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Electromagnetic Interaction of Dual Systems*

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We introduce the electromagnetic interaction of dual models by means of a generalized minimum-interaction principle. It leads to a conserved current and a gauge-invariant theory. Unfortunately, the strong gauge conditions are preserved only for on-mass-shell photons, which leads to unphysical poles in $k\cdot k'$.

INTRODUCTION

The gravest problem that confronts dual-resonance models is our lack of understanding of their interactions along conventional physical lines.[1] This may be due to the nonperturbative nature[2] of the interaction in the usual sense (for instance the nonanalyticity of the coupling constant in the width parameter[3]). Whatever the cause, one must understand how to implement the particle widths in a way that restores unitarity. On the positive side, we have a relativistic system which has a built-in mechanism[4] to eliminate the ghosts that arise from the Lorentz metric. This astounding faculty, shared by no other hadronic model, clearly suggests its importance if only as a formal tool in the immediate future. It is for this reason that we present a study of the interaction of free dual atoms with external fields. Most of the results are known in one form or another except for the case of the electromagnetic field interacting with a dual fermionic atom. We show our theory to have a conserved current and gauge invariance, and we present arguments for the existence of strong gauge conditions compatible with the interaction. However, it yields a structureless proton, which we believe to be a result of the zero-width approximation.

Section I reviews the formalism of free dual systems, bosonic and fermionic. In Sec. II, we present our interaction schemes and discuss the form factors while Sec. III is essentially a calculation of the Compton scattering amplitude.

I. FREE DUAL SYSTEMS

We devote this section to a review of the description of hadrons (bosons and fermions) in the zero-width approximation,[5] i.e., when all the possible states a hadron can occupy are stable. The theory is best understood in terms of an analogy with a relativistic quantum-mechanical system evolving along a "path" $Q_\mu(\tau)$. The departure from the conventional description of a point particle is achieved by generalizing the path functions $Q_\mu(\tau)$ to include an infinite number of degrees of freedom which generate an internal motion periodic in its time parameter τ. This generalized position operator is given by

$$Q_\mu(\tau)\equiv e^{-i\tau H_B}\left(\sum_{n=0}^{\infty} q_\mu^{(n)}\right)e^{i\tau H_B} \tag{1.1}$$

while its conjugate, the generalized momentum coordinate, is

$$P_\mu(\tau)\equiv e^{-i\tau H_B}\left(\sum_{n=0}^{\infty} p_\mu^{(n)}\right)e^{i\tau H_B}\,, \tag{1.2}$$

where the generator of internal motions is the Nambu Hamiltonian[6]

$$H_B=\tfrac{1}{2}\sum_{n=0}^{\infty}\left[p^{(n)}\cdot p^{(n)}+(\omega_0+2n\pi/T)^2q^{(n)}\cdot q^{(n)}\right] \tag{1.3}$$

and the normal mode coordinates satisfy

$$\begin{aligned}&[q_\alpha^{(n)},q_\beta^{(m)}]=[p_\alpha^{(n)},p_\beta^{(m)}]=0\,,\\ &[q_\alpha^{(n)},p_\beta^{(m)}]=-ig_{\alpha\beta}\delta_{n,m},\quad n,m=0,1,2,\ldots;\end{aligned} \tag{1.4}$$

here $g_{\alpha\beta}=(1,-1,-1,-1)$ is the Lorentz metric, while T is the period of the internal motion.

We then postulate that the conventional observables of the system are just the averages of their generalized counterparts over the internal space when $\omega_0\to 0$, i.e., when the lowest mode becomes translational. Define the average of an operator $A(\tau)$ by

$$\langle A\rangle\equiv\frac{1}{T}\int_{-T/2}^{+T/2}d\tau A(\tau)\,. \tag{1.5}$$

Thus the position and momentum of the particle are given by

$$x_\mu=\langle Q_\mu\rangle\,, \tag{1.6a}$$

$$p_\mu=\langle P_\mu\rangle\,, \tag{1.6b}$$

respectively. These are consistent with the commutation relations

$$[Q_\mu(\tau),P_\nu(\tau')]=-ig_{\mu\nu}\delta((\tau-\tau')/T),\quad \mathrm{mod}(T) \tag{1.7}$$

obtained from Eqs. (1.1) through (1.4). It then follows that the equation of motion of the bosonic system is just the generalization of the Klein-Gordon equation, namely

$$[:\langle P\cdot P\rangle:-m^2]\,|\,\phi\rangle=0\,. \tag{1.8}$$

Note the normal ordering of the internal modes to eliminate the zero-point energy. The solutions of this equation lie on a family of parallel straight-line Regge trajectories whose slope is identified with the period of the internal motion.[7] The remarkable feature of this equation, however, is its compatibility with an infinite number of subsidiary conditions obtained by applying the correspondence principle to the elimination of the ghosts appearing in the theory. These are

$$L_n^B|\,\phi\rangle=0,\quad n=1,2,\ldots \tag{1.9}$$

where

$$L_n^B=:\langle e^{i2\pi n\tau/T}P\cdot P\rangle:,\quad n=0,\pm1,\pm2,\ldots \tag{1.10}$$

are the Virasoro operators.[4] These conditions can be derived in a less heuristic manner by requiring that the internal space be spurious, i.e., that τ not be observable.[8] An amusing way of seeing this is to consider the matrix element of the operator $P_\mu(\tau)P^\mu(\tau)$. The necessary condition that this matrix element be independent of τ is that the states obey the constraints (1.9) since

$$\langle\phi\,|P_\mu(\tau)P^\mu(\tau)|\,\phi'\rangle=\langle\phi\,|\,L_0^B|\,\phi'\rangle+\sum_{n=1}^{\infty}(\langle\phi\,|\,L_{-n}^B|\,\phi'\rangle e^{i2\pi n\tau/T}+\langle\phi\,|\,L_n^B\,|\phi'\rangle e^{-i2\pi n\tau/T})\,. \tag{1.11}$$

This is equivalent to a super gauge condition,[9] which, as far as it can be calculated,[10] eliminates the ghost states from the theory. In this respect, the dual theory stands alone since it describes an infinite number of relativistic states without ghosts. The big problem to be faced later is of course the introduction of interactions compatible with the constraints.

The generalization of the Dirac equation evolves along similar lines by introducing super Dirac matrices, $\Gamma_\mu(\tau)$, which obey *local* anticommutation relations in the internal space and reduce to the ordinary Dirac matrices when averaged over the internal motions, i.e.,

$$\langle\Gamma_\mu(\tau)\rangle=\gamma_\mu\,, \tag{1.12a}$$

$$\{\Gamma_\mu(\tau),\Gamma_\nu(\tau')\}=2g_{\mu\nu}\delta((\tau-\tau')/T),\quad \mathrm{mod}(T) \tag{1.12b}$$

from which

$$\Gamma_\mu(\tau)=\gamma_\mu+i\sqrt{2}\,\gamma_5\sum_{n=1}^{\infty}(b_\mu^{(n)\dagger}e^{i2n\pi\tau/T}+b_\mu^{(n)}e^{-i2n\pi\tau/T})\,, \tag{1.13}$$

where the b's obey the anticommutation relations

$$\begin{aligned}&\{b_\mu^{(n)},b_\nu^{(m)}\}=0\,,\\&\{b_\mu^{(n)},b_\nu^{(m)\dagger}\}=-g_{\mu\nu}\delta_{n,m},\quad n,m=1,2,\ldots\,.\end{aligned} \tag{1.14}$$

It is then straightforward to write the generalized Dirac equation:

$$[\langle\Gamma\cdot P\rangle-m]\,|\,\psi\rangle=0\,. \tag{1.15}$$

Its solutions lie on straight-line trajectories. In particular, the mother trajectory has parity doublets *above* the ground state.[11] Again, the remarkable thing is that it is compatible with this equation to impose subsidiary conditions on its solutions. In analogy with the Rarita-Schwinger formalism, these are

$$F_n|\,\psi\rangle=0\,,\quad n=1,2,\ldots \tag{1.16}$$

where

$$F_n=\langle e^{i2\pi n\tau/T}\Gamma\cdot P\rangle\,,\quad n=0,\pm1,\pm2,\ldots\,. \tag{1.17}$$

These, together with (1.15), imply that

$$(L_n^B+L_n^F)\,|\,\psi\rangle=0\,,\quad n=1,2,\ldots \tag{1.18}$$

with

$$L_n^F=-\frac{i}{4}:\left\langle e^{i2n\pi\tau/T}\Gamma\cdot\left(\frac{d\Gamma}{d\tau}\right)\right\rangle:. \tag{1.19}$$

The important point in the description of a free dual system is that it does not seem to have ghost states even though it describes relativistically an infinite number of stationary states. Another feature of the approximation is that it treats the ground state as a point particle, i.e., the ground-state fermion obeys the ordinary Dirac equation. Although it would be tempting to interpret this state as a quark,[12] we choose to think of it as a bare proton on purely esthetic grounds, rather than to face the unthinkable three-body problem.

II. DUAL INTERACTIONS

The identification of $Q_\alpha(\tau)$ as the position operator of the dual theories, together with the correspondence principle,[5] allows for obvious generalizations of the coupling schemes that appear in the usual treatments of hadron interactions. Before tackling the electromagnetic case, it is amusing to consider in this light the coupling with an external scalar field. In spirit with the previous remarks, the interaction will take place at the generalized coordinate $Q_\alpha(\tau)$, averaged over the internal space with a suitable scalar density, $\rho_S(\tau)$, which will lead to the new equation of motion

$$[:\langle P\cdot P\rangle: - m^2 + g:\langle \rho_S \phi(Q)\rangle:]\,|\phi\rangle = 0\,. \tag{2.1}$$

In the case of unit scalar density we show that this equation leads to the Veneziano amplitude in the tree approximation. To see this, consider the interaction vertex in momentum space,

$$V(k) = \frac{1}{T}\int_{-T/2}^{+T/2} d\tau : e^{ik\cdot Q(\tau)}:\,, \tag{2.2}$$

and perform a Taylor expansion of the integrand around $\tau = 0$. All the terms of the series save the first one can be expressed as a commutator with $:\langle P^2\rangle:(=H_B)$ by means of the Heisenberg equations of motion in the internal space; these will accordingly decouple from the physical states. Thus we can write

$$V(k) \approx :e^{ik\cdot Q(0)}:, \tag{2.3}$$

which is the familiar scalar vertex.[6] This peculiarity of the model is very significant and comes from the fact that the propagator of space-time motion coincides with that of the internal space.

A more recent example is the interaction of a pseudoscalar field with the dual fermion system, giving rise to the equation

$$[\langle \Gamma\cdot P\rangle - m + g\langle \Gamma_5\phi(Q)\rangle]\,|\psi\rangle = 0\,, \tag{2.4}$$

where Γ_5 is the generalization of γ_5 of Neveu and Schwarz[13] and of Thorn[14]:

$$\Gamma_5 = \gamma_5(-1)^{\beta}\,, \tag{2.5}$$

where

$$\beta = \sum_{n=1}^{\infty} b^{(n)\dagger}\cdot b^{(n)}\,.$$

This theory has been investigated by the abovementioned authors and leads in bosonic channels to the Neveu-Schwarz model.[15]

We touched earlier on the problem of consistency of these interactions with the subsidiary conditions. It is the sad fate of these models that such a consistency exists only where the external fields have an imaginary rest mass.[16]

Similarly, the electromagnetic interaction of these systems should be obtained by some generalization of the usual minimal interaction,[17] which we take to be

$$P_\mu(\tau) \to P_\mu(\tau) - e:A_\mu(Q(\tau)):\,. \tag{2.6}$$

Its insertion into the generalized Klein-Gordon equation yields

$$[:\langle P\cdot P\rangle: - m^2 - e(\langle P\cdot :A:\rangle + \langle :A:\cdot P\rangle) + e^2\langle :A:\cdot :A:\rangle]\,|\phi\rangle = 0\,, \tag{2.7}$$

corresponding to an electromagnetic current in momentum space[18]

$$j_\mu^B(k) = \frac{1}{T}\int_{-T/2}^{+T/2} d\tau \{P_\mu(\tau), :e^{ik\cdot Q(\tau)}:\}\,. \tag{2.8}$$

It is, however, more interesting to consider the dual fermion electrodynamics. The equation of motion becomes

$$[\langle \Gamma\cdot P\rangle - m - e:\langle \Gamma\cdot A\rangle:]\,|\psi\rangle = 0\,, \tag{2.9}$$

from which we obtain the following current:

$$j_\mu^F(k) = \frac{1}{T}\int_{-T/2}^{+T/2} d\tau\, \Gamma_\mu(\tau) : e^{ik\cdot Q(\tau)}:\,. \tag{2.10}$$

Notice the close analogy with Dirac electrodynamics. Current conservation follows simply from the identity

$$k_\mu : e^{ik\cdot Q(\tau)}: = -[P_\mu(\tau), :\langle e^{ik\cdot Q}\rangle:]\,, \tag{2.11}$$

which is a consequence of Eq. (1.7). Thus,

$$k_\mu j_\mu^F(k) = -[\langle \Gamma\cdot P\rangle, :\langle e^{ik\cdot Q}\rangle:]\,, \tag{2.12}$$

leading to the vanishing of the matrix elements of the current divergence between states obeying the free generalized Dirac equation. Similarly, one can show the same for the boson case, namely,

$$k_\mu j_\mu^B(k) = -[\langle P^2\rangle, :\langle e^{ik\cdot Q}\rangle:]\,. \tag{2.13}$$

This coupling scheme of the electromagnetic potential $A_\mu(x)$ with the dual boson system implies the identification of the vector meson lying on the leading trajectory as being the photon.[19] It is nice that in this case, consistency with the subsidiary conditions is attained when the mass of the vector particle is zero, i.e., in our interpretation, for on-mass-shell photons. As the boson theory is already well known, we concentrate on the electrodynamics of dual fermions for the remainder of this paper.

The conservation of our current insures that of the three-point function. A trivial calculation shows that the matrix element of our current between the ground-state protons leads to a unit form factor. This means that in our approximation the proton is still treated as a point particle, which is not amazing when one realizes that the free dual theory is obtained by treating the ground state as a point particle. Rather, we choose to think that the strong corrections will successfully account for the odd electromagnetic properties of the proton.

The current can, however, cause transitions between the stationary states of the system; also the higher mass states of the theory have nontrivial form factors. Those are difficult to build because the general form of the stationary states is not known.

There remains to check that our interaction preserves electromagnetic gauge invariance on the one hand, while it does not lead to the production of ghostlike states. In the absence of a Lagrangian formalism, the only way to test electromagnetic gauge invariance is to build the Compton amplitude (and maybe others) and explicitly check that it is divergenceless – this we shall do in Sec. III. The second point, which we call strong gauge invariance, requires more thought since the ghosts must be absent from both the integer-spin and half-integer-spin channels. The best way to check the latter is to start from the wave equation (2.9) which is first order in momentum. The fact that F_0 and F_n have the same anticommutation relations with the vertex evaluated at $\tau=0$ suggests that there are no mass constraints to the applicability of some subsidiary conditions in fermion channels. The same is not true for bosonic channels where the equation is second order in momentum and where the commutator of the vertex with L_0 and L_n differs by a term proportional to the dual spin of the vertex, which, in turn, depends on the mass. In fact, in our theory, there are gauge conditions in the boson channel only when the photon mass is zero. We will expand on this point in the next section.

III. COMPTON AMPLITUDE

We now consider in detail the case of the elastic scattering of a photon and a fermionic dual system in its ground state, i.e., the proton Compton scattering amplitude (see Fig. 1). The kinematics are described by

$$\gamma(k)+p(p_i)\rightarrow\gamma(k')+p(p_f)\,, \tag{3.1}$$

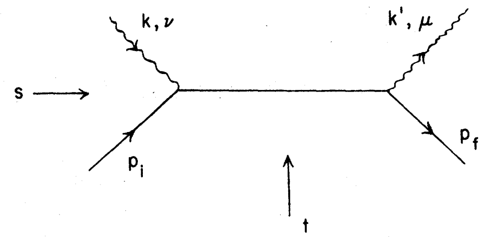

FIG. 1. Compton scattering amplitude.

which corresponds to an amplitude

$$S_{\mu\nu}^{(C)}=\bar{u}(p_f)\Big\langle 0,-p_f\Big|\, j_\mu^F(-k')\frac{1}{F_0-m}j_\nu^F(k)\Big|0,p_i\Big\rangle u(p_i)\,, \tag{3.2}$$

where $u(p_i)$ and $u(p_f)$ are Dirac spinors and

$$|0,p_{i,f}\rangle=e^{ip_{i,f}\cdot q^{(0)}}|0\rangle\,. \tag{3.3}$$

The first step is to take the projection operator out of the propagator,

$$\frac{1}{F_0-m+i\epsilon}=-\frac{F_0+m}{L_0+m^2}\,, \tag{3.4}$$

where we have used

$$L_0=-F_0F_0\,. \tag{3.5}$$

It is crucial to note that L_0 is the generator of internal motions in τ. Thus we can write

$$j_\nu^F(k)=\frac{1}{T}\int_{-T/2}^{+T/2}d\tau\, e^{-i\tau L_0}\,\Gamma_\nu(0):e^{ik\cdot Q(0)}:e^{i\tau L_0}\,. \tag{3.6}$$

It follows that

$$S_{\mu\nu}^{(C)}=\frac{-1}{T^2}\int d\tau\int d\lambda\,\bar{u}(p_f)\Big\langle 0,-p_f\Big|\,\Gamma_\mu(0):e^{-ik'\cdot Q(0)}:\frac{F_0+m}{L_0+m^2}e^{i(\tau-\lambda)(L_0+m^2)}:\Gamma_\nu(0)e^{ik\cdot Q(0)}:\Big|0,p_i\Big\rangle u(p_i)\,. \tag{3.7}$$

We see that the extra dependence on the internal space we had noticed before goes away when on the pole. Another way to say this is to identify the states, generated by this dependence, with spurious states which decouple from the ground states of the theory.[20] Then, we write

$$S_{\mu\nu}^{(C)}=\bar{u}(p_f)\Big\langle 0,-p_f\Big|\,\Gamma_\mu(0):e^{-ik'\cdot Q(0)}:\frac{F_0+m}{L_0+m^2}\Gamma_\nu(0):e^{ik\cdot Q(0)}:\Big|0,p_i\Big\rangle u(p_i)\,. \tag{3.8}$$

We now use the equation

$$\{F_0,\Gamma_\nu(0):e^{ik\cdot Q(0)}:\}=[2P_\nu(0)+\Gamma_\nu(0)k\cdot\Gamma(0)]:e^{ik\cdot Q(0)}: \tag{3.9}$$

and the fact that the ground state obeys the free Dirac equation to rewrite our amplitude as

$$S_{\mu\nu}^{(C)}=-\bar{u}(p_f)\Big\langle 0,-p_f\Big|\Gamma_\mu(0):e^{-ik'\cdot Q(0)}:\frac{1}{L_0+m^2}[2P_\nu(0)+\Gamma_\nu(0)k\cdot\Gamma(0)]:e^{ik\cdot Q(0)}:\Big|0,p_i\Big\rangle u(p_i)\,. \tag{3.10}$$

Then, by writing the denominator of the propagator as

$$\frac{1}{L_0+m^2}=\int_0^1 dx\, x^{L_0+m^2-1} \tag{3.11}$$

and by using the Heisenberg equations of motion, one obtains

$$S_{\mu\nu}^{(C)} = -\int_0^1 dx\, x^{m^2-1-s}$$
$$\times\, \bar{u}(p_f)\langle 0, -p_f|\Gamma_\mu(0) : e^{-ik'\cdot Q(0)} : [2P_\nu(-i\ln x) + \Gamma_\nu(-i\ln x)k\cdot\Gamma(-i\ln x)] : e^{ik\cdot Q(-i\ln x)} : |0, p_i\rangle u(p_i)\,. \tag{3.12}$$

Next, we separate the a from the b modes, and proceed with care because each contribution gives rise to an infinite term which cancel one another at the end.

The Compton amplitude is then seen to reduce to

$$S_{\mu\nu}^{(C)} = -\int_0^1 dx\, x^{-1-\alpha_s}(1-x)^{2k\cdot k'}\bar{u}(p_f)\left[\gamma_\mu\gamma_\nu \not{k} - 2\gamma_\mu(p_i+k)_\nu + \frac{2x}{1-x}(g_{\mu\nu}\not{k} - \gamma_\mu k'_\nu - \gamma_\nu k_\mu)\right]u(p_i)\,, \tag{3.13}$$

with

$$\alpha_s = s - m^2\,. \tag{3.14}$$

It is clear from this form that poles in the t channel are generated only when the two photons are on their mass shell, since only then do we have

$$2k\cdot k' = -t\,. \tag{3.15}$$

Before checking gauge invariance it would be natural to include the term with the photon lines interchanged, in analogy with Dirac electrodynamics. However, in our case, the (s, t) amplitude is gauge-invariant by itself.[21] Indeed, we find that

$$k_\nu S_{\mu\nu}^{(C)} = \int_0^1 dx\, x^{-1-\alpha_s}(1-x)^{2k\cdot k'-1}[2k\cdot k'x + \alpha_s(1-x)]\bar{u}(p_f)\gamma_\mu u(p_i)\,. \tag{3.16}$$

Use of the identities

$$\int_0^1 dx\, x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \tag{3.17a}$$

and

$$\Gamma(a+1) = a\Gamma(a) \tag{3.17b}$$

leads to

$$k_\nu S_{\mu\nu}^{(C)} = 0\,. \tag{3.18}$$

It can be checked that $k'_\mu S_{\mu\nu}^{(C)} = 0$ as well.

We must therefore conclude that gauge invariance does not enter in the perturbative treatment of the electromagnetic force with dual systems. We point out that the spurious states which we have eliminated by neglecting the dependence on $(\tau - \lambda)$ would have spoiled the gauge invariance. It would then have been necessary to add some seagull term to restore it because the addition of the crossed diagram would not have been sufficient.

This peculiarity may be seen as a consequence of the removal of some spurious states, together with current conservation.

This amplitude yields for the deep-inelastic structure functions

$$W_2(k^2, s) = m\left\{\sum_{l=0}\binom{2k^2-1}{l}(-1)^l[\delta(l-\alpha_s) + \delta(l+1-\alpha_s)] - \delta(\alpha_s)\right\} \tag{3.19a}$$

and

$$W_1(k^2, s) = \frac{k\cdot p}{m}\sum_{l=1}\binom{2k^2}{l}\delta(l-\alpha_s)\,. \tag{3.19b}$$

As we do not know yet how to properly smear the imaginary part of the propagator, we prefer to leave these results as they are.

We would like to conclude this section with some comments concerning the t-channel behavior of our amplitude. As we remarked earlier, it has t-channel poles only for on-mass-shell photons. However the correct spectrum in this channel can be obtained by considering the n-photon process and then factorizing in t. The theory proceeds in the same way as that of Neveu and Schwarz[13] and of Thorn.[14] For example, the photon vertex is of the form given by Eq. (3.9) (with the Γ_μ replaced by the Neveu-Schwarz H_μ) and allows for subsidiary conditions only when the photon mass is zero.

In the fermion channel, we cannot show any dependence of the subsidiary conditions on the masses. Nevertheless, consider the state

$$|\psi'\rangle = \Gamma_\mu(0) : e^{ik\cdot Q(0)} : |\psi\rangle , \qquad (3.20)$$

where $|\psi\rangle$ satisfies Eqs. (2.15) and (2.16). Then, it follows that

$$(F_0 - F_n)|\psi'\rangle = -\Gamma_\mu(0) : e^{ik\cdot Q(0)} : (F_0 - F_n)|\psi\rangle$$
$$= -m\Gamma_\mu(0) : e^{ik\cdot Q(0)} : |\psi\rangle . \qquad (3.21)$$

This suggests that the states created by a photon on a proton seem to obey certain subsidiary conditions in which the photon mass does not enter.

CONCLUSION

Our theory has many shortcomings, in particular the structurelessness of the proton in this order of the strong interactions.[22] On the other hand, the ever-so-important strong gauge conditions seem to be compatible with it. This is highly nontrivial in the sense that we have coupled, in an electromagnetic gauge-invariant manner, the photon field with a system enjoying a non-Abelian gauge group. Also there is factorization in the t channel for on-mass-shell photons. The lack of duality off the mass shell does not appear as such a loss as long as the strong gauges are really preserved. It is also sad that we cannot compare with experiment our deep-inelastic structure functions since the dependence on the smearing scheme is probably more important than the form of the unsmeared expression. This difficulty can only be remedied by a better understanding of the unitarity question, i.e., by inclusion of strong corrections. Finally, the problem of saturation of the current algebra can be undertaken only when the inclusion of internal quantum numbers is successfully tackled; however, this theory may be a serious first step if indeed there are no ghosts.

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Complex Pomeranchukon Singularities and Asymptotic Behavior*

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We assume that the partial-wave amplitude of the scattering of two scalar bosons, in the t channel, has n complex-conjugate pairs of Pomeranchukon singularities of the form $\alpha(t)=1 \pm a\sqrt{t}$. We consider the cases where the complex singularities appear in an additive way and in a multiplicative way in the partial-wave amplitude. We obtain for the scattering amplitude an asymptotic expansion of the form

$$F(s,t) \underset{s\to\infty;\, t<0}{\sim} \sum_{k=1}^{n} a_k(t)(\ln s)^k g(z),$$

where $g(z)$ is an entire function of the argument $z=\sqrt{-t}\ln s$. We discuss properties of the forward spike in the differential cross section, and the properties of the amplitude in connection with the Froissart bound and the violation of the Pomeranchuk theorem.

We consider the two-body scalar-boson collision $1+2\to 3+4$. There is a single scattering amplitude $F(s,t)$, and the crossing symmetry gives $F(-s+i\epsilon) = F^*(s+i\epsilon)$. We construct the symmetrical amplitude $F_s(s,t)=\frac{1}{2}(F_1+F_2)$ and the antisymmetrical amplitude $F_a(s,t)=\frac{1}{2}(F_1-F_2)$, where $F_1(s,t)$ and $F_2(s,t)$ are the scattering amplitude for the processes $1+2\to 3+4$ and $1+\overline{3}\to\overline{2}+4$, respectively. The crossing then gives

$$F_s(se^{i\pi}) = F_s^*(s),$$
$$F_a(se^{i\pi}) = -F_a^*(s).$$

The optical theorem relates the total cross section σ_T to the imaginary part of the forward elastic amplitude; thus we have

$$\mathrm{Im}F_i(s,0) = 2k\, s^{1/2}\sigma_T(s), \quad i=1,2 \tag{1}$$

where σ_T is bounded by the Froissart bound.[1] By definition we have

$$\sigma_{el} = \frac{1}{16\pi}\int_{4m^2-s}^{0} \frac{|F(s,t)|^2 dt}{s(s-4m)^2} = \frac{1}{16\pi}\int_{4m^2-s}^{0} \frac{d\sigma}{dt}dt. \tag{2}$$

The unitarity relation provides that

$$\sigma_{el} < \sigma_T, \tag{3}$$

so as the energy increases the integrated elastic cross section and the total cross section remain bounded. The forward spike of the scattering is defined by its width $[k(s)]^{-1}$ and height $(d\sigma/dt)_{t=0}$ which are exhibited by the parametrization $d\sigma/dt = (d\sigma/dt)_{t=0}e^{2kt}$. We shall also use the asymptotic phase of the amplitude which is defined by the ratio $\mathrm{Re}F^{\pm}(s,0)/\mathrm{Im}F^{\pm}(s,0)$.

The high-energy scattering amplitude is constructed in the usual way by means of the Sommerfeld-Watson transformation:

$$F^{\pm}(s,t) \underset{s\to\infty;\, t<0}{\sim} \frac{1}{2i\pi}\int_{c_j} s^j\, \xi^{\pm}(j)a^{\pm}(j,t)\,dj, \tag{4}$$

where $\xi^{\pm}(j)$ is the usual signature factor

$$\frac{1\pm e^{-i\pi j}}{\sin(\pi j)}.$$

$\xi^+(j)$ is purely imaginary at $t=0$ and $\xi^-(j)$ exhibits a right-signature pole at $j=1$, so we shall use

$$\xi^-(j) \sim i - \frac{2}{\pi}\frac{1}{j-1}.$$

$a^{\pm}(j,t)$ is the partial-wave amplitude in the t channel which can have all singularities allowed