known tip of the cut theorem of J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967), where, of course, at issue there was the analytic structure of an *isolated* J-plane cut.

[23]I. M. Gel'fand and G. E. Shilov, *Generalized Functions* (Academic, New York, 1964), Vol. I, pp. 48–50. We thank B. Lee for interesting discussions on generalized function theory.

[24]The symmetry in β follows from our discussion of the symmetry: $\mathcal{G}(M_s^2)=\mathcal{G}(M_u^2)$.

[25]Retaining nonleading terms here appears inconsistent with our assumption that the Regge boxes are governed by the leading Regge singularity (see Ref. 13).

[26]D. Gordon and G. Veneziano, Phys. Rev. D 3, 2116 (1971); M. A. Virasoro, *ibid.* 3, 2834 (1971); C. E. DeTar, K. Kang, C.-I Tan, and J. H. Weis, *ibid.* 4, 425 (1971); R. C. Brower and R. E. Waltz, Nuovo Cimento (to be published); C. E. DeTar and J. Weis, Phys. Rev. D 4, 3141 (1971).

[27]C. E. DeTar, D. Z. Freedman, and G. Veneziano, Phys. Rev. D 4, 906 (1971).

[28]H. D. Abarbanel, G. F. Chew, M. L. Goldberger, and L. M. Saunders, Phys. Rev. Letters 26, 937 (1971); Phys. Rev. D 4, 2958 (1971).

[29]Although the results quoted are not explicity stated in the sum rules (identical to those of Ref. 27) of T. T. Chou and C. N. Yang, Phys. Rev. Letters 25, 1072 (1970), we believe they must be implicitly present. We further remark that a general formulation of the inclusive sum rules has been recently given by E. Predazzi and G. Veneziano, Lett. Nuovo Cimento 2, 749 (1971); see also S.-H. H. Tye, *ibid.* 2, 1271 (1971).

[30]G. F. Chew and D. Snyder, Phys. Rev. D 3, 420 (1971). See H. D. Abarbanel, G. F. Chew, M. L. Goldberger, and L. M. Saunders, Ref. 28, for a discussion of the schizophrenic Pomeranchukon in the context of the helicity-pole limit.

[31]H. D. I. Abarbanel and M. B. Green, Phys. Letters 38B, 90 (1972).

[32]J. B. Bronzan, Phys. Rev. D 4, 1097 (1971).

[33]J. B. Bronzan and C. S. Hui, Phys. Rev. D 5, 964 (1972).

[34]V. N. Gribov, Zh. Eksperim. i Teor. Fiz. 53, 654 (1971) [Soviet Phys. JETP 26, 414 (1968)].

[35]V. N. Gribov and A. A. Migdal, Zh. Eksperim. i Teor. Fiz. 55, 1498 (1968) [Soviet Phys. JETP 28, 784 (1969)]; Yadern. Fiz. 8, 1002 (1968); 8, 1213 (1968) [Soviet J. Nucl. Phys. 8, 583 (1969); 8, 703 (1969)].

[36]J. Scherk, Nucl. Phys. B31, 222 (1971).

[37]The fact that this result arises in our opinion is truly amazing owing to the quite different combinatorial aspects of the dual perturbation expansion and the Feynman-Dyson expansion.

[38]The position of the Pomeranchukon intercept is in fact, dual-model-dependent, i.e., different dual factorizable, and possibly ghost-free models yield different results. For a detailed discussion of the Pomeranchukon of the dual model and its relation to the Freund-Harari conjecture we refer the interested reader to G. Veneziano, in *Proceedings of the International Conference on Duality and Symmetry in Hadron Physics,* edited by E. Gutsman (Weizmann Science Press of Israel, Jerusalem, 1971), p. 179.

[39]H. Harari and Y. Zarmi, Phys. Rev. 187, 2230 (1969), and H. Harari, Ann. Phys. (N.Y.) 63, 432 (1971).

[40]P. G. O. Freund, Phys. Rev. Letters 20, 235 (1968), and H. Harari, *ibid.* 20, 1395 (1968).

[41]Presumably t-channel unitarity should provide some

information on this question.

[42]When we refer to ladders, we do *not* include ladder graphs in which, e.g., 2 or more ladders are welded together at their sides and with their rungs alternating.

[43]X, in fact, includes a bit more than the spectral function; indeed, we also include the computable cut structure that emerges at the junction of the three Regge black boxes.

[44]See I. T. Drummond, P.V. Landshoff, and W. J. Zakrzewski, Nucl. Phys. B11, 383 (1969), Eq. (3.13), for an example of the convoluted form.

[45]One can indeed argue that the product of a known function times an unknown function, whose only property we make use of is its superconvergence behavior, could yield anything one desires. We are, of course, assuming that the spectral functions are not so perverse as to cancel the effects generated from the known functions. If one believed in such a happenstance then one can ignore Ref. 4 and the conclusions of this paper.

[46]We note here the significant new development in dual-loop theory (Ref. 47) by V. Alessandrini, D. Amati, and B. Morel, Nuovo Cimento 7A, 797 (1972), in which the asymptotic limits of the orientable nonplanar box diagrams have been rigorously calculated and found to be convergent in the right-half complex s plane. Thus we feel an asymptotic evaluation of, e.g., the graphs discussed in Ref. 48, and a test of our second conjecture should be possible in the near future.

[47]See for example, C. Lovelace, Phys. Letters 32B, 703 (1970); V. Alessandrini and D. Amati, Nuovo Cimento 4A, 793 (1971); M. Kaku and Loh-ping Yu, Phys. Rev. D 3, 2992 (1971); 3, 3007 (1971); 3, 3020 (1971), and A. D. Karpf, Institute für Theoretische Physik, Univ. Freiburg, Germany, Reports No. 71-805 and No. 71-806 (unpublished).

[48]D. Gordon and G. Veneziano, Ref. 26; see in addition G. Veneziano, Lett. Nuovo Cimento 1, 681 (1971), and for a confrontation of the experimental data, S.-H. H. Tye and G. Veneziano, Phys. Letters 38B, 30 (1972), which tends to support our seven-component generalization of the Freund-Harari conjecture.

[49]A. H. Mueller and T. L. Trueman, following paper, Phys. Rev. D 5, 2115 (1972).

[50]M. Ademollo and E. Del Giudice, Nuovo Cimento 63A, 639 (1969).

[51]G. Veneziano, private communication.

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Nonplanar Couplings in the Triple-Regge Vertex*

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Recently zeros have been found at $\alpha_V(0) = 2\alpha(t) - 1,\ 2\alpha(t) - 2, \ldots$ in the triple-Regge vertex involving $\alpha_V(0)$-$\alpha(t)$-$\alpha(t)$. Such zeros were found both in a dual-resonance model and in certain classes of Feynman graphs. We have examined this question in a model of nonplanar Feynman graphs and found zeros at $\alpha_V(0) = 2\alpha(t) - 2,\ 2\alpha(t) - 4, \ldots$ but not at $\alpha_V(0) = 2\alpha(t) - 1,\ 2\alpha(t) - 3, \ldots$. In particular, the zero involving the triple-Pomeranchukon coupling at $t = 0$ is not present.

I. INTRODUCTION

The question of the size of the Pomeranchukon-Pomeranchukon-Pomeranchukon coupling (the triple-Pomeranchukon vertex) is intimately connected with the problem of the self-consistency of a factorizable Pomeranchuk trajectory at, or near, $J = 1$. This problem seems to have occurred first in the calculations of Kajantie and Finkelstein[1] who showed that a multiperipheral diagram including many Pomeranchukon exchanges [with the vacuum trajectory intercept $\alpha_V(0) = 1$] gives a total cross section which increases with energy. This same effect is evident in the models of Chew-Pignotti[2] and Chew-Frazer[3-5] where it was found that unlimited exchange of a Pomeranchuk trajectory can be consistent only if $\alpha_V(0) < 1$. As $1 - \alpha_V(0)$ goes to zero the Pomeranchukon must decouple from all other particles and trajectories. This led naturally to the idea that $1 - \alpha_V(0)$ might be different from zero.[2]

While the above arguments seemed to contain much truth they always referred to calculations within the multiperipheral and multi-Regge models. However, a model-independent statement arose which seemed to express the difficulty involved in