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Comparison of Approximate Methods for Multiple Scattering in High-Energy Collisions*

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We compare the efficacy of several approximate methods for dealing with multiple-scattering processes by applying these methods to a simple, soluble model. The model consists of the scattering of a particle by a pair of like particles with motion of all particles constrained to one dimension. Two cases are considered: (1) the target particles have a fixed separation and (2) the target particles are in a deuteron-like bound state with each other. The interaction of the incident particle with each target particle is the same and is chosen to be a zero-range potential. The potential binding the two target particles in case 2 is equal to this same zero-range interaction. The methods compared are the Born approximation for the T matrix, the impulse approximation for the T matrix, the WKB approximation, the Glauber approximation, second-order multiple-scattering theory for the T matrix, Born approximation for the K matrix, impulse approximation for the K matrix, and second-order multiple-scattering theory for the K matrix. For the fixed-separation case the K-matrix formalisms work much better than the T-matrix formalisms including Glauber theory. Second-order multiple-scattering theory is superior to Glauber theory for fixed separation. For case 2 the aforementioned approximate methods are used in conjunction with the adiabatic approximation. Comparison of the calculated transmission probabilities with the exact transmission probability shows the Glauber theory doing much better than any of the other approximate methods. The success of the Glauber theory results from its suppression of a second-order term in the multiple-scattering expansion which is inadequately damped by the adiabatic approximation.

I. INTRODUCTION

The Glauber theory[1] for the scattering of a high-energy particle by a many-body target has proved to be very successful in predicting high-energy scattering cross sections. In order to gain some added insight into the reasons for the success of the Glauber theory we have applied it to a simple, soluble one-dimensional scattering model. For comparison we have applied other approximate scattering theories to the same model.

The scattering model consists of a particle constrained to move along a straight line on which are located two identical scattering centers separated by a distance b. The interaction of the incident particle with each scattering center (or target particle) is given by a zero-range potential. The probabilities for transmission and reflection are analogous to the differential cross sections for forward and backward scattering in three dimensions. Two cases are considered: (1) the target-particle separation b is assumed to be fixed, and (2) the target-particle separation b is assumed to vary in time in accordance with an exponential form factor. Case 2 corresponds to the two target particles being in a bound state created by a zero-range potential interaction between them.

Approximate expressions for the transmission and reflection probabilities are constructed by means of taking the Born approximation, the impulse approximation, and the second-order multiple-scattering approximation for either the T matrix or the K matrix. In addition, the WKB approximation and the Glauber approximation for the T matrix are used.

For the case where the target-particle separation is held fixed we find that all the K-matrix methods give results superior to all the T-matrix methods. Also the second-order multiple-scattering approximations do better than the others. The Glauber theory does rather poorly. For this case multiple scatterings involving one or more backward scatterings play a non-negligible part. The K-matrix methods seem to be very successful in handling this kind of an effect.

For the case where the two target particles are in a deuteron-like bound state the aforementioned approximate methods are inserted into an adiabatic-approximation treatment of the problem. The approximate transmission probabilities are compared to the exact result for this one-dimensional three-body problem. The Glauber theory proves to be superior to any of the other approximate methods. The reason for this result appears to be the Glauber-theory neglect of a second-order term in the multiple-scattering approximation which is insufficiently damped in the adiabatic-approximation treatment.

In Sec. II we present the model and present a version of formal scattering theory appropriate to the peculiar geometry of the model. The derivation of the formalism is contained in Appendix A. The scattering by single and double zero-range potentials is described in Sec. III, and the Born approximation is introduced. Section IV is devoted to the WKB and Glauber approximations. Then in Sec. V we present the Watson multiple-scattering expansion which is derived in Appendix B. From this we get the impulse approximation in first order and the "second-order multiple-scattering approximation" by retaining the first- and second-order terms. In Sec. VI we show how the previous results must be modified to account for a time-varying target-particle separation. The results of numerical calculations are presented in Sec. VII, and Sec. VIII is devoted to some concluding remarks.

II. ONE-DIMENSIONAL SCATTERING THEORY

Suppose we have a particle constrained to move along a straight line. This particle can move freely except for its interaction with a finite-range scatterer. The Schrödinger equation for this system is

$$\left(\frac{d^2}{dx^2}+w(x)+k^2\right)\psi(x)=0\,, \tag{1a}$$

$$w(x)=0 \quad \text{for } |x|>c\,. \tag{1b}$$

We require the asymptotic behavior of the wave function to be appropriate to a scattering process.

$$\psi=e^{ikx}+f_-e^{-ikx} \quad (x<-c) \tag{2a}$$

$$=e^{ikx}+f_+e^{ikx} \quad (x>c)\,. \tag{2b}$$

Then the conservation of flux imposes the following requirement on the scattering amplitudes:

$$|1+f_+|^2+|f_-|^2=1\,. \tag{3}$$

This model has been used by Kujawski[2] to test Glauber theory.

The task of scattering theory is to provide expressions for the scattering amplitudes $f_\pm$ in terms of the scattering potential $w(x)$ and the energy k^2. These expressions are derived in Appendix A. Their structure is somewhat different from what we find in the three-dimensional case. We derive two relations for $f_\pm$ – a T-matrix expression and a K-matrix expression.

The T-matrix expression is

$$f_\pm=\frac{i}{2k}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\; e^{ikx}T(x,y)e^{\mp iky}\,, \tag{4a}$$

$$T(x,y)=w(x)\delta(x-y)-w(x)G(x,y)^{(+)}w(y)\,, \tag{4b}$$

$$\left(\frac{d^2}{dx^2}+w(x)+k^2\right)G(x,y)^{(+)}=\delta(x-y)\,. \tag{4c}$$

Here $T(x,y)$ is the T matrix and $G(x,y)^{(+)}$ is the "outgoing-wave" or "retarded" Green's function.

The relationship between the scattering amplitudes and the K matrix can be presented in the form of a pair of linear algebraic equations.

$$f_+(1-iK_{ss})+f_-(-1-K_{cs})=K_{cs}+iK_{ss}\,, \tag{5a}$$

$$f_+(1+K_{cs})+f_-(1-iK_{cc})=K_{cs}+iK_{cc}\,, \tag{5b}$$

$$K_{sc}=\frac{1}{k}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\sin kx\,K(x,y)\cos ky\,, \tag{5c}$$

$$K_{cs}=\frac{1}{k}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,\cos kx\,K(x,y)\sin ky\,, \tag{5d}$$

$$K(x,y)=w(x)\delta(x-y)-w(x)G(x,y)^{(0)}w(y)\,, \tag{5e}$$

$$\left(\frac{d^2}{dx^2}+w(x)+k^2\right)G(x,y)^{(0)}=\delta(x-y)\,. \tag{5f}$$

Here $K(x,y)$ is the K matrix and $G(x,y)^{(0)}$ is the "standing wave" or "principal value" Green's function.

$G^{(+)}$ and $G^{(0)}$ are solutions of the same differential equation but display different asymptotic behaviors. For the outgoing-wave Green's function we require

$$G(x,y)^{(+)}\sim e^{ikx} \quad \text{for } x>y,\,c \tag{6a}$$

and

$$G(x,y)^{(+)}\sim e^{-ikx} \quad \text{for } x<y,\,-c\,. \tag{6b}$$

For the standing-wave Green's function we require

$$G(x,y)^{(0)}\sim \cos kx \quad \text{for } x>y,\,c \tag{7a}$$

and

$$G(x,y)^{(0)}\sim \sin kx \quad \text{for } x<y,\,-c\,. \tag{7b}$$

The Green's functions are symmetric.

$$G(x,y)=G(y,x)\,. \tag{8}$$

III. SCATTERING BY SINGLE AND DOUBLE ZERO-RANGE POTENTIALS. EXACT AMPLITUDES AND BORN-APPROXIMATION AMPLITUDES

The Born approximation consists in neglecting terms that are higher than first order in powers of the interaction potential. Thus $K(x,y)^B=T(x,y)^B=w(x)\delta(x-y)$ is the Born-approximation K matrix and T matrix.

Now let us consider the simple case where w is a zero-range interaction; that is,

$$w_1(x)=-\nu\delta(x-a)\,. \tag{9a}$$

In this case it is easy to construct the solutions of the Schrödinger equation, Eq. (1), and show that the scattering amplitudes are

$$f_+^{(1)}=\Lambda,\quad f_-^{(1)}=\Lambda e^{i2ka}\,, \tag{9b}$$

$$\Lambda=-i\gamma/(1+i\gamma),\quad \gamma=\nu/2k\,. \tag{9c}$$

Now if we use the Born approximation for the T matrix we find

$$f_+^{1TB}=-i\gamma,\quad f_-^{1TB}=-i\gamma e^{i2ka}\,, \tag{10}$$

whereas if we use the Born approximation for the K matrix we find

$$f_+^{1KB}=\frac{\Lambda}{1-\Lambda\sin 2ka}\,,\quad f_-^{1KB}=\frac{\Lambda e^{i2ka}}{1-i\Lambda\sin 2ka}\,. \tag{11}$$

Note that for $a=0$ the Born approximation for the K matrix matches the exact result.

Next let us address ourselves to the case of scattering by two identical zero-range potentials. We take the scattering potential to be

$$w_2(x)=-\nu\delta(x-\tfrac{1}{2}b)-\nu\delta(x+\tfrac{1}{2}b)\,. \tag{12a}$$

By constructing the solution to the Schrödinger equation and examining its asymptotic form one can find the following expressions for the scattering amplitudes:

$$f_+^{(2)}=(2\Lambda+\Lambda^2+\Lambda^2B^2)/(1-\Lambda^2B^2)\,, \tag{12b}$$

$$f_-^{(2)}=(\Lambda B^{-1}+\Lambda B+2\Lambda^2B)/(1-\Lambda^2B^2)\,, \tag{12c}$$

$$B=e^{ikb}\,. \tag{12d}$$

The various terms in these expressions can be interpreted on the basis of a very simple multiple-scattering representation of the scattering process. The factor Λ is contributed by each scattering while the factor B is a phase factor contributed by each additional length of path b traveled by the scattered particle. In the numerator of the transmission amplitude $f_+^{(2)}$ we can identify four terms corresponding to the four elementary single- and double-scattering processes contributing to transmission. Similarly, in the numerator of the expression for the reflection amplitude $f_-^{(2)}$ there are four terms that can be identified with four distinct single- and double-scattering processes leading to reflection. The denominator $(1-\Lambda^2B^2)^{-1}=\sum_0^\infty(\Lambda^2B^2)^n$ produces a sequence of multiple-scattering contributions from each of the four single- and double-scattering terms. In Figs. 1 and 2 are displayed diagramatic representations of all the terms of the multiple-scattering expansion up through third order.

When we make the Born approximation for the T matrix in the double zero-range potential case we find

$$f_+^{2TB}=-2i\gamma,\quad f_-^{2TB}=-2i\gamma\cos kb\,. \tag{13}$$

The Born approximation for the K matrix yields the following amplitudes:

$$f_+^{2KB}=\frac{2\Lambda+\Lambda^2+\Lambda^2\cos 2kb}{1-\Lambda^2\cos 2kb}\,, \tag{14a}$$

$$f_-^{2KB}=\frac{2\Lambda\cos kb+2\Lambda^2(\cos kb-\sin^2 kb)}{1-\Lambda^2\cos 2kb}\,. \tag{14b}$$

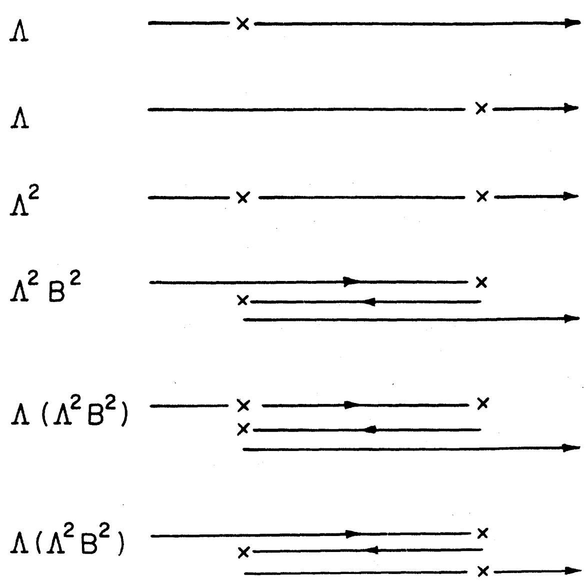

FIG. 1. Diagrams depicting terms in the multiple-scattering series for transmission processes. This is for a particle in one dimension scattering from two fixed scattering centers. All processes up to through third order are included.

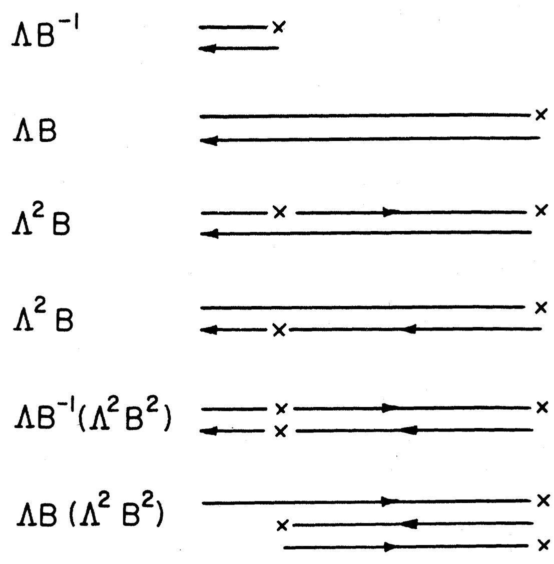

FIG. 2. Diagrams depicting terms in the multiple-scattering series for reflection processes. This is for a particle in one dimension scattering from two fixed scattering centers. All processes up to through third order are included.

IV. THE WKB APPROXIMATION AND GLAUBER THEORY

The T matrix may be regarded as the product of the interaction potential $w(x)$ with the wave matrix $\Omega(x, y)$.

$$T(x, y) = w(x)\Omega(x, y), \tag{15a}$$

$$\Omega(x, y) = \delta(x-y) - G(x, y)^{(+)}w(y). \tag{15b}$$

The wave matrix has the property that when it operates on the incident wave e^{ikx} the result is the corresponding exact solution of the Schrödinger equation, Eq. (1).

$$\psi(x) = \int dy\, \Omega(x, y)\, e^{iky}. \tag{16}$$

The wave-matrix concept is thus useful for generating an expression for the T matrix from an approximate wave function.

The high-energy limit $[k^2 \gg w(x)]$ of the WKB approximation[3] for the wave function of our particle moving in one dimension is

$$\psi^W = e^{ikx} \exp\left(\frac{i}{2k}\int_{-\infty}^{x} dy\, w(y)\right). \tag{17a}$$

Thus the wave matrix in the WKB approximation is just

$$\Omega(x, y)^W = \delta(x-y)\exp\left(\frac{i}{2k}\int_{-\infty}^{x} dz\, w(z)\right), \tag{17b}$$

and the T matrix is

$$T(x, y)^W = w(x)\delta(x-y)\exp\left(\frac{i}{2k}\int_{-\infty}^{x} dz\, w(z)\right). \tag{17c}$$

When this expression is used in Eq. (4) to calculate the transition amplitudes for the single zero-range potential case, Eq. (9), we find

$$f_+^{1TW} = (e^{-i\gamma} - 1), \tag{18a}$$

$$f_-^{1TW} = e^{i2ka}(e^{-i\gamma} - 1). \tag{18b}$$

Ordinarily, one would not expect the WKB approximation to work very well for a singular potential, but the result shown in Eq. (18) does not look too bad. At any rate, in the high-energy limit it agrees with the Born approximation.

Now let us repeat the WKB calculation for the double zero-range potential case, Eq. (12). The result is found to be

$$f_+^{2TW} = 2\Gamma + \Gamma^2, \tag{19a}$$

$$f_-^{2TW} = 2\Gamma \cos kb + \Gamma^2 e^{ikb}, \tag{19b}$$

$$\Gamma = e^{-i\gamma} - 1. \tag{19c}$$

We note that the quantity Γ is just the WKB approximation for the interaction with one of the scattering potentials. The exact expression for the interaction with a single-scattering center is just the quantity Λ given by Eq. (9c). The Glauber theory results from replacing the WKB single-interaction expression by the exact single-interaction expression in the WKB expression for the transition amplitude for the complex system. Thus Γ is replaced by Λ in Eq. (19).

$$f_+^{2TG} = 2\Lambda + \Lambda^2, \tag{20a}$$

$$f_-^{2TG} = \Lambda B^{-1} + \Lambda B + \Lambda^2 B. \tag{20b}$$

Comparison of the Glauber theory, Eq. (20), with the exact result, Eq. (12), reveals that Glauber theory contains the two first-order terms of the multiple-scattering series and one of the two second-order terms. The Glauber theory is characterized by the neglect of all contributions to the multiple-scattering series for the transmission amplitude which contain any backscatterings; for the reflection amplitude all terms of the multiple-scattering series are dropped which contain more than one backscattering.

V. THE SECOND-ORDER MULTIPLE-SCATTERING APPROXIMATION AND THE IMPULSE APPROXIMATION

The multiple-scattering expansion for the double zero-range potential case is derived in Appendix B. There it is shown that the T matrix and K matrix have the form

$$X(x, y) = [\delta(x-\tfrac{1}{2}b)\delta(y-\tfrac{1}{2}b)\beta_1 + \delta(x+\tfrac{1}{2}b)\delta(y+\tfrac{1}{2}b)\beta_2 - \delta(x+\tfrac{1}{2}b)\delta(y-\tfrac{1}{2}b)\beta_2 g\beta_1 - \delta(x-\tfrac{1}{2}b)\delta(y+\tfrac{1}{2}b)\beta_1 g\beta_2]\sum_{n=0}^{\infty}(g^2\beta_1\beta_2)^n, \tag{21a}$$

$$g = G_0(\tfrac{1}{2}b, -\tfrac{1}{2}b), \tag{21b}$$

$$\beta_1 = -\nu[1-\nu G_0(\tfrac{1}{2}b, \tfrac{1}{2}b)]^{-1}, \tag{21c}$$

$$\beta_2 = -\nu[1-\nu G_0(-\tfrac{1}{2}b, -\tfrac{1}{2}b)]^{-1}. \tag{21d}$$

X is the T matrix if $G_0 = G_0^{(+)}$, where

$$G_0(x, x')^{(+)} = \frac{e^{ikx_>}e^{-ikx_<}}{2ik} \tag{22}$$

and X is the K matrix if $G_0 = G_0^{(0)}$, where

$$G_0(x, x')^{(0)} = \frac{\cos kx_> \sin kx_<}{-k}. \tag{23}$$

Truncation of the multiple-scattering expansion to include only terms first order in β, the single-scattering terms, constitutes the impulse approximation. The transition amplitudes resulting from the impulse approximation to the T matrix are

$$f_+^{2TI} = 2\Lambda, \quad f_-^{2TI} = \Lambda B^{-1} + \Lambda B. \tag{24}$$

The transition amplitudes resulting from the impulse approximation to the K matrix are

$$f_+^{2KI} = \frac{2\Lambda + 2\Lambda^2 \cos^2 kb}{1 - \Lambda^2 \cos^2 kb}, \tag{25a}$$

$$f_-^{2KI} = \frac{2\Lambda \cos kb + 2\Lambda^2 \cos kb}{1 - \Lambda^2 \cos^2 kb}. \tag{25b}$$

If both single-scattering and double-scattering terms are retained in the multiple-scattering expansion, the result is the second-order multiple-scattering approximation. The transition amplitudes that result from the second-order multiple-scattering approximation to the T matrix are

$$f_+^{2TM} = 2\Lambda + \Lambda^2 + \Lambda^2 B^2, \tag{26a}$$

$$f_-^{2TM} = 2\Lambda \cos kb + 2\Lambda^2 B. \tag{26b}$$

If the same approximation is made for the K matrix, we find the transition amplitudes to be

$$f_+^{2KM} = \frac{2\Lambda + \Lambda^2 + \Lambda^2 - Q}{1 - \Lambda^2 B^2 - \Lambda^2 \sin^2 kb + Q}, \tag{27a}$$

$$f_-^{2KM} = \frac{2\Lambda \cos kb + 2\Lambda^2 B - Q}{1 - \Lambda^2 B^2 - \Lambda^2 \sin^2 kb + Q}, \tag{27b}$$

$$Q = \frac{2\Lambda^4 \sin^4 kb}{(1+\Lambda)^2 + \Lambda^2 \sin^2 kb}. \tag{27c}$$

There is a third type of second-order multiple-scattering approximation that is sometimes used. This type of approximation results from using an approximate form for $G_0^{(+)}$ in the second-order multiple-scattering approximation for the T matrix. The unperturbed Green's function $G_0^{(+)}$ defined in Eq. (22) may be written

$$\begin{aligned} G_0(x, x')^{(+)} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \, \frac{e^{iq(x'-x)}}{k^2 - q^2 + i\epsilon} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \, e^{iq(x'-x)} \left(\frac{k^2 - q^2}{(k^2-q^2)^2 + \epsilon^2} - \frac{i\epsilon}{(k^2-q^2)^2+\epsilon^2}\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \, e^{iq(x'-x)} \left(\frac{P}{k^2-q^2} - i\pi\delta(k^2-q^2)\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dq \, e^{iq(x'-x)} \frac{P}{k^2-q^2} + \frac{\cos k(x-x')}{2ik}, \end{aligned} \tag{28}$$

where ϵ is a positive infinitesimal and P denotes principle value. The second term on the right-hand side is called the "on-the-energy-shell part of the Green's function" which we denote by

$$G_0^E(x, x')^{(+)} = \frac{\cos k(x-x')}{2ik}. \tag{29}$$

If $G_0^{(+)}$ is replaced by $G_0^{E(+)}$ in the second-order multiple-scattering approximation for the T matrix, the transition amplitudes are found to be

$$f_+^{2TE} = 2\Lambda + 2\Lambda^2 \cos^2 kb, \tag{30a}$$

$$f_-^{2TE} = 2\Lambda \cos kb + 2\Lambda^2 \cos kb. \tag{30b}$$

For the cases we subjected to numerical evaluation we found that $|f_+^{2TE} + 1|^2$ was very nearly equal in value to $|f_+^{2TM} + 1|^2$.

VI. APPLICATION TO THE SYMMETRIC ONE-DIMENSIONAL THREE-BODY PROBLEM

Consider a system consisting of three identical, distinguishable particles constrained to move along a straight line. Suppose the particles interact pairwise via a zero-range potential. Then the Schrödinger equation for the system is

$$\left[E + \frac{\hbar^2}{2M}\left(\frac{\partial^2}{\partial r_1^2} + \frac{\partial^2}{\partial r_2^2} + \frac{\partial^2}{\partial r_3^2}\right) + A[\delta(r_1 - r_2) + \delta(r_2 - r_3) + \delta(r_3 - r_1)]\right]\psi(r_1, r_2, r_3) = 0. \tag{31}$$

Now introduce a new set of coordinates

$$x = (\tfrac{2}{3})^{1/2}[r_1 - \tfrac{1}{2}(r_2 + r_3)], \tag{32a}$$

$$y = (\tfrac{1}{2})^{1/2}(r_2 - r_3), \tag{32b}$$

$$z = (\tfrac{1}{3})^{1/2}(r_1 + r_2 + r_3). \tag{32c}$$

Then Eq. (31) becomes

$$\left\{K^2 + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + a\left[\delta(y) + \frac{2}{3^{1/2}}\,\delta\left(x - \frac{y}{3^{1/2}}\right) + \frac{2}{3^{1/2}}\,\delta\left(x + \frac{y}{3^{1/2}}\right)\right]\right\}\psi(x, y, z) = 0, \tag{33a}$$

$$K^2=2ME/\hbar^2, \quad (33b)$$

$$a=2MA/\hbar^2. \quad (33c)$$

Henceforth we will ignore the center-of-mass coordinate z.

Suppose particle 1 is incident on a bound state of particles 2 and 3. Then the wave function for the incident beam is

$$\psi_0(x,y)=e^{ikx}\tfrac{1}{2}a\,e^{-a|y|/2}, \quad (34a)$$

$$k^2=K^2+\tfrac{1}{4}a^2. \quad (34b)$$

This problem has been solved exactly by McGuire[4] who finds that the transmission probability is $|1+f_+|^2$, where

$$f_+=4\Lambda/(2-\Lambda), \quad (35a)$$

$$\Lambda=-i\gamma/(i+i\gamma), \quad (35b)$$

$$\gamma=\nu/2k=a/k3^{1/2}. \quad (35c)$$

He finds there is no reflection and no breakup. Those particles which are not transmitted participate in a knock-out process in which the incident particle exchanges roles with one of the bound particles.

We will make an analysis of this problem based on the adiabatic approximation. The adiabatic approximation prescribes that one first solves the problem as if the separation of the two bound particles were fixed. Thus we must solve the Schrödinger equation

$$\left(k^2+\frac{d^2}{dx^2}+\nu[\delta(x-\tfrac{1}{2}b)+\delta(x+\tfrac{1}{2}b)]\right)\psi(x)=0, \quad (36a)$$

$$\nu=2a/3^{1/2}, \quad (36b)$$

$$b=2y/3^{1/2}. \quad (36c)$$

Then, according to the adiabatic approximation, one averages the T matrix or K matrix over the various possible values of the separation b using the square of the bound-state wave function as a weighting factor:

$$X^{(s)}(x,y)=\frac{3\nu}{4}\int_0^\infty db\,e^{-3\nu b/4}X^{(2)}(b,x,y), \quad (37a)$$

$$X=T \text{ or } K. \quad (37b)$$

It might be said that this is the scattering from a pair of fixed particles whose separation varies in time in accordance with an exponential weight factor.

We have carried out the "smearing procedure" indicated by Eq. (37) on the T matrices and K matrices provided by the various methods described in the previous sections of this article for the double zero-range potential case. The resulting transition amplitudes are identified by a superscript S replacing the superscript 2. The transition amplitudes that result from applying the adiabatic approximation to the approximate T matrices for the double zero-range potential problem are presented in Table I. The adiabatic approximation for the K matrix is presented in Table II. Table III contains the elements of the K matrix for the fixed scattering case.

VII. NUMERICAL COMPARISON

Although most of the quantities we have discussed can be presented in simple algebraic form, it is still difficult to assess how well the approxi-

TABLE I. Transition amplitudes resulting from averaging the T matrix over a variation in the separation in the double zero-range potential with an exponential weighting factor.

Born approximation
$f_+^{STB}=-2i\gamma$
$f_-^{STB}=-2i\gamma C_1$
Impulse approximation
$f_+^{STI}=2\Lambda$
$f_-^{STI}=2\Lambda C_1$
WKB approximation
$f_+^{STW}=2\Gamma+\Gamma^2$
$f_-^{STW}=2\Gamma C_1+\Gamma^2E_1$
Glauber approximation
$f_+^{STG}=2\Lambda+\Lambda^2$
$f_-^{STG}=2\Lambda C_1+\Lambda^2E_1$
Second-order multiple-scattering approximation with G_0 on the energy shell
$f_+^{STE}=2\Lambda+\Lambda^2+\Lambda^2C_2$
$f_-^{STE}=2\Lambda C_1+2\Lambda^2C_1$
Second-order multiple-scattering approximation
$f_-^{STM}=2\Lambda+\Lambda^2+\Lambda^2E_2$
$f_-^{STM}=2\Lambda C_1+2\Lambda^2E_1$
Exact T matrix
$f_+^{S}=\sum_{n=0}^{\infty}[(2\Lambda+\Lambda^2)\Lambda^{2n}E_{2n}+\Lambda^{2n+2}E_{2n+2}]$
$f_-^{S}=\sum_{n=0}^{\infty}[\Lambda^{2n+1}E_{2n-1}+(\Lambda+2\Lambda^2)\Lambda^{2n}E_{2n+1}]$
$\Lambda=-i\gamma(1+i\gamma)^{-1}$
$\gamma=\nu/2k=a/k\,3^{1/2}$
$E_n=C_n+iS_n=\dfrac{\alpha^2}{\alpha^2+n^2}+\dfrac{i\alpha n}{\alpha^2+n^2}$
$\alpha=\tfrac{3}{2}\gamma$

TABLE II. Elements of the K matrix resulting from averaging the K matrix over a variation in the separation in the double zero-range potential with an exponential weighting factor.

Born approximation

$$X_{cc}^{SB}=-2\gamma(1+C_1)$$

$$X_{ss}^{SB}=-2\gamma(1-C_1)$$

$$X_{cs}^{SB}=0$$

Impulse approximation

$$S_{cc}^{SI}=-2\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma^{2n}\begin{bmatrix}2n\\ m\end{bmatrix}\times(C_{2n-2m}+\tfrac{1}{2}C_{2n-2m+1}+\tfrac{1}{2}C_{2n-2m-1})$$

$$X_{ss}^{SI}=-2\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma^{2n}\begin{bmatrix}2n\\ m\end{bmatrix}\times(C_{2n-2m}-\tfrac{1}{2}C_{2n-2m+1}-\tfrac{1}{2}C_{2n-2m-1})$$

$$X_{cs}^{SI}=2\sum_{n=1}^{\infty}\sum_{m=0}^{n}\gamma^{2n}\begin{bmatrix}2n\\ m\end{bmatrix}C_{2n-2m}$$

$$\begin{bmatrix}2n\\ m\end{bmatrix}=2^{-2n}(2-\delta_{nm})(-1)^{m+n}(2n)!/[m!(2n-m)!]$$

Second-order multiple-scattering approximation

$$X_{cc}^{SM}=-2\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma^{2n}\begin{bmatrix}2n\\ m\end{bmatrix}\times(C_{2n-2m}+\tfrac{1}{2}C_{2n-2m+1}+\tfrac{1}{2}C_{2n-2m-1})$$

$$-2\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma^{2n+1}\left\langle\begin{matrix}2n\\ m\end{matrix}\right\rangle\times(S_{2n-2m+1}+\tfrac{1}{2}S_{2n-2m+2}+\tfrac{1}{2}S_{2n-2m})$$

$$X_{ss}^{SM}=-2\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma^{2n}\begin{bmatrix}2n\\ m\end{bmatrix}\times(C_{2n-m}-\tfrac{1}{2}C_{2n-2m+1}-\tfrac{1}{2}C_{2n-2m-1})$$

$$+2\gamma\sum_{n=0}^{\infty}\sum_{m=0}^{n}\gamma^{2n+1}\left\langle\begin{matrix}2n\\ m\end{matrix}\right\rangle\times(S_{2n-2m+1}-\tfrac{1}{2}S_{2n+2m+2}-\tfrac{1}{2}S_{2n-2m})$$

$$X_{cs}^{SM}=2\sum_{n=1}^{\infty}\sum_{m=0}^{n}\gamma^{2n}\begin{bmatrix}2n\\ m\end{bmatrix}C_{2n-m}$$

$$\left\langle\begin{matrix}2n\\ m\end{matrix}\right\rangle=2^{-2n}(-1)^{m+n}(2n+1)!/[m!(2n+1-m)!]$$

$$C_n+iS_n=\frac{\alpha^2}{\alpha^2+n^2}+\frac{i\alpha n}{\alpha^2+n^2}$$

$$\alpha=\tfrac{3}{2}\gamma$$

mate transition probabilities conform to the exact values. Therefore we have evaluated these quantities numerically for a particular case. We have chosen the strength ν of the zero-range potential to be such that two particles of mass 1 amu interacting via such a potential would have a bound state at -2.2 MeV. The masses of the particles have all been set equal to 1 amu.

The transmission probability for the case of the two zero-range scattering centers in one dimension having a separation of $b=1.128$ F is plotted in Fig. 3. All the approximate transmission probabilities converge to the correct value at sufficiently large values of the energy. Below 100 MeV the Born approximation for the T matrix is extremely bad. The impulse approximation for the T matrix and the WKB approximation are poor at low energies but become very good above 50 MeV. The Glauber theory is fair below 100 MeV. The Born approximation for the K matrix, the impulse approximation for the K matrix, and the second-order multiple-scattering approximation for the T matrix are fair below 50 MeV and excellent above that energy. The second-order multiple-scattering approximation for the K matrix is the only approximation that works well at all energies for this case.

TABLE III. Elements of the K matrix for the double zero-range potential scattering problem.

Born approximation

$$X_{cc}^{2B}=-2\gamma(1+\cos kb)$$

$$X_{ss}^{2B}=-2\gamma(1-\cos kb)/(1-\gamma^2\sin^2 kb)$$

$$X_{cs}^{2I}=2\gamma^2\sin kb/(1-\gamma^2\sin^2 kb)$$

Impulse approximation

$$X_{cc}^{2I}=-2\gamma(1+\cos kb)/(1-\gamma^2\sin^2 kb)$$

$$X_{ss}^{2I}=-2\gamma(1-\cos kb)/(1-\gamma^2\sin^2 kb)$$

$$X_{cs}^{2I}=2\gamma^2\sin kb/(1-\gamma^2\sin^2 kb)$$

Second-order multiple-scattering approximation

$$X_{cc}^{2M}=-2\gamma(1+\cos kb)/(1-\gamma\sin kb)$$

$$X_{ss}^{2M}=-2\gamma(1-\cos kb)/(1+\gamma\sin kb)$$

$$X_{cs}^{2M}=2\gamma^2\sin kb/(1-\gamma^2\sin^2 kb)$$

Exact

$$X_{cc}^{(2)}=-2\gamma\frac{(1+\cos kb)(1-\gamma^2\sin^2 kb)}{(1-\gamma\sin kb)(1-2\gamma^2\sin^2 kb)}$$

$$X_{ss}^{(2)}=-2\gamma\frac{(1-\cos kb)(1-\gamma^2\sin^2 kb)}{(1+\gamma\sin kb)(1-2\gamma^2\sin^2 kb)}$$

$$X_{cs}^{(2)}=2\gamma^2\sin kb/(1-2\gamma^2\sin 2kb)$$

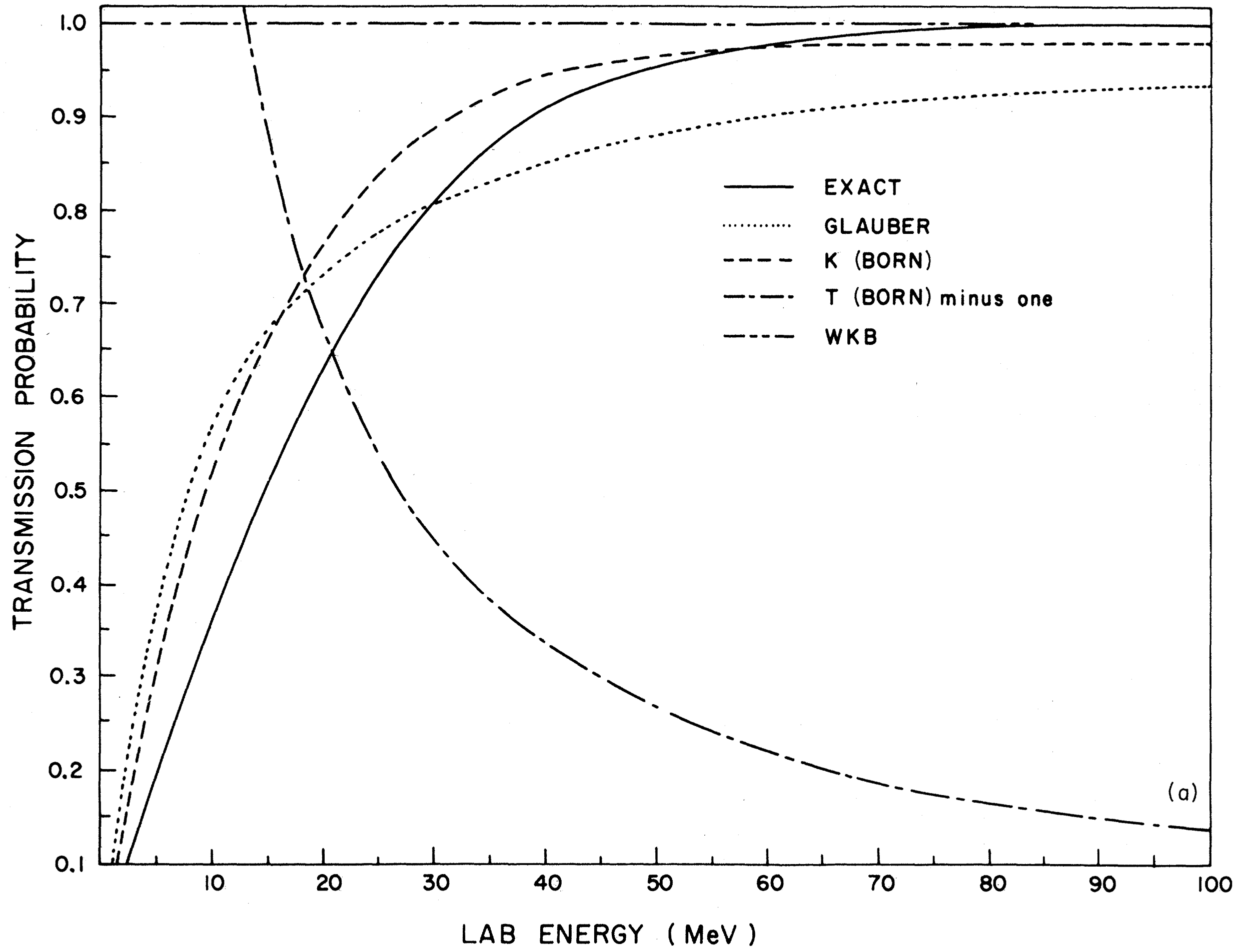

FIG. 3. *(Continued on following page)*

It appears that for treating the scattering of a particle from two fixed scattering centers the K-matrix methods work better than the T-matrix methods. Among the various approximations used, the second-order multiple-scattering approximation seems to work the best.

In Fig. 4 we plot the transmission probability for a particle in one-dimension scattering from two like particles in a bound state. The three particles are distinguishable, have mass of 1 amu each, and interact with each other via identical zero-range potentials of strength such as to bind a pair of the particles by −2.2 MeV. Plotted is the exact transmission probability together with transmission probabilities derived from inserting into the adiabatic approximation the transmission amplitude for scattering from two fixed scatterers. For the fixed scatterer amplitude we tried the Glauber approximation, the WKB approximation, and the exact fixed scatterer amplitude. In addition we used the Born approximation, the impulse approximation, and the second-order multiple-scattering approximation for the K matrix and the T matrix, respectively. The result of using the exact fixed scatterer amplitude in the adiabatic approximation is labeled "adiabatic."

It is remarkable that the Glauber theory works quite well even at very low energies. This is surprising because (a) the Glauber approximation does poorly for the fixed scatterer case and (b) the insertion of the exact fixed scatterer amplitude into the adiabatic approximation gives a poor result.

VIII. DISCUSSION

We find that the Glauber approximation of neglecting backscattering works relatively poorly for the one-dimensional scattering from two fixed scattering centers. However, when the Glauber approximation for the one-dimensional fixed scatterer problem is inserted into the adiabatic approximation for scattering by a bound state of two particles in one dimension one gets a very good result for the transmission probability. The re-

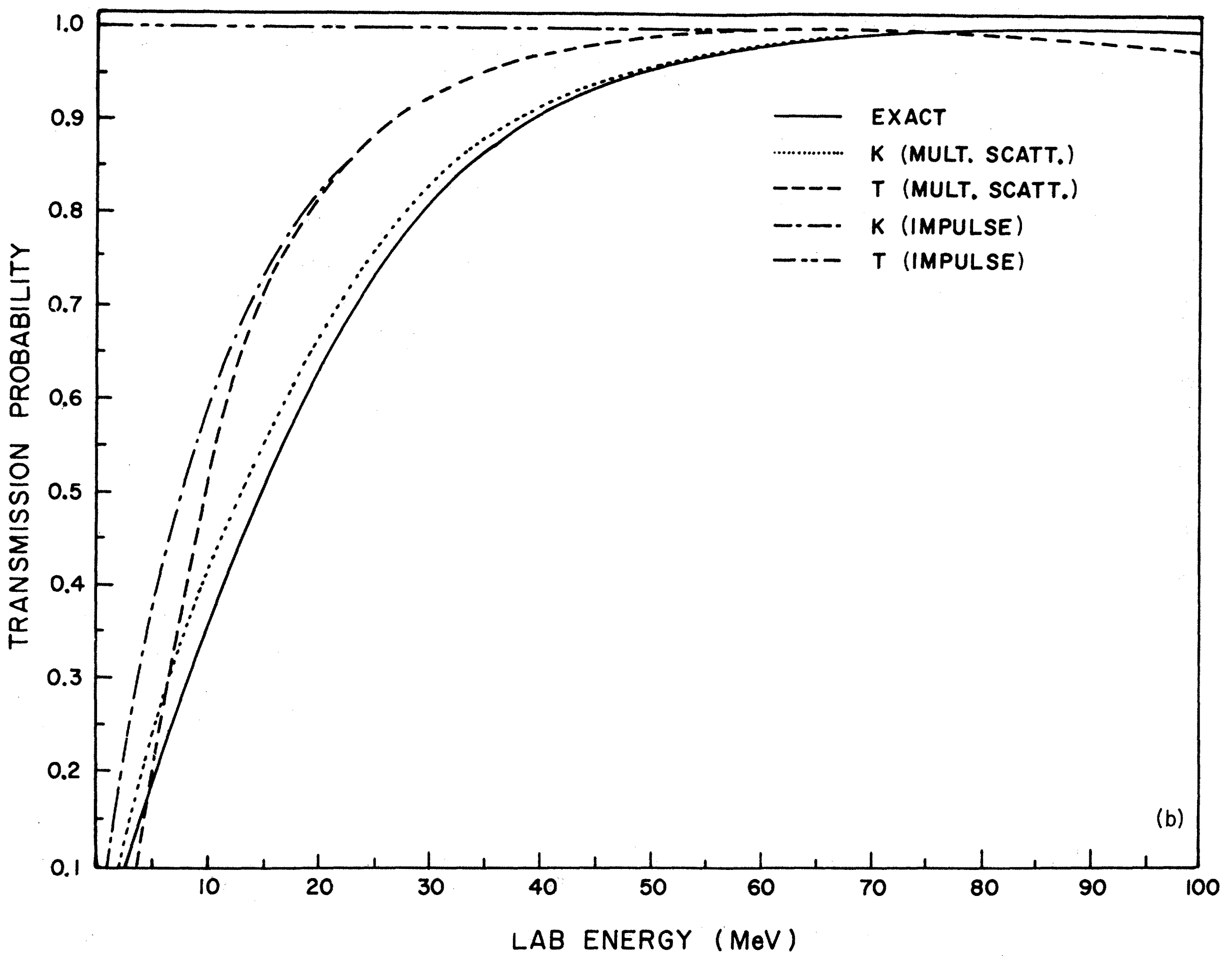

FIG. 3. Transmission probability for a particle in one dimension scattering from a pair of identical zero-range potentials as a function of the kinetic energy in the lab frame. The strength of each zero-range potential is such as to bind a particle of mass 0.5 amu with a binding energy of −2.2 MeV. The incident particle has a mass of 1 amu. The approximate calculations used the Glauber theory, the WKB approximation, the Born approximation for the K matrix and for the T matrix, and the second-order multiple-scattering approximation for the K matrix and the T matrix, respectively.

sult is in fact superior to what is found if the exact two-fixed-scattering-center amplitude is inserted into the adiabatic approximation.

Another numerical verification of the effectiveness of the Glauber theory is provided by Franchiotti and Osborn[5] who consider the scattering in three dimensions of a particle from two fixed overlapping Yukawa potentials.

The remarkable efficacy of Glauber theory has been discussed by Osborn,[6] Harrington,[7] and by Queen.[8] These authors suggest that in neglecting the second-order backscattering term Glauber theory is in fact taking into account the contributions of the higher-order terms in the multiple-scattering series which are neglected. Our results do not appear to be consistent with this view. For the fixed scatterer system the transmission amplitude is

$$f_+^{(2)} = 2\Lambda + \Lambda^2 + \Lambda^2 B^2 + 2\Lambda^3 B^2 + O(\Lambda^4)$$
$$= (2\Lambda + \Lambda^2 + \Lambda^2 B^2)/(1 - \Lambda^2 B^2)\,. \quad (38)$$

The Glauber approximation to this is

$$f_+^{2TG} = 2\Lambda + \Lambda^2\,. \quad (39)$$

The two differ in order Λ^2. In our numerical tests we have found that the Glauber theory gives a rather poor result for the fixed scatterer case. Thus we find nothing to support the idea that the neglect of the second-order backscattering term can be justified by taking into account contributions from higher-order terms in the multiple-scattering series.

On the other hand, when we insert the Glauber amplitude into the adiabatic approximation for

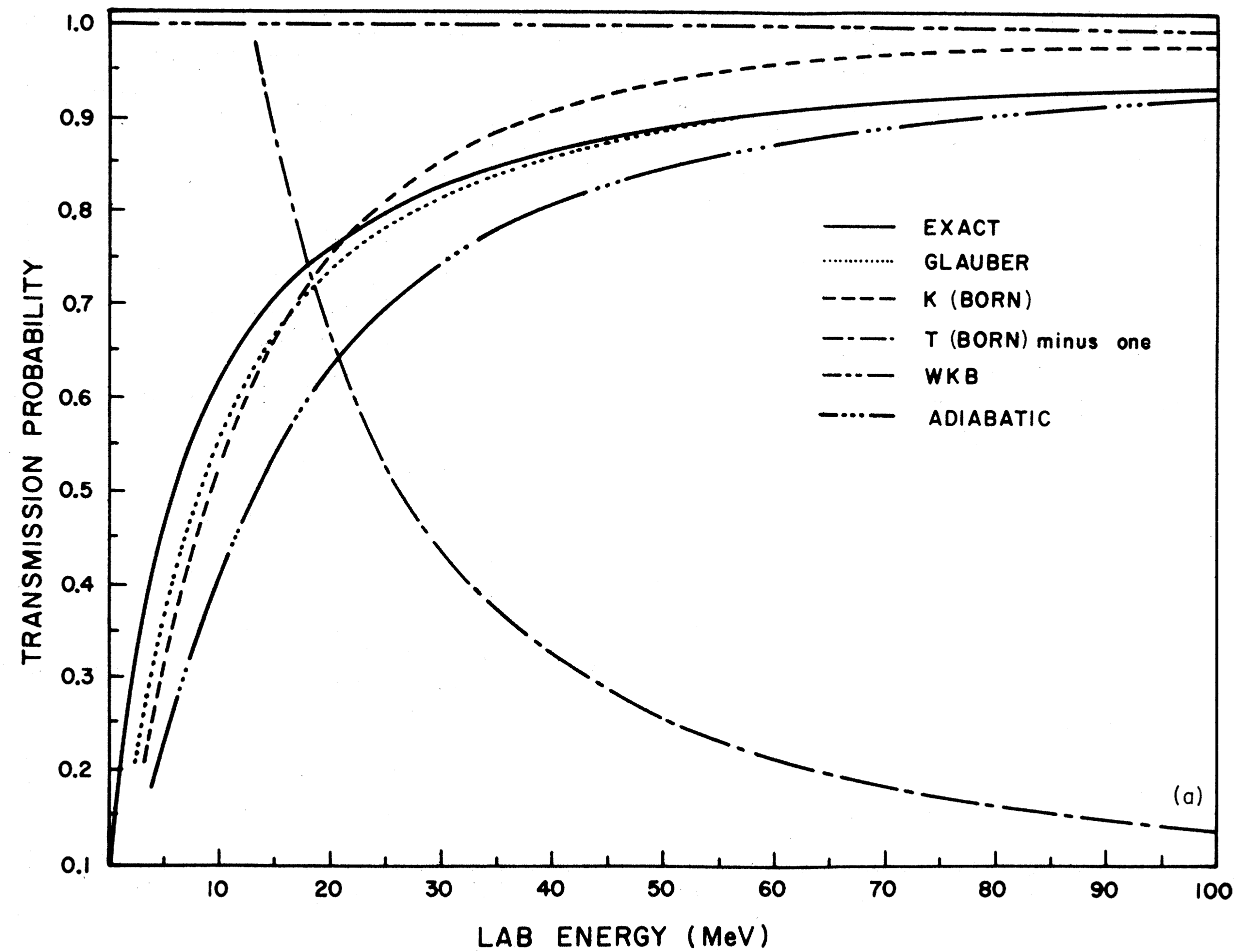

FIG. 4. *(Continued on following page)*

scattering by a pair of particles in a bound state we find the transmission amplitude to be

$$f_+^{STG} = \langle f_+^{2TG} \rangle = 2\Lambda + \Lambda^2 \tag{40}$$

as compared to the exact transmission amplitude

$$f_+ = 2\Lambda + \Lambda^2 + \tfrac{1}{2}\Lambda^3 + O(\Lambda^4). \tag{41}$$

The difference now occurs only in third order in Λ. In our numerical tests $|f_+^{STG}|^2$ becomes quite nearly equal to $|f_+|^2$ above 20 MeV incident energy.

If the exact transmission amplitude for two fixed scatterers is inserted into the adiabatic approximation for scattering by a pair of bound particles, one finds

$$\begin{aligned} f_+^S = \langle f_+^{(2)} \rangle &= 2\Lambda + \Lambda^2 + \Lambda^2 E_2 + O(\Lambda^4) \\ &= 2\Lambda + \Lambda^2 - \tfrac{3}{4}\Lambda^3 + O(\Lambda^4)\,. \end{aligned} \tag{42}$$

The discrepancy between this expression and the exact result is somewhat greater than that for the Glauber-theory expression: $-\frac{5}{4}\Lambda^3 + O(\Lambda^4)$ as compared to $\frac{1}{2}\Lambda^3 + O(\Lambda^4)$. In our numerical calculation this increased discrepancy has a marked effect: $|f_+^S|^2$ does not approach closely the value of $|f_+|^2$ until the incident energy reaches 100 MeV.

It appears from our results that the adiabatic approximation is a much less effective procedure than has been heretofore supposed. Neither is the Glauber theory a very effective approach to the fixed scattering center problem. But inserting the Glauber approximation into the adiabatic approximation gives a very good result. There must be a compensation of errors effect at work here.

From the multiple-scattering expansion for the fixed-scatterer problem we have seen that each backscattering picks up a phase factor $B^2 = e^{2ikb}$ while no such factor is introduced for a forward scattering. We can see that for a collection of bound scatterers the backscattering contributions must be considerably damped by destructive interference as a result of the time variation of the separations of the scatterers. The damping of backscattering introduced by the adiabatic approx-

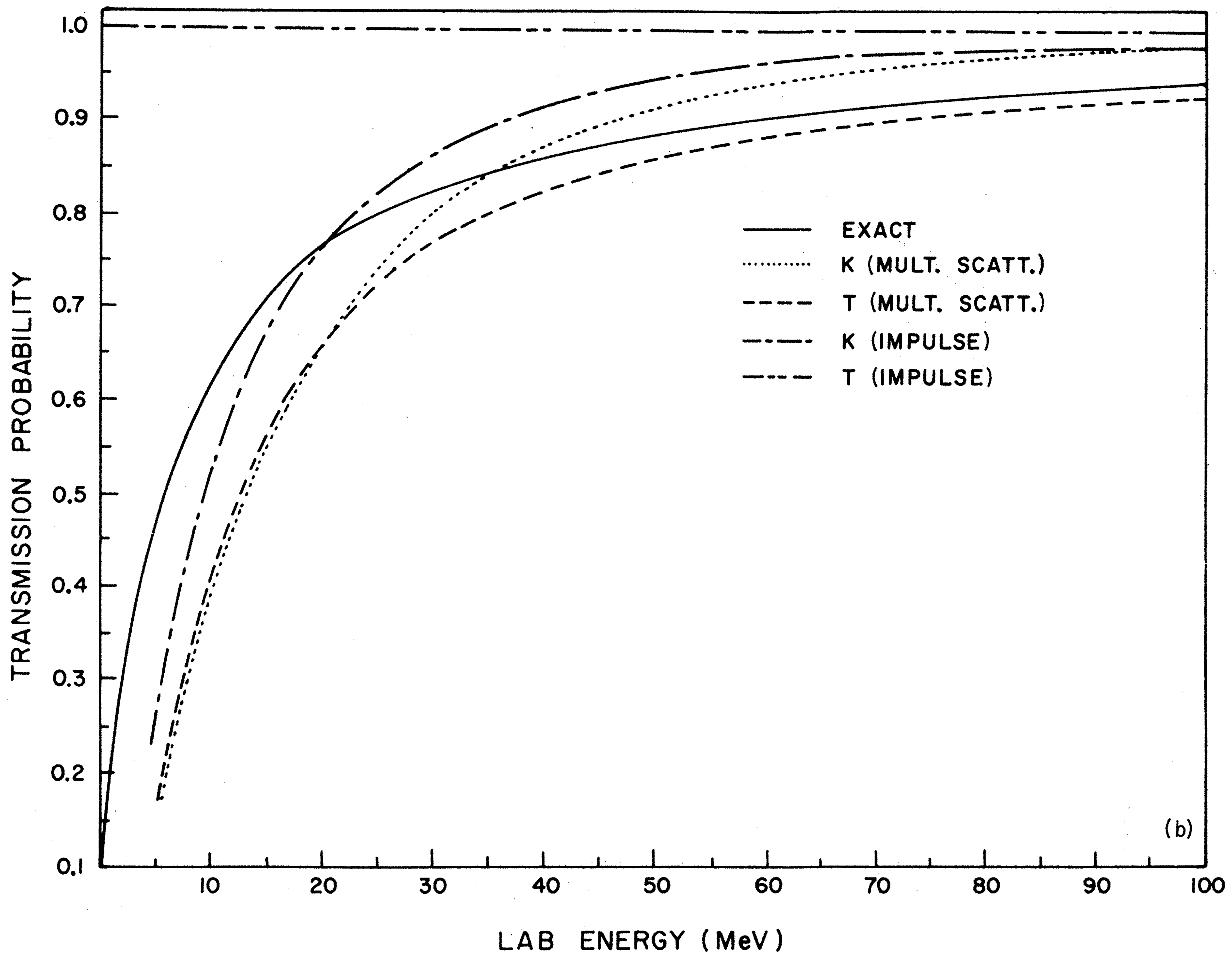

FIG. 4. Transmission probability for a particle in one dimension scattering from two others in a bound state. The three particles are distinguishable, have mass of 1 amu each, and interact with each other via identical zero-range potentials of strength such as to bind a pair of the particles by −2.2 MeV. The approximate cross sections employ the adiabatic approximation. The approximate calculations used the Glauber theory, the WKB approximation, the Born approximation for the K matrix and for the T matrix, the impulse approximation for the K matrix and the T matrix, and the second-order multiple-scattering approximation for the K matrix and the T matrix, respectively. The curve labeled "adiabatic" is the result of inserting the exact fixed scatterer amplitude into the adiabatic approximation.

imation is plainly inadequate. The weakness of the adiabatic approximation in damping the backscattering contributions is compensated for by using the Glauber approximation which neglects backscattering entirely.

ACKNOWLEDGMENT

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APPENDIX A: FORMAL SCATTERING THEORY IN ONE DIMENSION

Consider a particle free to move along a straight line. Suppose there is some finite-range scattering potential $w(x)$ located near the origin. The Schrödinger equation for this system is

$$\left(\frac{d^2}{dx^2}+w(x)+k^2\right)\psi(x)=0\,, \tag{A1a}$$

$$w(x)=0, \quad |x|>c\,. \tag{A1b}$$

Let the asymptotic behavior of ψ be appropriate to a scattering process:

$$\psi(x)=e^{ikx}+f_-e^{-ikx} \quad (x<-c) \tag{A2a}$$

$$=e^{ikx}+f_+e^{ikx} \quad (x>c)\,. \tag{A2b}$$

We seek the relationship between the transition amplitudes f_+ and f_- and the scattering potential w and the energy k^2. The procedure we follow will be that of the extended R-matrix theory.[9]

Let $\{\phi_n(x), n=1, 2, \ldots\}$ be a complete, orthonormal set of functions on the interval $-\bar{c} \leqslant x \leqslant \bar{c}$, where $\bar{c} > c$.

$$\int_{-\bar{c}}^{\bar{c}} dx\, \phi_n(x)\phi_m(x) = \delta_{nm}\,, \tag{A3a}$$

$$\sum_{n=1}^{\infty} \phi_n(x)\phi_n(y) = \delta(x-y)\,. \tag{A3b}$$

On that interval $\psi(x)$ may be represented by an expansion in terms of the ϕ_n's.

$$\psi(x) = \sum_{n=1}^{\infty} A_n \phi_n(x)\,. \tag{A4}$$

Using Eqs. (A1) and (A4) we find

$$\begin{aligned}\int_{-\bar{c}}^{\bar{c}} dx\left(\psi \frac{d^2}{dx^2}\phi_n - \phi_n \frac{d^2}{dx^2}\psi\right) &= \int_{-\bar{c}}^{\bar{c}} dx\, \psi\left(k^2 + w(x) + \frac{d^2}{dx^2}\right)\phi_n \\ &= \sum_{m=1}^{\infty} A_m \int_{-\bar{c}}^{\bar{c}} dx\, \phi_m\left(k^2 + w(x) + \frac{d^2}{dx^2}\right)\phi_n \\ &= \sum_{m=1}^{\infty} A_m [k^2 + w + D^2]_{mn} \\ &= \left[\psi \frac{d}{dx}\phi_n - \phi_n \frac{d}{dx}\psi\right]_{-\bar{c}}^{\bar{c}}.\end{aligned} \tag{A5}$$

It follows that

$$\begin{aligned}\psi(x) &= \sum_{m=1}^{\infty} A_m \phi_m(x) \\ &= \sum_{m=1}^{\infty} \phi_m(x) \sum_{n=1}^{\infty} \left[\psi \frac{d}{dx}\phi_n - \phi_n \frac{d}{dx}\psi\right]_{-\bar{c}}^{\bar{c}} \\ &\qquad \times [(k^2 + w + D^2)^{-1}]_{nm}\,,\end{aligned} \tag{A6a}$$

where

$$\sum_{m=1}^{\infty} [k^2 + w + D^2]_{lm}[(k^2 + w + D^2)^{-1}]_{mn} = \delta_{ln}\,. \tag{A6b}$$

This expression may be rewritten to read

$$\psi(x) = \left[\psi(y)\frac{d}{dy}G(y, x) - G(y, x)\frac{d}{dy}\psi(y)\right]_{y=-\bar{c}}^{y=\bar{c}}\,, \tag{A7a}$$

where

$$G(y, x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_n(y)[(k^2 + w + D^2)^{-1}]_{nm}\phi_m(x)\,, \tag{A7b}$$

and is valid for $|x| < \bar{c}$.

We choose the ϕ_n's to satisfy homogeneous boundary conditions at $\pm\bar{c}$. Then the Green's function $G(y, x)$ will be symmetric. By means of the Green's function we have established a relationship between the asymptotic behavior of ψ and the dynamics of the system in the interaction region.

Now we set $x = \pm c$ and combine Eqs. (A2) and (A6). The result is two linear equations for f_+ and f_-.

$$f_+ Z_{++} + f_- Z_{-+} = z_+\,, \tag{A8a}$$

$$f_+ Z_{+-} + f_- Z_{--} = z_-\,, \tag{A8b}$$

$$Z_{++} = e^{ikc} - e^{ik\bar{c}}\left(\frac{d}{d\bar{c}} - ik\right)G(-\bar{c}, c)\,, \tag{A8c}$$

$$Z_{-+} = e^{ik\bar{c}}\left(\frac{d}{d\bar{c}} - ik\right)G(-\bar{c}, c)\,, \tag{A8d}$$

$$Z_{--} = e^{ikc} - e^{ik\bar{c}}\left(\frac{d}{d\bar{c}} - ik\right)G(-\bar{c}, -c)\,, \tag{A8e}$$

$$Z_{+-} = -e^{ik\bar{c}}\left(\frac{d}{d\bar{c}} - ik\right)G(\bar{c}, -c)\,, \tag{A8f}$$

$$z_+ = -Z_{++} + e^{-ik\bar{c}}\left(\frac{d}{d\bar{c}} + ik\right)G(-\bar{c}, c)\,, \tag{A8g}$$

$$z_- = -Z_{+-} = -e^{-ikc} + e^{-ik\bar{c}}\left(\frac{d}{d\bar{c}} + ik\right)G(-\bar{c}, -c)\,. \tag{A8h}$$

Equation (A8) is equivalent to the Peierls-Wigner form of R-matrix theory.[10] For our purposes it proves convenient to modify these expressions somewhat.

Define the unperturbed Green's function to be

$$G_0(y, x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \phi_n[(k^2 + D^2)^{-1}]_{nm}\phi_m(x)\,. \tag{A9}$$

Then one can easily establish the relationship

$$G(y, x) = G_0(y, x) - \int dz\, G_0(y, z)w(z)G(z, x)\,, \tag{A10a}$$

which in operator notation reads

$$G = G_0 - G_0 w G\,. \tag{A10b}$$

Another relation which can be derived is

$$G = G_0 - G w G_0\,. \tag{A10c}$$

Combining Eqs. (A10b) and (A10c) gives

$$G = G_0 - G_0 X G_0\,, \tag{A11a}$$

$$X(x, y) = w(x)\delta(x-y) - w(x)G(x, y)w(y)\,. \tag{A11b}$$

We will substitute Eq. (A11) into Eq. (A8) and find an expression for $f_\pm$ in terms of X rather than G. The relationship given in Eq. (A8) relates $f_\pm$ to the values and derivatives of $G(x, y)$ for x and y being in the asymptotic region. By expressing the $f_\pm$'s in terms of X instead of directly in terms G, we end up being required to calculate an average of G over the interaction region rather than the value and derivative of G at c and $\bar{c}$.

To carry out our program we need an explicit, simple representation of G_0. Actually, G and G_0 depend on the boundary conditions satisfied by the

ϕ_n's at $x=\pm\bar{c}$, and these boundary conditions may be chosen arbitrarily. We will consider only two of the infinity of possibilities. One choice corresponds to what are called "outgoing wave" or "causal" boundary conditions:

$$G_0(x, x') \equiv G_0(x, x')^{(+)} = \frac{e^{ikx_>} e^{-ikx_<}}{2ik}, \tag{A12a}$$

$$X(x, x') \equiv X(x, x')^{(+)} = T(x, x'). \tag{A12b}$$

For this choice we will call $X^{(+)} \equiv T$ the T matrix. The other choice corresponds to what are called "standing wave" or "principal value" boundary conditions:

$$G_0(x, x') \equiv G_0(x, x')^{(0)} = \frac{\cos kx_> \sin kx_<}{-k}, \tag{A13a}$$

$$X(x, x') \equiv X(x, x')^{(0)} \equiv K(x, x'). \tag{A13b}$$

For this choice we will call $X^{(0)} \equiv K$ the K matrix.

Substitution of Eq. (A11) into Eq. (A8) using the definitions of Eq. (A12) gives

$$f_\pm = \frac{i}{2k}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, e^{ikx} T(x, y)\, e^{\mp iky}, \tag{A14a}$$

$$T(x, y) = w(x)\delta(x-y) - w(x)G(x, y)^{(+)}w(y), \tag{A14b}$$

$$G(x, y)^{(+)} = G_0(x, y)^{(+)} - \int_{-\infty}^{\infty} dz\, G_0(x,z)^{(+)} w(z) G(z, y)^{(+)}. \tag{A14c}$$

On the other hand, if the definitions of Eq. (A13) are used in Eqs. (A8) and (A11), we find

$$f_+(1 - iK_{ss}) + f_-(-1 - K_{cs}) = K_{cs} + iK_{ss}, \tag{A15a}$$

$$f_+(1 + K_{cs}) + f_-(1 - iK_{cc}) = -K_{cs} + iK_{cc}, \tag{A15b}$$

$$K_{ss} = \frac{1}{k}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \sin kx\, K(x, y) \sin ky, \tag{A15c}$$

$$K_{cc} = \frac{1}{k}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \cos kx\, K(x, y) \cos ky, \tag{A15d}$$

$$K(x, y) = w(x)\delta(x-y) - w(x)G(x, y)^{(0)} w(y), \tag{A15e}$$

$$G(x, y)^{(0)} = G_0(x, y)^{(0)} - \int_{-\infty}^{\infty} dz\, G_0(x, z)^{(0)} w(z) G(z, y)^{(0)}. \tag{A15f}$$

APPENDIX B: THE MULTIPLE-SCATTERING EXPANSION

To derive the multiple-scattering expansion we will employ an operator notation. Let us represent Eqs. (4b) and (5e) by

$$X(x, y) = w(x, y) - \int dx' \int dy' w(x, y') G(y', x') w(x', y), \tag{B1a}$$

where X is either the T matrix or the K matrix depending on whether G is $G^{(+)}$ or $G^{(0)}$. The quantity $w(x, y)$ is just $w(x)\delta(x-y)$. Our operator notation will consist in writing an equation like Eq. (B1a) in the form

$$X = w - wGw. \tag{B1b}$$

From the definition of G, Eqs. (4c) and (5f), we can write

$$G(x, y) = \left(\delta(x-y)\frac{d^2}{dx^2} + w(x, y) + k^2\delta(x-y)\right)^{-1} \tag{B2a}$$

or

$$G = (D^2 + w + k^2)^{-1}. \tag{B2b}$$

Now define the unperturbed Green's function.

$$G_0 = (D^2 + k^2)^{-1}. \tag{B3}$$

Note that

$$G_0^{-1} = G^{-1} - w \tag{B4a}$$

so that by operating on both sides of Eq. (B4a) with G_0 and G we find

$$G = G_0 - G_0 w G. \tag{B4b}$$

Substituting Eq. (B4b) into Eq. (B1b) gives

$$\begin{aligned} X &= w - wG_0 X \\ &= (1 + wG_0)^{-1} w. \end{aligned} \tag{B5}$$

For our two-potential case we can write

$$\begin{aligned} w(x, y) &= w_1(x, y) + w_2(x, y) \\ &= -\nu\delta(x - \tfrac{1}{2}b)\delta(x-y) - \nu\delta(x + \tfrac{1}{2}b)\delta(x-y). \end{aligned} \tag{B6}$$

We make a corresponding decomposition of the X matrix.

$$X = X_1 + X_2, \tag{B7a}$$

$$X_j = w_j - w_j G_0 (X_1 + X_2). \tag{B7b}$$

Equation (B7b) may be rearranged in the following manner:

$$(1 + w_j G_0) X_j = w_j - w_j G_0 (X - X_j) \tag{B8}$$

or

$$X_j = x_j - x_j G_0 (X - X_j), \tag{B9a}$$

$$\begin{aligned} x_j &= (1 + w_j G_0)^{-1} w_j \\ &= w_j - w_j G_0 x_j. \end{aligned} \tag{B9b}$$

The quantity x_j may be recognized as the X matrix for the case where all the terms on the right-hand side of Eq. (B6) vanish except w_j. Thus x_j is the exact solution of the one-potential problem. The Watson multiple-scattering formalism[11] for our two-potential problem may be summarized now as follows:

$$X = X_1 + X_2\,, \tag{B10a}$$

$$X_1 = x_1 - x_1 G_0 X_2\,, \tag{B10b}$$

$$X_2 = x_2 - x_2 G_0 X_1\,. \tag{B10c}$$

Successive iteration yields the multiple-scattering expansion.

$$X = x_1 + x_2 - x_1 G_0 x_2 - x_2 G_0 x_1 + x_1 G_0 x_2 G_0 x_1 + x_2 G_0 x_1 G_0 x_2 + \cdots\,. \tag{B11}$$

The zero-range interaction we use may be written in the form

$$w_j(x, y) = -\nu \delta(x - a_j)\delta(y - a_j)\,. \tag{B12}$$

From this form it is easy to verify that the solution to Eq. (B9b) is

$$x_j(x, y) = \delta(x - a_j)\beta_j \delta(y - a_j)\,, \tag{B13a}$$

$$\beta_j = -\nu/[1 - \nu G_0(a_j, a_j)]\,. \tag{B13b}$$

Substitution of Eq. (B13) into Eq. (B11) allows us to sum the multiple-scattering expansion.

$$\begin{aligned} X(x, y) = {} & \delta(x - \tfrac{1}{2}b)\delta(y - \tfrac{1}{2}b)\beta_1(1 - g\beta_2 g\beta_1)^{-1} \\ & + \delta(x + \tfrac{1}{2}b)\delta(y + \tfrac{1}{2}b)\beta_2(1 - g\beta_1 g\beta_2)^{-1} \\ & - \delta(x + \tfrac{1}{2}b)\delta(y - \tfrac{1}{2}b)\beta_2 g\beta_1(1 - g\beta_2 g\beta_1)^{-1} \\ & - \delta(x - \tfrac{1}{2}b)\delta(y + \tfrac{1}{2}b)\beta_1 g\beta_2(1 - g\beta_1 g\beta_2)^{-1}\,, \end{aligned} \tag{B14a}$$

$$g = G_0(\tfrac{1}{2}b, -\tfrac{1}{2}b) = G_0(-\tfrac{1}{2}b, \tfrac{1}{2}b)\,, \tag{B14b}$$

$$\beta_1 = -\nu[1 - \nu G_0(\tfrac{1}{2}b, \tfrac{1}{2}b)]^{-1}\,, \tag{B14c}$$

$$\beta_2 = -\nu[1 - \nu G_0(-\tfrac{1}{2}b, -\tfrac{1}{2}b)]^{-1}\,. \tag{B14d}$$

To evaluate these expressions explicitly we need G_0. For the $X = T$ case we have

$$G_0(x, x')^{(+)} = \frac{e^{ikx_>} e^{-ikx_<}}{2ik} \tag{B15a}$$

while for the $X = K$ case we have

$$G_0(x, x')^{(0)} = \frac{\cos kx_> \sin kx_<}{-k}\,. \tag{B15b}$$

Substituting these expressions into Eq. (15) gives

$$\begin{aligned} T(x, y) = {} & [\delta(x - \tfrac{1}{2}b)\delta(y - \tfrac{1}{2}b) + \delta(x + \tfrac{1}{2}b)\delta(y + \tfrac{1}{2}b)]\, 2ik\Lambda(1 - B^2\Lambda^2)^{-1} \\ & - [\delta(x + \tfrac{1}{2}b)\delta(y - \tfrac{1}{2}b) + \delta(x - \tfrac{1}{2}b)\delta(y + \tfrac{1}{2}b)] 2ik\Lambda^2 B(1 - B^2\Lambda^2)^{-1}\,, \end{aligned} \tag{B16a}$$

$$\begin{aligned} K(x, y) = {} & \left(\frac{\delta(x - \tfrac{1}{2}b)\delta(y - \tfrac{1}{2}b)}{1 + \gamma \sin kb} + \frac{\delta(x + \tfrac{1}{2}b)\delta(y + \tfrac{1}{2}b)}{1 - \gamma \sin kb}\right) \frac{(-2k\gamma)(1 - 2\gamma^2 \sin^2 kb)}{1 - \gamma^2 \sin^2 kb} \\ & - [\delta(x + \tfrac{1}{2}b)\delta(y - \tfrac{1}{2}b) + \delta(x - \tfrac{1}{2}b)\delta(y + \tfrac{1}{2}b)]\, \frac{(2k\gamma^2 \sin kb)(1 - 2\gamma^2 \sin^2 kb)}{1 - \gamma^2 \sin^2 kb}\,. \end{aligned} \tag{B16b}$$

$$\gamma = \nu/2k\,, \tag{B16c}$$

$$\Lambda = -i\gamma/(1 + i\gamma)\,, \tag{B16d}$$

$$B = e^{ikb}\,. \tag{B16e}$$

Substitution of Eq. (B16a) into Eq. (4a) and the substitution of Eq. (B16b) into Eqs. (5a)–(5d) both produce the exact result of Eq. (12).

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