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High-Energy Scattering at Backward Angles

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The previous attempt by Schiff to describe large-angle potential scattering is shown to be inaccurate for strong potentials. The reason for this is that a small ripple in partial-wave amplitude, t_l, of the order of $(ka)^{-1}(-1)^l t_l$ is neglected in the eikonal approximation. This ripple may produce a strong coherent effect in the backward direction. To overcome this difficulty the partial-wave sum for the second Born approximation is carried out exactly, and it is shown that a stationary phase, hitherto neglected, occurs in the oscillatory integrals. An application to the square-well potential shows that the error in the eikonal approximation is of order $(ka)^{-1/2}$ rather than $(ka)^{-1}$ as previously thought. Even more important is the fact that this occurs in the second Born term and that the cross section is now increased by a factor of $(V/E)^2(ka)$. Investigation of higher-order terms shows that one is, of course, describing the square-well glory, but it is quite clear that these effects persist for potentials other than those with discontinuities. Finally, in a numerical check, a marked improvement in the fit to the correct differential cross section is observed when second- and third-order contributions are added to Schiff's essentially first-order result.

I. INTRODUCTION

It is the purpose of this paper to obtain a simple prescription for describing high-energy scattering at backward angles. In principle, the scattering problem for any spherically symmetric potential can be solved numerically on a computer by utilizing an expansion in partial waves. Though in practice it is more difficult than the authors realized to obtain accurate results at backward angles in this manner, the aim here is not to replace such calculations by analytic methods. Rather it is to gain deeper physical and mathematical insight into the problem so that in intractable situations, not amenable to numerical methods, one may be aware of the important factors to guide necessary approximations. Thus much attention has been focused on analytic methods for high-energy scattering, the most notable being Glauber's[1] application of the Molière[2] phase method for scattering at forward angles.

Many authors have sought to modify this method so as to extend its applicability to large angles. Bassichis, Feshbach, and Reading[3] replaced the Glauber phase,

$$\chi = -\frac{i}{2k}\int_{-\infty}^{z} U(b,z')\,dz', \tag{1}$$

by the WKB phase,

$$\chi = \int_{-\infty}^{z}\{[k^2 - U(b,z')]^{1/2} - k\}\,dz'. \tag{2}$$

This was tested for a square-well potential and although some improvement was found in the scattering amplitude at small angles, the method was unreliable at backward angles. The approximate results were often several orders of magnitude too small. The agreement was greatly improved when absorptive potentials were considered.

The problem of extending the Glauber method to backward angles was approached as early as 1956 by Schiff.[4] He was able to sum approximately the Born series for $f(\pi)$ by assuming that the main contribution to backward scattering from the nth Born term arose from $n-1$ forward or soft scatterings, amenable to Glauber approximation, and one hard scattering through 180°. (This will be discussed further in Sec. IV.) Unfortunately, the resulting expression offered no improvement when compared to the exact solution for a square-well potential. (See Sec. VI.) Sugar and Blankenbecler[5] pointed out the importance of double-scattering events in which the particle is turned around by two potentials. Hahn[6] has investigated the accuracy of Schiff's method and this refinement and found, in comparison with exact calculations, that with Gaussian and Yukawa potentials there is some improvement, but that the method is still unreliable.

Weingarten[7] calculated each phase shift in the WKB approximation and converted the sum,

$$f(\theta) = \frac{1}{k}\sum_l (2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l\ ,$$

to an integral over the Watson-Sommerfeld contour. For purely absorptive Yukawa potentials this led to a reliably accurate expression for $f(\theta)$ at high energies. Finally, Wallace[8] has presented a systematic method which improves the accuracy at larger angles.

In the present paper a different approach is discussed which yields a rather reliable expression for $f(\pi)$ and has the virtue of affording physical insight into the problem. The particular potential to which the method is applied is a square well which in some ways is the most difficult to handle because of the discontinuity. The approach is different from that of Sugar and Blankenbecler, but contains the same feature in that double-scattering events are emphasized. Furthermore, it is shown to be essential to include higher-order events if reliability is to be achieved.

In Sec. II, that feature of backward scattering which makes this a difficult problem will be isolated. In Sec. III the first and second Born approximations for backward scattering are investigated. In Sec. IV, higher-order terms are included and in Sec. V the intrinsically third-order term is calculated. The results of numerical tests of the method are given in Sec. VI.

II. THE SOURCE OF DIFFICULTY

The exact expression for the backward scattering amplitude is given by

$$f(\pi)=\frac{1}{2ik}\sum_{l=0}^{\infty}(-1)^l(e^{2i\delta_l}-1)(2l+1)$$
$$=-\frac{1}{2ik}\sum_{l\text{ odd}}(2l+1)\left(e^{2i\delta_l}-\frac{e^{2i\delta_{l-1}}+e^{2i\delta_{l+1}}}{2}\right)+\frac{e^{2i\delta_0}-1}{4ik}. \tag{3}$$

At high energies, it is tempting (and sometimes correct) to identify the angular momentum quantum number with kb, where b is the impact parameter. Following Glauber one may then approximate the phase shifts by

$$\delta_l\equiv\delta_{kb}\approx-\frac{1}{4k}\int_{-\infty}^{\infty}U(b,z)\,dz. \tag{4}$$

Using the second form for $f(\pi)$ in Eq. (3) one may then consider l as a continuous variable and write

$$f(\pi)\approx+\frac{e^{2i\delta_0}-1}{4ik}-\frac{1}{2ik}\sum_{l\text{ odd}}(l+\tfrac{1}{2})\frac{\partial^2}{\partial l^2}e^{2i\delta_l}. \tag{5}$$

Substitution of δ_l from Eq. (7) into this form of $f(\pi)$ leads to a simple expression for the scattering amplitude at backward angles. For the present purpose it is sufficient to note that the resulting $f(\pi)$ is of the form

$$f(\pi)\sim f(0)[\,\mathcal{O}(1/ka)+\mathcal{O}(U/k^2)]. \tag{6}$$

Thus for typical high-energy scattering, where $ka\gg1$ and $U/k^2\ll1$, $f(\pi)$ is small. Thus the neglect of terms $\mathcal{O}(1/ka)$ or $\mathcal{O}(U/k^2)$, which may be quite valid for small angles, might constitute a serious error at backward angles.

At this point it is appropriate to consider the partial-wave amplitudes as calculated exactly. A typical case is shown in Fig. 1 where $t_l=e^{i\delta_l}\sin\delta_l$ is plotted as a function of l for 300-MeV nucleons scattering off of a 20-MeV square well of radius 3 F. The important feature to be noted is the "ripple" superimposed on the smoothly varying amplitudes. The Glauber expression for the phase shift can reproduce the smoothly varying part, but certainly will be unable to describe the "ripple" which oscillates like $(-1)^l$. This effect is small, of the order of $(ka)^{-1}$, and will not be important in the forward direction. In the backward direction, however, the $(-1)^l$ factor in $f(\pi)$ will lead to a cancellation of the smoothly varying part but a *coherent sum* of the oscillating part. Because $f(\pi)$ is small, this coherent sum could, in fact, be the dominant term in the summation. This renders Eq. (5) useless since $e^{i\delta_l}$ is not a smoothly varying function of l. It is the failure to realize this fact that has led other approaches to this problem into difficulties.

It is straightforward to demonstrate the origin of these ripples for small l waves. Each term in the Born series for the phase shift has, from the Green's function, products of Bessel functions such as $j_l^2(kr)$ or $j_l(kr)n_l(kr)$. For example, in the Born approximation

$$\sin\delta_l\approx-k\int_0^{\infty}j_l^2(kr)U(r)r^2\,dr. \tag{7}$$

For small l, $l<ka$, the j_l's may be approximated so that

$$\sin\delta_l\approx-\frac{1}{k}\int_0^{\infty}\cos^2[kr-\tfrac{1}{2}(l+1)\pi]U(r)\,dr$$
$$=-\frac{1}{k}\int_0^{\infty}[1+\cos(2kr)(-1)^{l+1}]U(r)\,dr. \tag{8}$$

The cosine term is of the order of $1/2ka$ relative to the leading term and might then be considered negligibly small. For backward scattering, because of the $(-1)^l$ factor, this term gives the dominant contribution. In the usual eikonal treatments this ripple has been neglected for all terms except the first Born term.

The above argument holds, of course, only for small l so that if there were some mechanism for reducing the small l contribution, the neglect of the ripple might not be a serious error. Such a mechanism is provided if the potential is highly absorptive, in which case the eikonal methods are reliable even at large angles.[3] Finally note that for a potential singular at the origin such as a Yukawa potential, the integral in Eq. (8) is divergent and the ripple argument suspect.

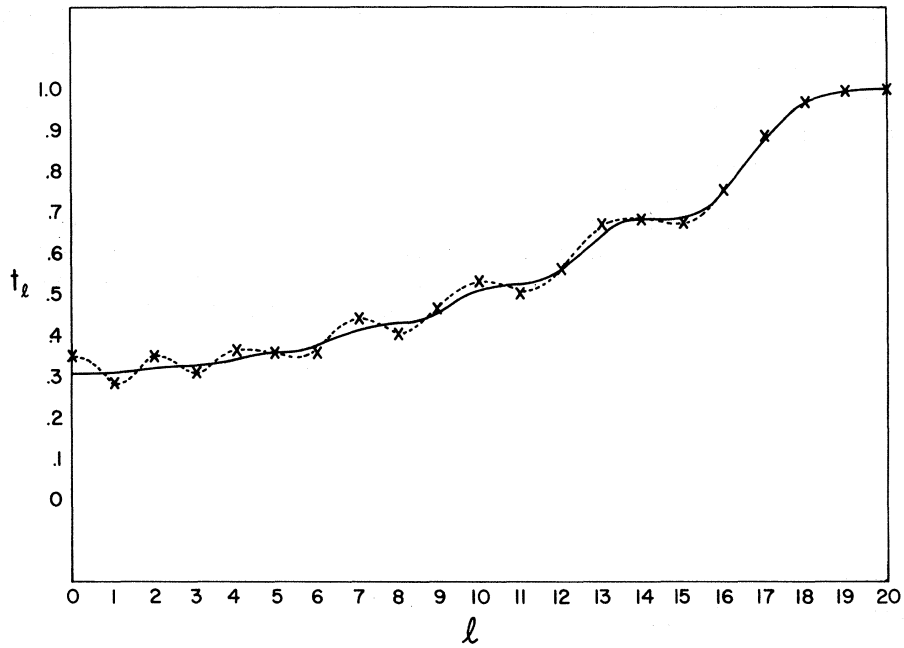

FIG. 1. The "ripple." The partial amplitude $t_l = e^{i\delta_l}\sin\delta_l$ is plotted as a function of l. It is obvious that for small l there exists a part which is varying like $(-1)^l$. This will produce a coherence for backward scattering.

To summarize, it has been pointed out that

(1) $f(\pi)$ is small compared to $f(0)$,

(2) subtle cancellation and enhancement effects are present,

(3) the situation may be different for absorptive and nonabsorptive potentials, and

(4) singularities at the origin may be important.

These considerations will determine the proper approach to the problem.

III. THE FIRST AND SECOND BORN APPROXIMATIONS

The scattering amplitude for backward scattering,

$$f(\pi) = -\sum_{l=0}^{\infty}(-1)^l(2l+1)\int_0^\infty j_l(kr)U(r)R_l(r)r^2\,dr\,, \tag{9}$$

can be written, using the Born series for R, as

$$\begin{aligned} f(\pi) = -\sum_l (-1)^l(2l+1)\int_0^\infty j_l^{\,2}(kr)U(r)r^2\,dr - ik\int_0^\infty j_l^{\,2}(kr)U(r)r^2\,dr\int_r^\infty h_l^{(1)}(kr')j_l(kr')U(r')r'^2\,dr' \\ - ik\int_0^\infty j_l(kr)h_l^{(1)}(kr)U(r)r^2\,dr\int_0^r j_l^{\,2}(kr')U(r')r'^2\,dr' + \cdots \\ = f_{B1}(\pi) + f_{B2}(\pi) + \cdots . \end{aligned} \tag{10}$$

For the present it will be assumed that the energy is sufficiently high, relative to the potential strength, so that the first two Born terms dominate. If in the second Born term the factors $h_l^{(1)}(kr)j_l(kr')$ and $j_l(kr)h_l^{(1)}(kr')$ are grouped together, the sum of l can immediately be done and one recovers the usual expression

$$f_{B2}(\pi) = \frac{1}{(4\pi)^2}\int d^3r\, e^{i\vec{k}\cdot\vec{r}}\,U(r)\int d^3r'\,\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}e^{i\vec{k}\cdot\vec{r}'}U(r')\,. \tag{11}$$

This form constitutes the usual starting point for studies of this problem. Schiff,[4] for example, noted that by inserting $e^{i\vec{k}\cdot\vec{r}'}e^{-i\vec{k}\cdot\vec{r}'}$ one obtains

$$f_{B2}(\pi) = \frac{1}{(4\pi)^2}\int d^3r'\,U(r')e^{i\vec{q}\cdot\vec{r}'}\,d^3\rho\,U(\vec{\rho}+\vec{r}')\frac{e^{i\vec{k}\cdot\vec{\rho}+ik\rho}}{\rho}\,, \tag{12}$$

where $\vec{\rho}=\vec{r}-\vec{r}\,'$ and $\vec{q}=2\vec{k}$. He then argues that the main contribution to the integral in brackets comes from $\vec{\rho}$ antiparallel to $\vec{k}$ and performs the angular integration by parts. He further notes that one could also insert $e^{i\vec{k}\cdot\vec{r}}e^{-i\vec{k}\cdot\vec{r}}$ and obtain a similar expression with the roles of $\vec{r}$ and $\vec{r}\,'$ interchanged. This will in fact lead to a factor of 2 if the regions where the main contributions to the integrals originate are distinct. Thus Schiff obtains two contributions to $f_{B2}(\pi)$ corresponding to a scattering of 180° occurring at $\vec{r}$ or $\vec{r}\,'$ with the other scattering being at forward angles.

With the above approach to the problem it is difficult, but not impossible (see Sec. IV), to properly treat the "ripple" on the partial amplitudes because the l dependence has disappeared. Since it has been shown that this might constitute a serious error, a different approach is necessary. Thus instead of the grouping of the Bessel (and Hankel) functions as above, functions with the same arguments are grouped together. It was this grouping that led to the $(-1)^l$ factor. One can then use the formulas[9]

$$j_l^2(kr)=\int_0^\pi d\alpha\,\sin\alpha\,P_l(\cos\alpha)\frac{\sin(2kr\sin\frac{1}{2}\alpha)}{4kr\sin\frac{1}{2}\alpha}$$

and (13)

$$j_l(kr)h_l^{(1)}(kr)=-i\int_0^\pi d\beta\,\sin\beta\,P_l(\cos\beta)\frac{e^{2ikr\sin\frac{1}{2}\beta}}{4kr\sin\frac{1}{2}\beta}\,.$$

Substituting these into Eq. (10) and noting that

$$\sum_l(2l+1)(-1)^lP_l(x)P_l(x')=2\delta(x+x')\,, \tag{14}$$

then leads to

$$\alpha+\beta=\pi$$

and

$$f_{B1}(\pi)=-\frac{1}{2k}\int_0^\infty\sin(2kr)U(r)\,r\,dr \tag{15}$$

(as usual) and

$$\begin{aligned}f_{B2}(\pi)&=\frac{1}{4k}\int_0^\pi d\beta\int_0^\infty dr\,rU(r)e^{2ikr\sin\frac{1}{2}\beta}\int_0^r dr'\,r'U(r')\sin(2kr'\cos\tfrac{1}{2}\beta)\\&\quad+\frac{1}{4k}\int_0^\pi d\beta\int_0^\infty dr\,rU(r)\sin(2kr\cos\tfrac{1}{2}\beta)\int_r^\infty dr'\,r'U(r')e^{2ikr'\sin\frac{1}{2}\beta}\\&=\frac{1}{k}\int_0^{\pi/2}d\beta\int_0^\infty dr\,rU(r)e^{2ikr\sin\beta}\int_0^r dr'\,r'U(r')\sin(2kr'\cos\beta)\,.\end{aligned} \tag{16}$$

By this technique the six-dimensional integral has been reduced to a three-dimensional integral with, as yet, no approximations. Thus the coherence of the $(-1)^l$ terms remains intact.

It is unfortunately not possible to perform the angular integration on β in terms of simple functions. One is forced to make approximations or, at this point, insert a specific potential and perform the r and r' integrations. It can now be shown, however, that the procedure which other authors have adopted is suspect for any potential.

What is normally assumed in Eq. (11) is that in a large region of the phase space

$$kr>kr'\gg1\,. \tag{17}$$

One then attempts to develop a power-series expansion for the scattering amplitude in powers of k^{-1} by integrating by parts. Thus the contributions to the integration arise from the end points of the range. However, consideration of Eq. (11) in the form obtained after exactly performing three of the integrations [Eq. (16)] shows that the largest contribution to the β integral comes rather from a stationary phase at β_0, where

$$r\cos\beta_0=r'\sin\beta_0\,. \tag{18}$$

Assuming the inequality in Eq. (17) one then obtains

$$I=\int_0^{\pi/2} d\beta\, e^{2ikr\sin\beta}\sin(2kr'\cos\beta)\approx I_{SP}+I_{EP}, \tag{19}$$

where the stationary-phase contribution is

$$I_{SP}=\frac{1}{2i}\left(\frac{\pi}{k(r^2+r'^2)^{1/2}}\right)^{1/2} e^{2ik(r^2+r'^2)^{1/2}-i\pi/4}, \tag{20}$$

and the end-point contribution is

$$I_{EP}=\frac{e^{2ikr}}{2kr'}+i\frac{\sin 2kr'}{2kr}. \tag{21}$$

It is significant to note that although I_{SP} is still oscillatory in both r and r', its amplitude is a factor of $k^{1/2}$ larger than that of I_{EP}. It would certainly be incorrect to ignore this term.

It is tempting now to use the above expression for the angular integral in evaluating $f_{B2}(\pi)$ [Eq. (16)]. This however could be quite incorrect since the condition assumed in the evaluation of I is definitely *not* valid over the entire range of integration on r and r'. The behavior of the potential when r (or r') is near zero is of crucial importance in determining the proper procedure. This is, of course, just a manifestation of the well-known fact that backward scattering is very sensitive to the details of the potential near the origin. To illustrate the correct procedure and the pitfalls of the straightforward procedure referred to above, the specific case of a square-well potential of radius a and strength U_0 will now be considered.

For the square well, substitution of Eqs. (19)–(21) into Eq. (16) leads to

$$f_{B2}(\pi)\approx\frac{U_0^2}{k}\int_0^a r\,dr\int_0^r r'\,dr'\left[\frac{1}{2i}\left(\frac{\pi}{k(r^2+r'^2)^{1/2}}\right)^{1/2} e^{2ik(r^2+r'^2)^{1/2}-i\pi/4}+\frac{e^{2ikr}}{2kr'}+\frac{i\sin 2kr'}{2kr}\right]. \tag{22}$$

An integration by parts, to develop a power series in k^{-1}, yields

$$f_{B2}(\pi)\approx\frac{U_0^2}{k}\int_0^a dr\left[\frac{-\pi^{1/2}}{4k^{3/2}}\left(2^{1/4}r^{3/2}e^{2ik\sqrt{2}r-i\pi/4}-r^{3/2}e^{2ikr-i\pi/4}\right)+\frac{r^2e^{2ikr}}{2k}+\frac{ir\cos 2kr}{(2k)^2}\right]. \tag{23}$$

The last term will be negligible after the subsequent integration on r. The other term arising from I_{EP}, the third term above, is now the leading term with the stationary-phase contribution being down by $k^{-1/2}$. The difficulty lies in the second term which results from evaluating I_{SP} at $r'=0$. This term should, in fact, *not be present*. One can show this by integrating on r' exactly first and then examing the β integration. Alternatively one can reexamine Eq. (19) paying particular attention to the region where the inequality (17) is not satisfied. Thus consider again

$$I=\int_0^{\pi/2}\frac{1}{2i}\left(e^{2ikr\sin\beta+2ikr'\cos\beta}-e^{2ikr\sin\beta-2ikr'\cos\beta}\right)d\beta. \tag{24}$$

The first integral has the stationary phase, as noted, when $\beta=\beta_0$ where $\tan\beta_0=r/r'$. The second integral also has a stationary phase when $\beta=-\beta_0$. This was ignored since β_0 is outside of the range of integration when (17) is satisfied. But when β_0 is zero, i.e., when r' is zero, this stationary phase occurs just at the end point and exactly cancels the contribution of the stationary phase of the first integral.

Thus, for a square well, the contribution of I_{SP} at $r'=0$ will be spurious. This problem would not arise for potentials which are zero and analytic at the origin but would, in fact, require even more careful consideration for potentials which are singular. The contribution of I_{SP} at $r=r'$ does persist in any case and represents an important correction. As noted it is no longer the leading term. The reason for its being previously overlooked in Eq. (11) is that it now appears as a "second-order" stationary phase.

For the square well the r integration may now be performed to obtain

$$f_{B2}(\pi)=\frac{U_0^2a^2}{4ik^3}\left[e^{2ika}-\frac{\pi^{1/2}e^{2ik\sqrt{2}a-i\pi/4}}{2^{1/4}2(ka)^{1/2}}\right]. \tag{25}$$

The first term arose from I_{EP} and is in exact agreement with the Schiff result. The second term arises from I_{SP} and for large ka is negligible compared to the first. This comparison is, however, misleading as will be shown in Sec. IV. It should be further noted that the error in neglecting the higher-order terms in the k^{-1} series is of order $(ka)^{-1/2}$, not $(ka)^{-1}$ as indicated by Schiff. This result has only been derived for the square-well potential and is not general.

The stationary-phase term described above represents a different contribution to the scattering than that calculated by Schiff. It arises from a double-scattering event turning the particle around rather than a single scattering event. Such a possibility is included in the work of Sugar and Blankenbecler[5] with a different approach. This stationary phase term is present for all potentials and could be more important than the Schiff contribution if the potential is real and sufficiently strong. The latter condition is a result of it being intrinsically second order in the potential while the Schiff term, as will be shown, is always first order. (The higher-order terms calculated by Schiff merely add a phase to the Born contribution.) There will also be present triple-scattering events, and so on, whose importance will increase for increasingly strong potentials. When complex (absorptive) potentials are considered the small-l partial waves will be suppressed and the stationary phase, or "ripple," contribution will be decreased. Hence, the approach of Weingarten,[7] which assumes the WKB, smoothly varying phase shifts should be valid and is for an absorptive potential. But it would fail for real potentials.

In Sec. IV higher-order terms are considered in a manner similar to the Schiff approach and the above arguments are mathematically confirmed.

IV. HIGHER-ORDER TERMS

Schiff summed the Born series by noting that the soft (forward) collisions before and after the hard collision could be treated in the eikonal approximation. He thus derived an expression for backward scattering of the form

$$\begin{aligned} f_{\text{Schiff}}(\pi) \equiv f_s(\pi) &= -\frac{1}{4\pi}\int e^{2ikZ}U(r)\exp\left(-\frac{i}{k}\int_{-\infty}^{Z}U(b,Z')\,dZ'\right)dZ\,d^2b \\ &= -\frac{1}{4\pi}\int e^{2ikr\cos\theta}U(r)\exp\left(-\frac{i}{k}\int_{-\infty}^{r\cos\theta}U(r\sin\theta,Z')\,dZ'\right)d\phi\,\sin\theta\,d\theta\,r^2dr \\ &\approx -\frac{1}{4ik}\int_0^\infty dr\,rU(r)\left[e^{2ikr}\exp\left(-\frac{i}{k}\int_{-\infty}^{r}U(0,Z')\,dZ'\right)-e^{-2ikr}\exp\left(-\frac{i}{k}\int_{-\infty}^{-r}U(0,Z')\,dZ'\right)\right]. \end{aligned} \tag{26}$$

The phase or distortion entering this expression is twice that obtained in the Glauber approximation as the wave is distorted both going in and coming out of the potential as is illustrated in Fig. 2(a).

For a square-well potential $f_s(\pi)$ may be approximately evaluated by integrating by parts to obtain

$$f_s(\pi)\approx\frac{U_0a}{8k^2}\left[e^{2ika-iU_02a/k}-e^{-2kia}\right]. \tag{27}$$

This result can be represented schematically as is shown in Fig. 2(b). There are two reflections, both at impact parameter zero, one at the first surface, the other at the back with, consequently, two traversals of the potential. If the potential were strongly absorptive only the first reflection would contribute and

$$f_s(\pi)\approx\frac{-U_0a}{8k^2}e^{-2ika}. \tag{28}$$

In fact, if the absorption is sufficiently strong, practically any calculation of the phase, be it WKB, Glauber, or Schiff gives the same result of Eq. (28).

If the exponential in Eq. (27) is expanded the second Born term obtained is identical to the end-

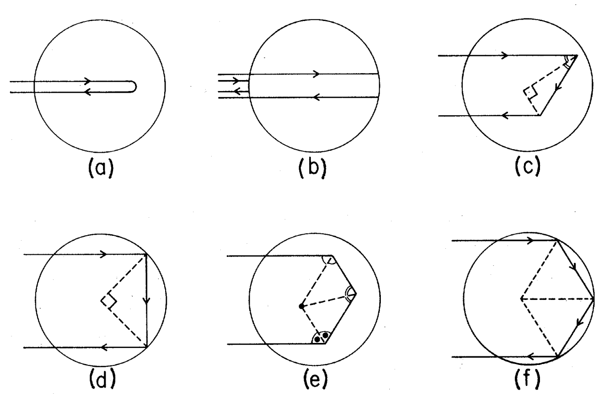

FIG. 2. (a) The process described by the Schiff term for an arbitrary potential wherein the projectile is turned around by a single collision. (b) This intrinsically single scattering for a square-well potential. (c) The process described by the stationary phase in the second Born term for an arbitrary potential. Note that the radius vector bisects the scattering angle at each hard collision. (d) The second-order process for the square-well potential. (e) The process described by the stationary phase in the third-order Born term. Again the scattering angles are bisected. (f) The third-order process for the square well.

point contribution of Eq. (25). Thus the effect of the higher-order terms is to add a distorting phase due to traveling to and from the single hard collision. In this way the single hard-scattering nature of the process is essentially unaltered by the addition of higher-order terms. In a similar manner the effect of incorporating higher-order terms into the stationary-phase term, which describes a process wherein two hard collisions occur, is also to add a distorting phase. This phase results from the traversing of the path from $-\infty$ to $\vec{r}_1$ in the Z (incoming) direction, from $\vec{r}_1$ to $\vec{r}_2$ and from $\vec{r}_2$ to $-\infty$ in the $-Z$ (outgoing) direction. (Here the locations of the two hard collisions are r_1 and r_2.) The proof of this statement is sketched in the Appendix. There it is shown that

$$f(\pi) = f_{B1}(\pi) + \tfrac{1}{2}\int r_1^2 dr_1 \sin\theta_{12} d\theta_{12}\, r_2^2 dr_2\, e^{i\chi_1} e^{i\Theta_{12}} e^{i\chi_2} \frac{\sin|k_1\vec{r}_1 + k_2\vec{r}_2|}{|k_1\vec{r}_1 + k_2\vec{r}_2|} U(r_2) \frac{e^{ik|\vec{r}_2-\vec{r}_1|}}{|\vec{r}_2 - \vec{r}_1|} U(r_1), \tag{29}$$

where θ_{12} is the angle between $\vec{r}_1$ and $\vec{r}_2$ and the χ's and Θ_{12} are distortions of the type calculated by Schiff [see Eq. (A10)]. An integration by parts on θ_{12} would lead back to Schiff's approximation. Guided by the previous analysis, a stationary phase is sought and indeed one exists when $\theta_{12} = \pi/2$. It is thus concluded that a *significant contribution arises from a process which is an intrinsically double, hard scattering, or a two-potential collision, when $\vec{r}_1$ and $\vec{r}_2$ are perpendicular, independent of the particular potential employed*. This process is illustrated in Fig. 2(c).

The caution described earlier must again be exercised when using the stationary-phase contribution in performing the subsequent integrations in Eq. (29). For the square-well potential the previous arguments obtain and the final result is

$$f(\pi) = f_s + f_2 = f_s - \frac{\pi^{1/2}}{2^{1/4}} \frac{U_0^2 a^2}{ik^3} \frac{e^{2ik\sqrt{2}a}}{8(ka)^{1/2}} e^{-i(U_0/2k)3\sqrt{2}a} e^{-i\pi/4}. \tag{30}$$

This new contribution arises from the process illustrated in Fig. 2(d). It corresponds to the glory.[10] However, it emerges in a straightforward way from this analysis and since the stationary phase that is responsible for the effect occurs for all potentials there will be, in general, some effect on the backward scattering. This holds even though the optical interpretation, such as is illustrated in Fig. 2(d) may not apply.

It can now be seen that for real potentials the amplitude for single potential backward scattering, f_s, and the double potential scattering have the ratio

$$|f_2|^2/|f_s|^2 \approx (ka)(U_0/k^2)^2 = (ka)(V/E)^2. \tag{31}$$

Thus if ka is greater than $(E/V)^2$ the double-scattering term contributes more to the total backward amplitude than the single term.

V. THIRD-ORDER SCATTERING

The question naturally arises as to the significance of processes which are intrinsically triple hard scattering, i.e., those in which the projectile is turned around in three steps. It is straightforward if a little tedious to calculate the resulting contribution to the scattering amplitude. A sketch of the procedure will be given here.

Beginning with the third Born term,

$$f_{B3} = \left(\frac{-1}{4\pi}\right)^3 \int e^{i\vec{k}_i\cdot\vec{r}_3} U(\vec{r}_3) \frac{e^{ik|\vec{r}_3-\vec{r}_2|}}{|\vec{r}_3 - \vec{r}_2|} U(\vec{r}_2) \frac{e^{ik|\vec{r}_2-\vec{r}_1|}}{|\vec{r}_2 - \vec{r}_1|} U(\vec{r}_1) e^{i\vec{k}_0\cdot\vec{r}_1} d^3r_1 d^3r_2 d^3r_3, \tag{32}$$

the integration on the angle between $\vec{r}_3$ and $\vec{k}_0$ is performed, keeping the angles between $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ fixed. The result is

$$f_{B3} = -\left(\frac{1}{4\pi}\right)^2 \int \frac{\sin k|\vec{r}_3 + \vec{r}_1|}{k|\vec{r}_3 + \vec{r}_1|} U(r_3) \frac{e^{ik|\vec{r}_3-\vec{r}_2|}}{|\vec{r}_3 - \vec{r}_2|} U(r_2) \frac{e^{ik|\vec{r}_2-\vec{r}_1|}}{|\vec{r}_2 - \vec{r}_1|} U(r_1)\, d^3r_1 d^3r_2 r_3^2 dr_3. \tag{33}$$

Upon investigation it can be seen that this integral contains a stationary phase when the azimuthal angle, $\phi_2 - \phi_1$, is zero or π so that $\vec{r}_1$, $\vec{r}_2$, and $\vec{r}_3$ are coplanar. Then

$$f_{B3} = -\frac{1}{16\pi}\int \left(\frac{2\pi}{kr_2 r_1 \sin\theta_{13}\sin\theta_{23}}\right)^{1/2} \frac{\sin k|\vec{r}_3 + \vec{r}_1|}{k|\vec{r}_3 + \vec{r}_1|} U(r_3) \frac{e^{ik|\vec{r}_3-\vec{r}_2|}}{|\vec{r}_3 - \vec{r}_2|} \times \left[\frac{e^{ik|\vec{r}_2-\vec{r}_1|_- - i\pi/4}}{|\vec{r}_2 - \vec{r}_1|_-} + \frac{e^{ik|\vec{r}_2-\vec{r}_1|_+ + i\pi/4}}{|\vec{r}_2 - \vec{r}_1|_+}\right] U(r_2) U(r_1) \sin\theta_{13}\sin\theta_{23} r_1^2 r_2^2 r_3^2 dr_1 dr_2 dr_3, \tag{34}$$

where

$$|\vec{r}_2 - \vec{r}_1|_{\pm} = [r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_{13} \pm \theta_{23})]^{1/2} .$$

The θ_{13} and θ_{23} integrations can now be performed using the method of stationary phase. The resulting description of the third-order process is illustrated in Fig. 2(e). The striking feature is that at each hard collision the *radius vector bisects the angle through which the particle is scattered.* This was also characteristic of the double- (and of course single-) scattering terms and presumably holds in general. (This has not yet been proven.) For the particular choice of the square-well potential the r_1, r_2, and r_3 integrations may be performed first with the only significant contribution coming from $r = a$. Then it is clear that a stationary phase obtains at $\theta_{13} = 60°$ and $\theta_{23} = 120°$. This process is illustrated in Fig. 2(f). A simple calculation including distortion then leads to

$$f_3(\pi) = -\frac{a(6\pi ka)^{1/2}}{8}\left(\frac{U_0^3}{k^6}\right) e^{3ika} e^{-i2U_0 a/k} e^{-i\pi/4} . \quad (35)$$

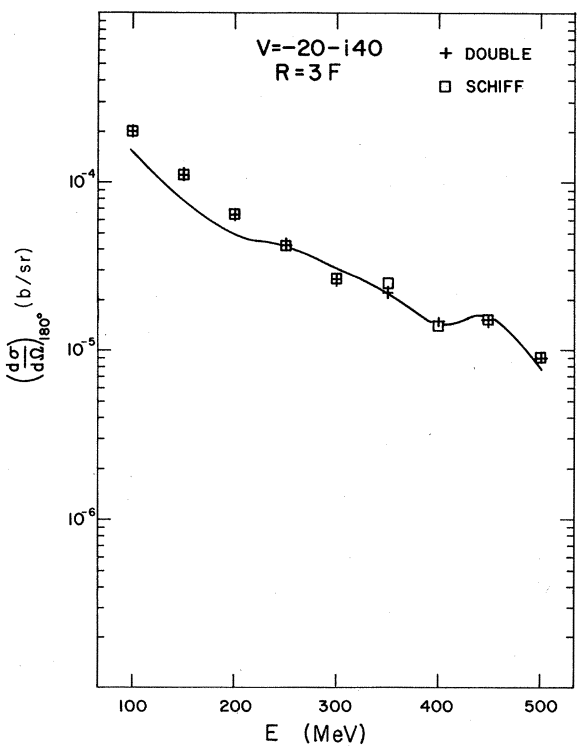

FIG. 3. Comparison of approximate and exact results for an absorptive potential. Since the higher-order terms are negligible only the Schiff and second-order results, which essentially overlap, have been included. For such a potential no significant flux reaches the "back" of the potential to be scattered so that any reasonable calculation of the phase inside will give the same result.

The fourth-order terms have not yet been investigated, but the general method is quite clear. One performs the integration exactly on the angle between one of the $\vec{r}$'s and $\vec{k}_0$, holding the angles between the $\vec{r}$'s fixed, and then searches for the points of stationary phase for the other angular integrations.

Finally it should be noted from Eq. (35) that

$$|f_3|^2/|f_2|^2 \approx (U_0/k^2)^4 ka = (V/E)^4 ka . \quad (36)$$

VI. NUMERICAL TEST OF APPROXIMATIONS

In this section the approximate expressions developed in the preceding sections are tested by comparing the resulting differential cross sections with a computer calculation of the correct results for a square-well potential. It should be stressed that the three expressions tested, given by Eqs. (27), (30), and (35), are simple analytic expressions. The calculations were performed with a nucleon incident on a target of 10^3 amu.

In Fig. 3 the comparison is made for a strongly absorptive potential. The assertion that such a situation does not constitute a stringent test is obviously borne out. The higher-order terms are

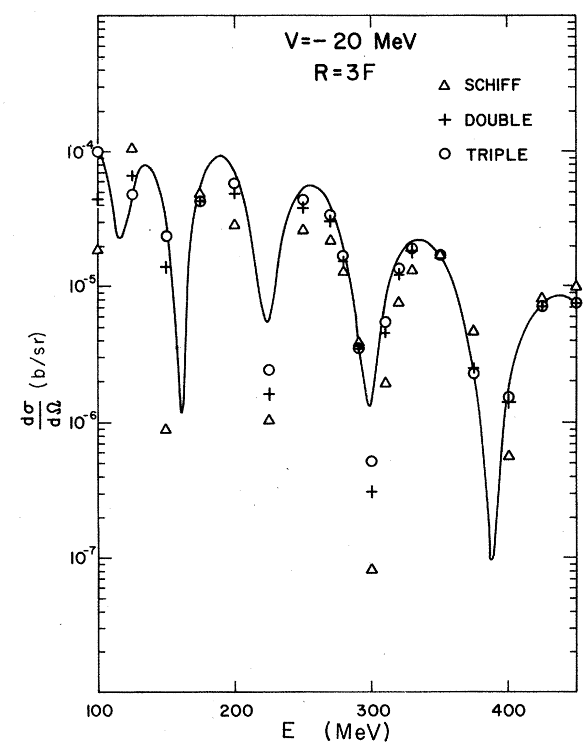

FIG. 4. Comparison of approximate and exact results for a real potential. Here it is apparent that the third-order contribution is necessary, but excluding higher-order terms is justified. Except at the minima the approximation proves quite reliable.

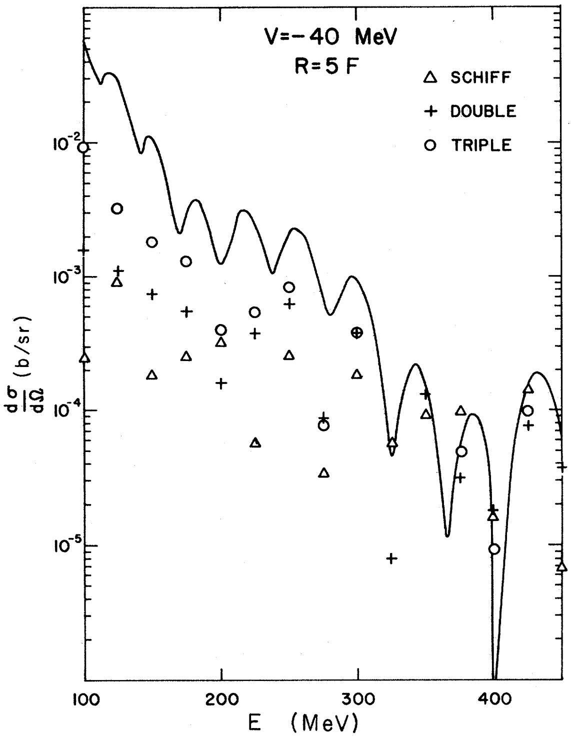

FIG. 5. Comparison of approximate and exact results for a stronger real potential. It is seen that at nearly every energy the second- and third-order terms lead to a significant improvement over the Schiff result. The best approximate result is still unreliable and obviously higher-order terms must not be neglected.

almost totally suppressed and the Schiff, second-order, and third-order results (not shown) nearly coincide for all the energies considered.

In Fig. 4 the comparison is made for a real potential with a depth of 20 MeV and a radius of 3 F. It is now apparent that the Schiff contribution, alone, is quite inaccurate except at extremely high energies. The inclusion of the double-scattering term is seen to improve the results for every energy, in some cases by more than an order of magnitude. As can be seen in this example, the approximation, including the third-order processes, agrees very well with the correct results and even at the lower energies the error is, except at the minima, a few percent at most. At those energies where the cross section is a minimum any approximation scheme will have difficulty.

As should be apparent from the method used to develop these approximations, there will be potentials whose strength or range is such that fourth- or higher-order processes must be included before agreement is obtained; i.e., *any approximation which treats only intrinsically multiple scattering processes of finite order will fail for some potential.* As an example of this consider the comparison made in Fig. 5 where a potential of depth 40 MeV and range 5 F is employed. Although at almost every energy the inclusion of the second-order processes constitutes an improvement over the Schiff result, and the third-order contribution improves the results again, even this is inadequate for this potential. Possibly the inclusion of the fourth-order processes would be sufficient, but what would obviously be desirable for a universally reliable description is a summation of the contributions from all orders. Alternatively, given a particular potential it is essential to ascertain that higher-order terms which are neglected are, in fact, negligible. If this is not recognized one has the situation which has been described in the literature of various authors testing arbitrary potentials with always some improvement but never achieving reliability.

The methods described here are immediately applicable to potentials which are analytic at the origin, but no attempt has been made to present a general, closed form expression. The reason for this is the necessity for treating each potential individually when the contribution of the stationary phase is considered in subsequent integrations. Backward scattering is a very sensitive probe of a potential so that a small variation in the potential can cause a large change in $f(\pi)$. In any case, the stationary phases and the processes attributed to them will be present for any potential.

Applications to angles smaller than π are rather straightforward and will be discussed in a later paper.

ACKNOWLEDGMENTS

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APPENDIX: EFFECT OF HIGHER-ORDER TERMS IN THE DOUBLE-SCATTERING PROCESS

It is the purpose of this Appendix to sketch a proof for the intuitively rather obvious assertion that there will be three contributions to the distortion in the double-scattering process. These correspond to summing the effect of all forward collisions before, between, and after the hard collisions.

Consider the contribution to the Nth Born term arising from hard scatterings at $\vec{r}_m$ and $\vec{r}_n$, all other $N-2$ scatterings at zero angle. This will be designated by f_{BN}^{nm} which is given by

$$f_{BN}^{nm} \approx \left(-\frac{1}{4\pi}\right)^{n-m+1} \int e^{i\vec{k}_0\cdot\vec{r}_n} \frac{(i\chi_n)^{N-m}}{(N-n)!} U(\vec{r}_n) G(n, n-1) U(\vec{r}_{n-1}) \cdots G(m+1, m) U(\vec{r}_m) \frac{(i\chi_m)^{m-1}}{(m-1)!} d^3r_n \cdots d^3r_m , \tag{A1}$$

where

$$\chi_q \equiv \frac{-1}{2k} \int_{-\infty}^{Z_q} U(\vec{b}_q, Z)\, dZ \tag{A2}$$

and

$$G(q, q-1) \equiv \frac{e^{ik|\vec{r}_q - \vec{r}_{q-1}|}}{|\vec{r}_q - \vec{r}_{q-1}|} . \tag{A3}$$

Defining $\vec{\rho}_j = \vec{r}_j - \vec{r}_m$, the integral

$$I = \frac{-1}{4\pi} \int G(m+2, m+1) U(\vec{r}_{m+1}) G(m+1, m)\, d^3r_{m+1}$$

can be written as

$$I = -\frac{1}{4\pi} \int \left[\frac{e^{ik|\vec{\rho}_{m+2} - \vec{\rho}_{m+1}|}}{|\vec{\rho}_{m+2} - \vec{\rho}_{m+1}|} U(\vec{\rho}_{m+1} + \vec{r}_m) \right] \frac{e^{ik\rho_{m+1}}}{\rho_{m+1}} d^3\rho_{m+1} . \tag{A4}$$

Assuming as usual a slowly varying potential one can integrate by parts on the angle between $\vec{\rho}_{m+2}$ and $\vec{\rho}_{m+1}$. Then

$$I \approx \frac{1}{2ik} \int [e^{ik|\rho_{m+2} - \rho|} U(\hat{k}_{m+2,m}\rho + \vec{r}_m) - e^{ik|\rho_{m+2}+\rho|} U(-\hat{k}_{m+2,m}\rho + \vec{r}_m)] \frac{e^{ik\rho}}{\rho_{m+2}} d\rho , \tag{A5}$$

where $k_{m+2,m}$ is the unit vector in the $(\vec{r}_{m+2} - \vec{r}_m)$ direction. The assumption that an integral containing an oscillatory exponential in k is negligible leads to

$$I \approx -\frac{i}{2k} \frac{e^{ik\rho_{m+2}}}{\rho_{m+2}} \int_0^{\rho_{m+2}} U(\hat{k}_{m+2,m}\rho + \vec{r}_m)\, d\rho . \tag{A6}$$

Using this procedure repeatedly in Eq. (A1) for all m and n, one finally arrives at the expression for $f(\pi)$:

$$f(\pi) = f_{B1}(\pi) + \frac{1}{(4\pi)^2} \int d^3r_1 d^3r_2\, e^{i\vec{k}_0\cdot\vec{r}_2 + i\chi_2} U(r_2) e^{i\Theta(\vec{r}_2, \vec{r}_1)} G(2,1) U(r_1) e^{i\chi_1} e^{i\vec{k}_0\cdot\vec{r}_1} , \tag{A7}$$

with

$$\Theta(\vec{r}_2, \vec{r}_1) = \frac{-1}{2k} \int_0^{|\vec{r}_2 - \vec{r}_1|} U(\hat{k}_{21}\rho + \vec{r}_1)\, d\rho . \tag{A8}$$

In order to proceed with the integration in Eq. (A7) the phases will temporarily be replaced by a function of differential operations, e.g.,

$$e^{i\chi_2} e^{ikr_2\cos\theta_2} = e^{i\chi((1+\partial^2/\partial k_2{}^2)^{1/2}, i\,\partial/\partial k_2)} e^{ik_2 r_2 \cos\theta_2} = e^{i\overline{\chi}} e^{ik_2 r_2\cos\theta_2} . \tag{A9}$$

After performing the differentiation k_2 will again be set equal to k. Now, keeping the angle between $\vec{r}_1$ and $\vec{r}_2$ fixed, an integration can be performed on the angle between $\vec{r}_1$ and $\vec{k}_0$. Then an integration on ϕ_{12} yields

$$f(\pi) = f_{B1}(\pi) + \tfrac{1}{2} \int r_1^2 dr_1 r_2^2 dr_2 \sin\theta_{12}\, d\theta_{12}\, e^{i\chi_1} e^{i\Theta_{12}} e^{i\chi_2} \frac{\sin|k_1\vec{r}_1 + k_2\vec{r}_2|}{|k_1\vec{r}_1 + k_2\vec{r}_2|} U(r_2) G(2,1) U(r_1) , \tag{A10}$$

where θ_{12} is the angle between $\vec{r}_2$ and $\vec{r}_1$. An integration by parts on θ_{12} would lead back to Schiff's approximation. Instead, guided by the previous analysis, a stationary phase is sought and indeed one exists when $\theta_{12} = \pi/2$. It is thus concluded that there is a significant contribution from a process which is intrinsically a double, hard scattering, or two-potential collision when $\vec{r}_1$ and $\vec{r}_2$ are perpendicular, independent of the particular potential employed. This situation is illustrated in Fig. 2(c).

The differential operators, $\partial/\partial k_1$ and $\partial/\partial k_2$, occurring in Eq. (A10) can be evaluated by noting the approximate eigenvalue relation

$$-i\frac{\partial}{\partial k_1}\frac{e^{i|k_1\vec{r}_1+k_2\vec{r}_2|}}{|k_1\vec{r}_1+k_2\vec{r}_2|}\approx\frac{kr_1^2+k(\vec{r}_1\cdot\vec{r}_2)}{|k_1\vec{r}_1+k_2\vec{r}_2|}\frac{e^{i|k_1\vec{r}_1+k_2\vec{r}_2|}}{|k_1\vec{r}_1+k_2\vec{r}_2|}. \tag{A11}$$

At the point of stationary phase then, where $\vec{r}_1\cdot\vec{r}_2=0$,

$$\chi(b_1, Z_1)\equiv\chi\left(\left(1+\frac{\partial^2}{\partial k_1^2}\right)^{1/2}, i\frac{\partial}{\partial k_1}\right)$$
$$\approx\chi\left(\frac{r_1r_2}{(r_1^2+r_2^2)^{1/2}}, \frac{r_1^2}{(r_1^2+r_2^2)^{1/2}}\right)$$
$$\equiv\chi(r_1\sin\beta_0, r_1\cos\beta_0). \tag{A12}$$

Similarly,

$$\chi(b_2, Z_2)\approx\chi(r_2\cos\beta_0, r_2\sin\beta_0),$$

where

$$\tan\beta_0=r_2/r_1, \tag{A13}$$

with a similar expression for Θ_{12}.

Again being careful about using this stationary-phase contribution in the subsequent integrals in Eq. (A10), one obtains for the square well

$$f(\pi)=f_s+f_2=f_s-\frac{\pi^{1/2}}{2^{1/4}}\frac{U_0^2a^2}{ik^3}\frac{e^{2ika\sqrt{2}}}{8(ka)^{1/2}}\times e^{-i(U_0/2k)3a\sqrt{2}}e^{-i\pi/4}. \tag{A14}$$

[1]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. I, p. 315.
[2]G. Molière, Z. Naturforsch. 2A, 133 (1947).
[3]W. H. Bassichis, H. Feshbach, and J. F. Reading, Ann. Phys. (N.Y.) 68, 462 (1971).
[4]L. I. Schiff, Phys. Rev. 103, 443 (1956).
[5]R. L. Sugar and R. Blankenbecler, Phys. Rev. 183, 1387 (1969).
[6]Y. Hahn, Phys. Rev. C 2, 775 (1970).
[7]D. Weingarten, Phys. Rev. D 2, 201 (1970).
[8]S. J. Wallace, Phys. Rev. Letters 27, 622 (1971).
[9]G. N. Watson, *Theory of Bessel Functions* (Cambridge Univ. Press, Cambridge, England, 1944).
[10]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw-Hill, New York, 1966).

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Current Algebra in Two-Particle Bases and Single-Particle Inclusive Processes*

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We discuss several problems involved in the current-algebra sum rules derived by Suzuki and his collaborators. In particular we discuss the ambiguities related to soft-current insertion to the external lines. Alternative prescriptions to get similar sum rules are also discussed. An application of the sum rule to the *W*-production process is briefly discussed.

I. INTRODUCTION

Suzuki and his collaborators[1,2] recently made interesting applications of current algebra[3] to single-particle inclusive processes, *W*-production, and also pion production processes. Their results seem to be in agreement with other predictions based on various other models of high-energy reactions; the basic assumptions involved are the following:

(i) the "low-energy theorem" of the soft-photon radiation;

(ii) simple analytic structure of the production amplitude (namely, dominance of the normal singularities at high energies);

(iii) duality arguments, which are used to motivate the neglect of 3-body scattering amplitudes; and

(iv) absence of the contact subtraction terms.

The purpose of the present note is to discuss the