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Complex Pomeranchukon Singularities and Asymptotic Behavior*

Roger Zaoui†
Center for Theoretical Physics, Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

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We assume that the partial-wave amplitude of the scattering of two scalar bosons, in the t channel, has n complex-conjugate pairs of Pomeranchukon singularities of the form $\alpha(t)=1\pm a\sqrt{t}$. We consider the cases where the complex singularities appear in an additive way and in a multiplicative way in the partial-wave amplitude. We obtain for the scattering amplitude an asymptotic expansion of the form

$$F(s,t)\underset{s\to\infty;\,t<0}{\sim}\sum_{k=1}^{n}a_k(t)(\ln s)^k g(z),$$

where $g(z)$ is an entire function of the argument $z=\sqrt{-t}\,\ln s$. We discuss properties of the forward spike in the differential cross section, and the properties of the amplitude in connection with the Froissart bound and the violation of the Pomeranchuk theorem.

We consider the two-body scalar-boson collision $1+2\to 3+4$. There is a single scattering amplitude $F(s,t)$, and the crossing symmetry gives $F(-s+i\epsilon)=F^*(s+i\epsilon)$. We construct the symmetrical amplitude $F_s(s,t)=\frac{1}{2}(F_1+F_2)$ and the antisymmetrical amplitude $F_a(s,t)=\frac{1}{2}(F_1-F_2)$, where $F_1(s,t)$ and $F_2(s,t)$ are the scattering amplitude for the processes $1+2\to 3+4$ and $1+\overline{3}\to\overline{2}+4$, respectively. The crossing then gives

$$F_s(se^{i\pi})=F_s^*(s),$$

$$F_a(se^{i\pi})=-F_a^*(s).$$

The optical theorem relates the total cross section σ_T to the imaginary part of the forward elastic amplitude; thus we have

$$\mathrm{Im}F_i(s,0)=2k\,s^{1/2}\sigma_T(s),\quad i=1,2 \tag{1}$$

where σ_T is bounded by the Froissart bound.[1] By definition we have

$$\sigma_{\rm el}=\frac{1}{16\pi}\int_{4m^2-s}^{0}\frac{|F(s,t)|^2dt}{s(s-4m)^2}$$

$$=\frac{1}{16\pi}\int_{4m^2-s}^{0}\frac{d\sigma}{dt}dt. \tag{2}$$

The unitarity relation provides that

$$\sigma_{\rm el}<\sigma_T, \tag{3}$$

so as the energy increases the integrated elastic cross section and the total cross section remain bounded. The forward spike of the scattering is defined by its width $[k(s)]^{-1}$ and height $(d\sigma/dt)_{t=0}$ which are exhibited by the parametrization $d\sigma/dt=(d\sigma/dt)_{t=0}e^{2kt}$. We shall also use the asymptotic phase of the amplitude which is defined by the ratio $\mathrm{Re}F^{\pm}(s,0)/\mathrm{Im}F^{\pm}(s,0)$.

The high-energy scattering amplitude is constructed in the usual way by means of the Sommerfeld-Watson transformation:

$$F^{\pm}(s,t)\underset{s\to\infty;\,t<0}{\sim}\frac{1}{2i\pi}\int_{c_j}s^j\,\xi^{\pm}(j)a^{\pm}(j,t)\,dj, \tag{4}$$

where $\xi^{\pm}(j)$ is the usual signature factor

$$\frac{1\pm e^{-i\pi j}}{\sin(\pi j)}.$$

$\xi^{+}(j)$ is purely imaginary at $t=0$ and $\xi^{-}(j)$ exhibits a right-signature pole at $j=1$, so we shall use

$$\xi^{-}(j)\sim i-\frac{2}{\pi}\frac{1}{j-1}.$$

$a^{\pm}(j,t)$ is the partial-wave amplitude in the t channel which can have all singularities allowed by the unitarity relation, namely poles and cuts.

Let us consider the analytic properties of the function $[a^{\pm}(j,t)]^{-1}$ in the (j,t) manifold near the point (1, 0). We shall use the same procedure as in Ref. 2. The position of the singularity of the amplitude in the angular momentum plane is given by the equation $[a^{\pm}(j,t)]^{-1}=0$ at $j=\alpha(t)$, and we investigate the structure of this singularity in the neighborhood of the point (1, 0). We assume that $[a^{\pm}(j,t)]^{-1}$ is regular in the neighborhood of $(t=0, j=1)$; by expanding it in a Taylor series about $t=0$, we obtain

$$[a_j^{\pm}(t)]^{-1}=\left(\frac{\partial}{\partial j}[a_j^{\pm}(0)]^{-1}\right)_{j=1}[\alpha(t)-1] +\left(\frac{\partial}{\partial t}[a_j^{\pm}(1)]^{-1}\right)_{t=0}t +\left(\frac{\partial^2}{\partial j^2}[a_j^{\pm}(0)]^{-1}\right)_{j=1}[\alpha(t)-1]^2+\cdots .$$

Now we require the vanishing of the first derivative in j of this function. If we do that (as we are free to do) then we obtain from the above expansion $\alpha(t)=1\pm a\sqrt{t}$. The a parameter is given by the square root of the ratio

$$\left(\frac{\partial}{\partial t}[a_1^{\pm}(t)]^{-1}\right)_{t=0}\Big/\left(\frac{\partial^2}{\partial j^2}[a_j^{\pm}(0)]^{-1}\right)_{j=1}.$$

The singular structure of $\alpha(t)$ is clearly a consequence of the analyticity of $[a^{\pm}(l,t)]^{-1}$. The Regge trajectory $\alpha(t)$ is singular and exhibits near the point (1, 0) a Riemann-sheet structure of the form $\alpha(t)=1\pm a\sqrt{+t}$, so in the physical s channel $(t<0)$, $\alpha(t)$ is a double-valued function, and we have two Pomeranchukon trajectories which cross at $t=0$, and $t=0$ is a branch point for both. This result suggests that Regge poles have left-hand branch cuts in t from $t=0$ to $t=-\infty$.

The above rough unconditional assumption about the j derivative of $[a^{\pm}(j,t)]^{-1}$ can only be justified by using the fact that such a similar singular structure has been deduced by using specific dynamic models.[3] In these models the consequences of the collision of a Regge pole with a Mandelstam branch cut are investigated. The main result of such a collision is the formation of a complex conjugate pair in the physical complex j plane, together with a left-hand cut in t in the Regge trajectory, of a square-root character.

The partial-wave amplitude $a^{\pm}(j,t)$ also contains cuts, and it is believed that at least some of these cuts can be thought of as generated by Regge poles. The location of the branch point $\alpha_c(t)$ of such cuts in the j plane can be written

$$\alpha_c(t)=\max[\alpha_1(t_1)+\alpha_2(t_2)-1],$$

in terms of two poles $\alpha_1(t_1)$ and $\alpha_2(t_2)$ which cause it, and the Regge cut will inherit the Riemann-sheet structure from the Regge poles which cause it; therefore a pair of Pomeranchukon trajectories has the property that it will intersect the cut at $t=0$. Then the cut itself together with another Pomeranchukon should by the same argument be expected to generate another cut with branch point at

$$\alpha_c^2(t)=\max[\alpha_c(t')+\alpha_P(t'')-1].$$

The latter cut has even a smaller slope than the first one, and at $t=0$ it satisfies $\alpha_c^2(c)=\alpha_1(0)$. If such an argument is repeated as we combine cuts and poles, then at $t=0$ we find a degeneracy between the Pomeranchukon cut and the Pomeranchukon pole with a condensation of an infinite number of complex singularities of the form $j=1\pm ia\sqrt{-t}$ at $j=1$ and $t=0$.

We consider, in this paper, the exchange of n complex Pomeranchukon singularities in the t channel of a two-body scalar collision. The Pomeranchukon singularities are parametrized in the physical region of the s channel as $\alpha_{\pm}(t)=1\pm ia\sqrt{-t}$. Let $\beta^{\pm}(\sqrt{t})$ be the Pomeranchukon residue which is analytic on the negative-t axis. We know that relation (3) together with the order and the nature of the singularity in the partial wave, in the forward direction, are in general enough for the determination of the properties of the forward spike in which we are interested (shrinking, zeros, oscillations, and asymptotic phase of the amplitude). Therefore a complete understanding of a diffraction scattering depends on the summation of all these singularities.

We consider first the case of complex conjugate pairs of singularities which appear in an additive way in the partial-wave amplitude. Using (4) we have in terms of Bessel functions with large arguments

$$\operatorname{Im}F^{\pm}(s,t)\underset{s\to\infty;\text{fixed }t}{\sim}\beta^{\pm}(t)s\frac{(\ln s)^{n-1}}{(n-1)!}\sqrt{\pi z}\,[J_0(z)-Y_0(z)], \tag{5a}$$

$$\operatorname{Re}F^{+}(s,t)\underset{s\to\infty;\text{fixed }t}{\sim}\beta^{+}(t)\sum_{k=0}^{n-1}s\frac{(\ln s)^{n-1-k}}{k!(n-1-k)!}\left(\frac{\pi z}{2}\right)^{1/2}\left[e^{i\pi/4}H_{\gamma}^{1}(z)(\cot\tfrac{1}{2}\pi j)^{(k)}_{j=\alpha_+}+e^{-i\pi/4}H_{\gamma}^{2}(z)(\cot\tfrac{1}{2}\pi j)^{(k)}_{j=\alpha_-}\right], \tag{5b}$$

$$\mathrm{Re}F^{-}(s,t)_{\substack{\sim\\ s\to\infty;\,\text{fixed}\,t}} -\frac{2s}{\pi}\beta^{-}(t)\left[E+\sum_{k=1}^{n}\frac{A_k}{(k-1)!}(\ln s)^{k-1}\left(\frac{\pi z}{2}\right)^{1/2}e^{i\pi/4}H_\gamma^1(z)+\sum_{k=1}^{n}\frac{B_k}{(k-1)!}(\ln s)^{k-1}\left(\frac{\pi z}{2}\right)^{1/2}e^{i\pi/4}H_\gamma^2(z)\right], \tag{5c}$$

where

$$H_\gamma^1(z)=J_0(z)+iY_0(z)\,,\quad H_\gamma^2(z)=J_0(z)-iY_0(z)\,.$$

$J_0(z)$ and $Y_0(z)$ are Bessel functions of argument $z=a\sqrt{-t}\ \ln s$ which have at large $\ln s$ the asymptotic behavior[4]

$$J_\gamma(z)=\left(\frac{2}{\pi z}\right)^{1/2}[P(\gamma,z)\cos x-Q(\gamma,z)\sin x],$$

$$Y_\gamma(z)=\left(\frac{2}{\pi z}\right)^{1/2}[P(\gamma,z)\cos x+Q(\gamma,z)\sin x],$$

where

$$x=z-(\tfrac{1}{2}\gamma+\tfrac{1}{4})\pi,\quad Q(\gamma,z)\sim\sum_{k=0}^{\infty}(-1)^k\frac{(\gamma,2k+1)}{(2z)^{2k+1}},\quad P(\gamma,z)\sim\sum_{k=0}^{\infty}(-1)^k\frac{(\gamma,2k)}{(2z)^{2k}}\,.$$

From (1) and (5a) we deduce an asymptotic expansion for the total cross section

$$\sigma_T^{\pm}=\beta^{\pm}(0)\frac{(\ln s)^{n-1}}{(n-1)!}\,.$$

The Froissart bound is saturated when three complex pairs of Pomeranchukon singularities are exchanged in the t channel. Equations (5a) and (5c) lead to an asymptotic phase for the antisymmetrical amplitude which increases logarithmically with the energy, so that in the forward direction the forward spike is dominated by the real part and, therefore, the Pomeranchuk theorem does not hold.[5] Using the constraint (2) we can see from (5a) and (5c) that the width of the forward peak depends upon the energy and upon the number of Pomeranchukon trajectories exchanged in the t channel. The dependence of the diffractive cone expected is $k(s)\sim(\ln s)^{n+1}$. For $n=1$ in the forward direction, this amplitude coincides exactly with the Finkelstein model[6] and has the same properties (shrinking, zeros, oscillations, asymptotic phase). Contrary to the negative signature amplitude the symmetrical amplitude has a phase which decreases logarithmically with the energy. In the forward direction, therefore, the imaginary part dominates on the real part, and for this amplitude the Pomeranchuk theorem is fulfilled.[5] For $n=1$ this amplitude coincides exactly with the amplitude given by Sugawara and Arafune[7,8]: For this amplitude the unitarity constraint (3), with the definition (2), (5a), and (5b), leads to an energy-independent diffraction spike. This result does not agree with axiomatic results of Eden and Kaiser,[9] who found that the behavior of the symmetrical amplitude is very similar to that of the antisymmetrical amplitude. Consequently we can conclude that n complex conjugate pairs of Pomeranchukon singularities which enter in the partial-wave amplitude of negative (positive) signature, in an additive manner, do (do not) lead to a violation of the Pomeranchuk theorem.

Let us now consider the case where Pomeranchukon singularities enter in the partial-wave amplitude in a multiplicative way. Using the properties of the Mellin integrals, we have

$$\mathrm{Im}F^{\pm}(s,t)\sim\mp\beta^{\pm}(t)s\left[\sum_{k\,\text{even}}(\ln s)^{2k-1}A_{2k}\sqrt{\pi z}\,[J_0(z)-Y_0(z)]+\sum_{k\,\text{odd}}(\ln s)^{2k-2}A'_{2k-1}(x)\sqrt{\pi z}\,[J_0(z)+Y_0(z)]\right], \tag{6a}$$

$$\mathrm{Re}F^{+}(s,t)\sim s\beta^{+}(t)\left[\sum_{k=1}^{n}\sum_{m=0}^{k-1}\frac{A_k}{m!(k-1-m)!}(\ln s)^{k-1-m}(\cot\tfrac{1}{2}\pi j)^{(m)}_{j=\alpha_+}\left(\frac{\pi z}{2}\right)^{1/2}e^{i\pi/4}H_0^1(z)\right.$$
$$\left.+\sum_{k=1}^{n}\sum_{m=0}^{k-2}\frac{B_k}{m!(k-1-m)!}(\ln s)^{k-1-m}(\cot\tfrac{1}{2}\pi j)^{(m)}_{j=\alpha_-}\left(\frac{\pi z}{2}\right)^{1/2}e^{-i\pi/4}H_0^2(z)\right], \tag{6b}$$

$$\mathrm{Re}F^{-}(s,t)\sim\frac{2s}{\pi}\beta^{-}(t)\left[\sum_{k=1}^{n}\frac{C_k}{(k-1)!}(\ln s)^{k-1}\left(\frac{\pi z}{2}\right)^{1/2}e^{i\pi/4}H_0^1(z)+\sum_{k=1}^{n}\frac{D_k}{(k-1)!}(\ln s)^{k-1}\left(\frac{\pi z}{2}\right)^{1/2}e^{-i\pi/4}H_0^2(z)+Es\right]. \tag{6c}$$

From (1) and (6a) we deduce an asymptotic expansion for the total cross section:

$$\sigma_T^{\pm} = \mp 2\beta^{\pm}(0)\left[\sum_k \frac{A_{2k}}{(2k-1)!}(\ln s)^{2k-2} + \sum_k \frac{A'_{2k-1}}{(2k-2)!}(\ln s)^{2k-1}\right].$$

The Froissart bound imposes $n=2$ and therefore only the exchange of two complex conjugate pairs of Pomeranchukon singularities is allowed. In this case, the total cross section for the symmetrical process for large s is $\sigma_T^{+} \simeq \beta^{\pm}(0)\ln s$ and the asymptotic phase grows logarithmically with the energy. Therefore the Pomeranchuk theorem does not hold. Such a component in σ_T^{+} which rises as $\ln s$ with s will give a rapid growth in σ_T^{+} at asymptotic energies, and therefore this term must be coupled very weakly at the total cross section in order to provide an explanation for the flatness of the Serpukhov data in the 35–65 GeV range.

To summarize:

(1) An asymptotic expansion of the form

$$F(s,t) \underset{s\to\infty;\, t<0}{\sim} \sum_{k=1}^{n} a_k(\ln s)^k g(z)$$

has been deduced for the scattering amplitude by using the exchange of n complex conjugate pairs of Pomeranchuk singularities in the t channel of a two-body scalar-boson scattering amplitude. [$g(z)$ is an entire function of the parameter $z=\sqrt{-t}\,\ln s$.] As Casella[10] has pointed out to us, for additive singularities, in the limit of a weak dependence of the Pomeranchukon residue, the symmetrical and the antisymmetrical amplitude can, for arbitrary large s, be put in the form

$$F^{\pm}(s,t) = F^{\pm}(s,0)\,g(z).$$

(2) The Froissart bound removes the degeneracy on the number of singularities which occurs in the partial-wave amplitude in the forward direction.

(3) The additive singularities lead to a class of antisymmetrical amplitudes which give a Pomeranchuk violation, while multiplicative singularities lead to a class of symmetrical amplitudes which give rise to the violation of the Pomeranchuk theorem.

(4) For three complex Pomeranchukon singularities we obtain $\sigma = (\ln s)^2$, and a vanishing asymptotic phase. Similar results have been obtained in other ways. Finkelstein and Zachariasen[11] predict by using the multiperipheral model $\sigma_T = (\ln s)^2$ and a purely absorptive amplitude. Andreev[12] also obtains $\sigma_T = (\ln s)^2$ and a vanishing asymptotic phase.

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†NATO Fellow, on leave of absence from Laboratoire de Physique Nucléaire, Orsay, France.

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