Efimov's Effect: A New Pathology of Three-Particle Systems. II

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(Received 4 October 1971)

By studying the eigenvalue spectrum of the Faddeev kernel in a certain singular limit, we give an independent proof of an effect recently deduced by Efimov: When three identical particles interact via short-range pairwise potentials, the number of three-body bound states grows without limit when the pairwise scattering length a becomes large. [The number of bound states is then roughly $(1/\pi)\ln(\Lambda|a|)$, where Λ is a momentum cutoff]. We extend our proof to the case where only two particles are identical and show that Efimov's effect persists in the special limiting cases with two heavy and one light particle, and with two light and one heavy particle.

I. INTRODUCTION

Recently Efimov demonstrated a remarkable and hitherto unsuspected property of three-body systems.[1,2] He showed that if three nonrelativistic identical bosons interact *via* short-range two-body potentials $gv(r)$, then as the coupling constant g increases to that value g_0 which can support a single two-body bound state at zero energy, the number of bound states of the three-particle system increases without limit, being roughly given by the formula

$$N \sim (1/\pi)\ln(\Lambda a), \tag{1}$$

where a is the two-body scattering length (which becomes infinite whenever there is a zero-energy s-wave two-body bound state) and Λ is a momentum cutoff determined by the range of the potential $v(r)$. This result surprised us because we did not expect that well-behaved short-range potentials could support infinitely many bound states (even in many-body systems); therefore Efimov's effect reflects yet another qualitative difference between two- and three-body scattering. A rigorous discussion of some things that can and cannot happen to many-body bound states, but not of the surprising Efimov effect, has been given by Simon.[3]

Since Efimov's discursive derivation[2] was not immediately available to us, and since the result was startling, we undertook an independent verification. Our efforts led to a rigorous proof, which turned out to be sufficiently different from that given by Efimov as to be interesting in its own right. An abbreviated account of our proof has appeared previously[4]; the purpose of this paper is to present our results in considerable detail, including their extension to the case of nonidentical particles. We will also discuss some of the physical aspects of Efimov's effect.

Our approach to the three-body bound-state problem is based on the integral form of the Schrödinger equation introduced by Faddeev[5]:

$$\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = G_0(E)\begin{pmatrix}0 & t_{23} & t_{23}\\ t_{13} & 0 & t_{13}\\ t_{12} & t_{12} & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}. \tag{2}$$

This is a system of three coupled homogeneous (because we are studying bound states) integral equations of Fredholm type, whose kernel has certain very desirable mathematical properties (i.e., it is "compact") if the two-body off-shell scattering amplitudes t_{ij} are derived from reasonably well-behaved short-range potentials (e.g., Yukawa potentials, Wheeler-Yamaguchi potentials, etc.).[6] For technical reasons connected with the fact that the trace of the kernel of Eq. (2) vanishes, it is difficult to treat the case of three nonidentical particles using our methods. However, if even two of the particles are identical, Eq. (2) can be put into the form of a single homogeneous Fredholm integral equation, the trace of whose kernel does not vanish identically and so can be studied in a relatively simple manner. As we shall see subsequently, Efimov's effect is a form of infrared divergence of certain traces, which appears when the three-body barycentric energy E is set to zero, and the two-body scattering length becomes infinite. (The order in which the limits are taken is immaterial.) If we consider any power of the kernel of our equation, we find that as $a \to \infty$ with $E=0$ its trace diverges as $\ln(\Lambda a)$, whereas if we let $E \to 0^-$ with $a=\infty$, the trace diverges as $\ln(-E)$. In Sec. II (which deals with the simplest situation, that of three identical bosons) we show by a power-counting argument that this divergence is indeed present in every power of the kernel. We then discuss the connection between this divergence and the number of bound states of the system. By confining our attention to the most singular part of the kernel, we obtain the leading contributions to certain traces, and thereby prove that the number of three-body bound states indeed grows in the manner indicated by Efimov. In Sec. III, we apply our method to the situation where only two particles are identical, confining ourselves to the two interesting limiting cases where the two are either very heavy or very light compared with the third particle, and taking into account the possibility that the pair scattering lengths may be different. In Sec. IV we discuss the physical interpretation of Efimov's effect. Finally, we have for didactic purposes relegated some of the more mathematical portions of the paper to appendixes.

II. IDENTICAL PARTICLES

In this section we show that when three identical bosons interact via short-range forces the number of three-body bound states grows like $\ln(\Lambda|a|)$ with increasing two-body scattering length a, where Λ is some momentum cutoff. When the particles are identical, the three coupled equations in (1) reduce to a single equation which can be written

$$\psi(\vec{p},\vec{k}) = \int\int d^3p'\, d^3k'\, K(\vec{p},\vec{k}|\vec{p}\,',\vec{k}\,')\psi(\vec{p}\,',\vec{k}\,'), \quad (3)$$

where ($\hbar = m = 1$)

$$K(\vec{p},\vec{k}|\vec{p}\,',\vec{k}\,') = \frac{2\langle\vec{k}+\frac{1}{2}\vec{p}|t(E-\frac{3}{4}p^2)|\vec{p}\,'+\frac{1}{2}\vec{k}\,'\rangle}{E-p^2-k^2-\vec{p}\cdot\vec{k}}\,\delta^3(\vec{p}-\vec{k}\,'), \quad (4)$$

t is the off-shell two-body scattering amplitude, and E is the total (barycentric) energy eigenvalue. True bound-state solutions of (3) must have $E<-b$, where b is the binding energy of the most tightly bound two-body bound state. If there are no two-body bound states, the three-body bound states correspond to $E<0$. The two-body scattering threshold also begins at $E=-b$. As we shall see, the singularity leading to the infinite number of bound states comes from the confluence of the three-body threshold ($E=0$), and the two-body threshold ($E=-b$), so that this singularity only occurs for the *first* two-body bound state ($b=0$).

We want to calculate the number of three-body bound states when g is smaller than, but close to g_0, i.e., as the scattering length a gets large. We can do this by studying the traces of various powers of the kernel of (3). Unfortunately, if the scattering amplitude in (4) comes from a local potential, the traces of the first and second powers of the kernel will diverge at large momenta independent of anything else. This difficulty can be circumvented by studying only higher powers of K, but such a procedure is technically much more complicated. Since the divergence connected with the Efimov effect arises from the small-momentum behavior of the kernel, it seems inappropriate to complicate the problem merely to deal with its (irrelevant) high-momentum behavior. We avoid the difficulty by using a two-body scattering amplitude derived from an s-wave separable potential,

$$\langle\vec{q}|t(\epsilon)|\vec{q}\,'\rangle = u(q)\tau(\epsilon)u(q'),$$
$$\tau(\epsilon) = -\left[\frac{1}{g} + \int d^3q\,\frac{u^2(q)}{\epsilon-q^2}\right]^{-1}, \quad (5)$$

where u is the separable potential vertex, normalized so that $u(0)=1$, and g is the potential strength. (Positive g corresponds to an attractive potential.) Introducing a separable t matrix into the Faddeev equation reduces it to the one-variable equation[7]

$$\phi(\vec{p}) = \int d^3p'\, K(\vec{p},\vec{p}\,')\phi(\vec{p}\,'), \quad (6a)$$

$$K(\vec{p},\vec{p}\,') = \frac{2u(\vec{p}+\frac{1}{2}\vec{p}\,')u(\vec{p}\,'+\frac{1}{2}\vec{p})\tau(E-\frac{3}{4}p^2)}{E-p^2-p'^2-\vec{p}\cdot\vec{p}\,'}. \quad (6b)$$

But near $\epsilon=0$, $\tau(\epsilon)$ is approximated by

$$\bar{\tau}(\epsilon) = -\frac{1}{2\pi^2}\left(\frac{1}{a}+\sqrt{-\epsilon}\right)^{-1} \quad (7)$$

(this is just the effective-range expansion)[8] so that when $E=0$ and a is very large compared with the range Λ^{-1} of u,

$$\mathrm{tr}(K) = \int d^3p\, K(\vec{p},\vec{p}) \to \frac{4}{3\pi}\int_0^\infty dp\,\frac{u^2(\frac{3}{2}p)}{1/a+p(\frac{3}{4})^{1/2}}$$
$$= (\tfrac{4}{3})^{3/2}\pi^{-1}\ln(\Lambda a) + \text{const}, \quad (8)$$

where the numerical value of the constant term depends on the exact form of $u(k)$. On the other hand, if $g>g_0$ so that the two-body scattering threshold begins at $-b<0$, we see that when $E\leq -b$, there is no singularity at the lower limit of the integral defining $\mathrm{tr}(K)$ [or $\mathrm{tr}(K^n)$, for that matter] so the trace remains finite and the divergence associated with the Efimov effect does not appear. If we now examine in like fashion the trace of the nth power of K for $E=0$, $a=\infty$,

$$\mathrm{tr}(K^n)=\int d^3p_1\int d^3p_2\cdots\times\int d^3p_n K(\vec{p}_1,\vec{p}_2)K(\vec{p}_2,\vec{p}_3)\cdots K(\vec{p}_n,\vec{p}_1) \qquad (9)$$

and employ hyperspherical coordinates in $3n$-dimensional space, we see by power-counting that the integral (9) has the form $C_1\int_0^{\epsilon>0}d\rho/\rho+C_2$, where C_1 and C_2 are constants and $\rho^2=\sum_{k=1}^n p_k^2$. That is, $\mathrm{tr}(K^n)|_{E=0}$ is logarithmically divergent at the origin in $3n$-dimensional momentum space and elsewhere convergent, so that $\mathrm{tr}(K^n)|_{E=0}\propto\ln(\Lambda a)$ for large a. Thus it is natural to separate the kernel into a (singular) low-momentum part $\tilde{K}$ and a nonsingular remainder K_R,

$$K=\tilde{K}+K_R, \qquad (10)$$

$$\tilde{K}(\vec{p},\vec{p}\,')=\frac{2\tilde{\tau}(E-\frac{3}{4}p^2)}{E-p^2-p'^2-\vec{p}\cdot\vec{p}\,'}\theta(\Lambda-p)\theta(\Lambda-p'), \qquad (11)$$

where $\tilde{\tau}$ comes from (7), and Λ is some momentum cutoff. [Equation (11) is in fact the singular part of the kernel for *any* two-body interaction. The separable potential is merely a convenient heuristic tool for arriving at this effectively model-independent decomposition while avoiding the complications associated with the large-momentum behavior of the t matrix.] By construction $\mathrm{tr}(K_R)$, $\mathrm{tr}(K_R^2)$, and $\mathrm{tr}(\tilde{K}K_R)$ remain finite when $E=0$, $a\to\infty$; and as we shall soon see, the decomposition (10) together with this good behavior of K_R is essential to the subsequent analysis.

We next establish the connection between the number of bound states and the mathematical properties of the kernel of Eq. (6). When $E<0$ and $g<g_0$, $K(E,g)$ represents an operator of the Hilbert-Schmidt type,[9] since it is obviously equivalent under similarity transformation to a real symmetric kernel, and since $\mathrm{tr}(K^2)<\infty$. Thus if we consider the eigenvalue problem

$$K(E,g)\phi_\nu=\eta_\nu(E,g)\phi_\nu, \qquad (12)$$

we see (1) that the eigenvalues $\eta_\nu(E,g)$ are real and discretely distributed when $E<0$, $g<g_0$; (2) that $\lim_{\nu\to\infty}\eta_\nu(E,g)=0$; and (3) that whenever some eigenvalue η_ν satisfies

$$\eta_\nu(E,g)=1, \qquad (13)$$

the corresponding eigenvector $\phi_\nu(E,g)$ satisfies Eq. (3), i.e., is a solution of the Schrödinger equation. From the above considerations we conclude that the number of three-body bound states for fixed g is just the number of times the eigenvalues of K become equal to unity as E increases from $-\infty$ to 0. We also note that since

$$\lim_{E\to-\infty}\mathrm{tr}(K^2(E,g))=0, \qquad (14)$$

there is for each g a most tightly bound state with finite binding energy $0<B(g)<\infty$ so that all the bound-state energy eigenvalues $E_n(g)$ lie between $-B(g)$ and 0. Now since each eigenvalue $\eta_\nu(E,g)$ is a real-valued continuous function of E and g for $E<0$, $g<g_0$ and therefore must pass through the value $\eta=1$ in order to become larger than unity, if we can show that an infinite number of η's exceed unity when $E=0$ and $g=g_0$, then we will have demonstrated the existence of an infinite number of bound states. In fact, we will show this by proving that the sequence $s_\nu=\eta_\nu(0,g_0)$, $\nu=1,2,3,\ldots$ of discrete eigenvalues of the limiting kernel $K(0,g_0)$[10] have a denumerable set of accumulation points $L_J\neq0$, $J=0,1,2,\ldots,$ where J is the total angular momentum quantum number (which is of course conserved because of the rotation invariance of the problem), and that $L_0>1$ but $L_J<1$ for $J>0$.

To prove the above statements, we use the decomposition

$$K(0,g_0)=\tilde{K}(0,g_0)+K_R(0,g_0) \qquad (15)$$

of the singular operator $K(0,g_0)$ into the sum of a compact operator $K_R(0,g_0)$ [recall $\mathrm{tr}(K_R^2(0,g_0))<\infty$] and a bounded but noncompact operator $\tilde{K}(0,g_0)$. As we are only interested in the limit points of the eigenvalues S_ν, it is enough to study the eigenvalue spectrum $\tilde{S}_\nu$ of the operator $\tilde{K}(0,g_0)$, since, by a theorem due to Weyl,[11] the limit points of the sequence $\{S_\nu\}_{\nu=1}^\infty$ and those of the sequence $\{\tilde{S}_\nu\}_{\nu=1}^\infty$ are identical. [In fact, it is just this theorem which makes our method applicable to any reasonable interaction, local or otherwise, since the difference between the actual kernel, Eq. (4), and its singular part, defined by the low-energy expansion of the two-body t matrix, is always a compact kernel.]

The partial-wave decomposition of the kernel $\tilde{K}$ is given by

$$\tilde{K}(\vec{p},\vec{p}\,')=\sum_{J=0}^{\infty}\sum_{M=-J}^{J}Y_{JM}(\hat{p})Y^*_{JM}(\hat{p}')\tilde{K}_J(p,p'), \qquad (16)$$

where

$$\tilde{K}_J(p,p')=\frac{4(-1)^J}{\pi pp'}\theta(\Lambda-p)\theta(\Lambda-p')\times Q_J\left(\frac{p^2+p'^2-E}{pp'}\right)\left[\frac{1}{a}+(\tfrac{3}{4}p^2-E)^{1/2}\right]^{-1}, \qquad (17)$$

and $Q_J(Z)$ is the Legendre function of the second kind. We note that the projections $\tilde{K}_J$ are operators of definite sign, $(-1)^J$, and moreover (see Appendix A), that the eigenvalues $\tilde{\eta}_\nu(J,E,g)$ of $\tilde{K}_J(E,g)$ obey the monotonicity conditions

$$\frac{1}{\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial E}\geq 0\,,$$
$$\frac{1}{\tilde{\eta}}\frac{\partial\tilde{\eta}}{\partial g}\geq 0\,, \tag{18}$$

when $E<0$, $g<g_0$. Unfortunately, the same statements cannot be made about the eigenvalues of $K_J(E,g)$; that is, they are in general neither of definite sign nor monotonic in E (although they *are* monotonic in g). This complicates matters, since it might be possible for an eigenvalue $\eta_\nu(J,E,g_0)$ to take on the value unity an even number of times as E varies from $-\infty$ to 0, and thus to be smaller than unity when $E=0$ so that it escapes being counted. Similarly, it might also be possible for an eigenvalue to pass through unity an odd number of times and thus generate more than one bound state even though it is counted but once. Eigenvalues which behave in either of these ways we call "peculiar." We conclude that the number of eigenvalues $\eta_\nu(J,0,g_0)\equiv S_\nu(J)$ which exceed unity in the limit $\nu\to\infty$ is only a lower bound to the number of bound states, although it is a faithful bound since the number of three-body bound states generated by peculiar eigenvalues is finite (see Appendix B), whereas the total number of bound states is infinite when $g=g_0$.

We now demonstrate the existence of the previously mentioned limit points L_J. Consider the ratio

$$R_J(E,g)=\frac{\mathrm{tr}(\tilde{K}_J^{\,2}(E,g))}{\mathrm{tr}(\tilde{K}_J(E,g))}\,; \tag{19}$$

since both the numerator and the denominator diverge in identical fashion as $E\to 0$ and $g\to g_0$ (i.e., as $a\to\infty$), $R_J(E,g)$ approaches a constant in this limit, call it L_J'. Clearly L_J' has the sign $(-1)^J$. Now $(-1)^J L_J'$ is a lower bound to the limit of the positive sequence $\{(-1)^J\tilde{S}_\nu(J)\}_{\nu=1}^{\infty}$,

$$(-1)^J L_J=\lim_{\nu\to\infty}(-1)^J\tilde{S}_\nu(J)\,, \tag{20}$$

because of the relations

$$(-1)^J R_J(0,g)=\frac{\sum_{\nu=1}^{\infty}\tilde{\eta}_\nu^{\,2}(J,0,g)}{\sum_{\nu=1}^{\infty}(-1)^J\tilde{\eta}_\nu(J,0,g)}$$
$$\leq\frac{\sum_{\nu=1}^{\infty}\tilde{\eta}_\nu(J,0,g)\tilde{S}_\nu(J)}{\sum_{\nu=1}^{\infty}\tilde{\eta}_\nu(J,0,g)(-1)^J} \tag{21}$$

which follow first from the fact that the trace of a positive, compact operator is the sum of its eigenvalues; and second from the monotonicity property of the eigenvalues $\tilde{\eta}_\nu(J,E,g)$ expressed in the form

$$(-1)^J\tilde{\eta}_\nu(J,E,g)\leq(-1)^J\tilde{S}_\nu(J),\quad E\leq 0,\quad g<g_0\,. \tag{22}$$

We now apply a theorem on divergent series[12] (whose proof is included for completeness in Appendix C): Given a sequence $\{A_\nu\}_{\nu=1}^{\infty}$ with a definite limit, l, and a set of positive weight functions $w_\nu(x)$, the function

$$f(x)=\sum_{\nu=1}^{\infty}w_\nu(x)A_\nu\Big/\sum_{\nu=1}^{\infty}w_\nu(x) \tag{23}$$

has the limit $l=\lim_{x\to\infty}f(x)\equiv\lim_{\nu\to\infty}S_\nu$ if and only if

$$\lim_{x\to\infty}\sum_{\nu=1}^{m}w_\nu(x)\Big/\sum_{\nu=1}^{\infty}w_\nu(x)$$

is zero for any positive integer m. Clearly the functions $(-1)^J\tilde{\eta}_\nu(J,0,g)$ play the role of the weights and $(-1)^J\tilde{S}_\nu(J)$ is the corresponding sequence whose limit point, $L_J(-1)^J$, we are interested in locating.[13] The excess of $(-1)^J L_J$ over $(-1)^J L_J'$ is a geometric factor, as can be seen from the example

$$\lim_{x\to\infty}\frac{\sum_{\nu=1}^{\infty}(e^{-\nu/x})^2}{\sum_{\nu=1}^{\infty}e^{-\nu/x}}=\lim_{x\to\infty}\frac{1-e^{-1/x}}{1-e^{-2/x}}=\frac{1}{2}\,,$$

whereas $\lim_{x\to\infty}e^{-\nu/x}=1$, $\nu=1,2,\ldots$.

To actually calculate the lower bounds $(-1)^J L_J'$, we must evaluate the traces $\mathrm{tr}(\tilde{K}_J^{\,2})$ and $\mathrm{tr}(\tilde{K}_J)$ and take their ratio as $E\to 0$, $g\to g_0$. These are given by

$$\mathrm{tr}(\tilde{K}_J(0,g))=(-1)^J\,\frac{4}{\pi}\left(\frac{4}{3}\right)^{1/2}Q_J(2)\int_0^{\Lambda}dp\left[p+\frac{1}{a}\left(\frac{4}{3}\right)^{1/2}\right]^{-1}$$
$$=\frac{8}{\pi\sqrt{3}}(-1)^J Q_J(2)\ln[1+\Lambda a(\tfrac{3}{4})^{1/2}] \tag{24}$$

and

$$\mathrm{tr}(K_J^{\,2}(0,g))=\frac{16}{\pi^2}\left(\frac{4}{3}\right)\int_0^{\Lambda a(3/4)^{1/2}}\frac{dp}{1+p}\int_0^{\Lambda a(3/4)^{1/2}}\frac{dp'}{1+p'}\,Q_J^{\,2}\left(\frac{p^2+p'^2}{pp'}\right)$$
$$=\frac{16}{\pi^2}\left(\frac{4}{3}\right)\ln[\Lambda a(\tfrac{3}{4})^{1/2}]\int_0^{\pi}\frac{d\theta}{\sin\theta}\,Q_J^{\,2}\left(\frac{2}{\sin\theta}\right)+\mathrm{const.} \tag{25}$$

Thus, letting $g \to g_0$ (i.e., $a \to \infty$)

$$(-1)^J L_J' \geq \frac{8}{\pi\sqrt{3}\,Q_J(2)} \frac{\Gamma^3(J+1)}{\Gamma^3(J+\frac{3}{2})} \frac{\pi^{3/2}}{16^{J+1}} > 0\,, \qquad (26)$$

where we have used a simple lower bound on $Q_J(Z)$ to obtain Eq. (26).[14] For $J=0$, we see

$$L_0' \geq \frac{8}{\pi\sqrt{3}\ \ln 3} > 1.$$

However, the operator $(-1)^J \tilde{K}_J(0, g_0)$ is bounded and positive and has norm less than $(8/\pi\sqrt{3})2^J Q_J(2) \times \Gamma^2(\frac{1}{2}J+\frac{1}{2})/\Gamma(J+1)$ (see Appendix D). This upper bound is a decreasing function of J; for $J=0$ it is $4(\ln 3)/\sqrt{3}$, but for $J=2$, it is $(4/\sqrt{3})Q_2(2) \simeq 0.049$. That is, *none* of the eigenvalues of $\tilde{K}_J(0, g_0)$ ever attain the value unity when $J>0$. On the other hand, we have shown above that $4(\ln 3)/\sqrt{3} \geq L_0 \geq L_0' > 1$, so that an infinite number of eigenvalues of $\tilde{K}_0(0, g_0)$ exceed unity. Since K_J differs from $\tilde{K}_J$ by a compact operator even in the singular limit, we have shown that L_0 is an accumulation point of the eigenvalues $S_\nu(0)$ of $K_0(0, g)$, and that in general, the points L_J, which are distinct from 0, are accumulation points of $S_\nu(J)$. [Incidentally, this proves that $\tilde{K}_J(0, g_0)$ is a bounded, *noncompact* operator.]

Thus, the number of three-body bound states is infinite when $g=g_0$, and they all have spin-parity $J^P = 0^+$. We can also ask how the number of bound states depends on the coupling when g is near g_0, i.e., a is very large. Since the number of bound states associated with "peculiar" eigenvalues of $\tilde{K}_0(0, g)$ is at most finite, we see that "most" of the eigenvalues behave exactly as do those of $\tilde{K}_0(0, g)$, so that asymptotically the number will be $N \sim \mathrm{tr}(\tilde{K}_0(0, g))/L_0$. Since L_0 lies between $4(\ln 3)/\sqrt{3}$ and $8/\pi\sqrt{3}\ \ln 3$, we have $N \sim k \ln(\Lambda a)$, where $1/\pi \leq k \leq \frac{1}{2}\ln^2 3$, in agreement with Efimov's result $k = 1/\pi$.

III. UNEQUAL MASSES

We now consider the case of distinguishable particles with masses and pairwise coupling constants arbitrary: The Faddeev equation for a bound state of such particles is given by (2). Clearly the trace of the kernel of Eq. (2) is zero. Even the singular part of the kernel, restricted to a particular partial wave, is not an operator with definite sign, in contrast with the identical-particle case. To apply the methods of Sec. II we should first have to separate the singular part of the partial-wave kernel into its positive and negative pieces, which would involve solving a cubic operator equation. The result would be a complicated operator-valued irrational function of the kernel components, whose traces would be hard to calculate or even to estimate. Since we do not know how to surmount this technical difficulty, and since the general case of three inequivalent particles all of whose pairwise scattering lengths simultaneously become infinite seems quite unlikely, we specialize to a slightly simpler case which we can solve.

When two particles (l) are identical and the third (u) is distinguishable from them, the Faddeev equation takes the form

$$\begin{pmatrix}\psi\\ \phi\end{pmatrix} = \begin{pmatrix}A & B\\ C & 0\end{pmatrix}\begin{pmatrix}\psi\\ \phi\end{pmatrix}. \qquad (27)$$

Whereas it might seem that [since the kernel of (27) has inherently indefinite sign] we have merely traded the difficulties associated with cubic operator equations for those of quadratic operator equations (which from a practical point of view are equally insoluble), in fact Eq. (27) can be reduced immediately to 1-dimensional form. That is,

$$\psi = (A + BC)\psi\,. \qquad (28)$$

The new kernel, $K = A + BC$, when we take its most singular part as in Sec. II, and partial-wave analyze it, exhibits the pleasant feature of being positive definite in states of even total angular momentum. The corresponding reduced equation is (for simplicity we set $E=0$)

$$f_J(p) = \int_0^\Lambda dp'\,p'^2 \tilde{K}_J(p, p') f_J(p')\,, \qquad (29)$$

where

$$\tilde{K}_J(p, p') = \theta(\Lambda - p)\theta(\Lambda - p')\left(\frac{1+\gamma}{\pi p p'}\right)\Bigg\{Q_J\left(\frac{p^2+p'^2}{2pp'/(1+\gamma)}\right)(-1)^J + \frac{4}{\pi\gamma}\int_0^\Lambda dp''\,Q_J\left(\frac{p^2+p''^2[(1+\gamma)/2\gamma]}{pp''}\right)Q_J\left(\frac{p'^2+p''^2[(1+\gamma)/2\gamma]}{p'p''}\right) \times\left[\frac{1}{a_{ll}} + p''\left(\frac{2+\gamma}{4\gamma}\right)^{1/2}\right]^{-1}\left[\frac{1}{a_{ul}} + p'\left(\frac{\gamma(2+\gamma)}{(1+\gamma)^2}\right)^{1/2}\right]^{-1}\Bigg\}, \qquad (30)$$

and where γ is the ratio m_u/m_l. The l-l and u-l pairwise scattering lengths are denoted by a_{ll} and

a_{ul}, respectively. Applying our previous analysis to the kernel $\tilde{K}_J$, we find that if a_{ll} becomes infinite while a_{ul} remains finite, the traces of all powers of $\tilde{K}_J$ remain finite, and so, therefore, does the number of bound states. We only have interesting results when a_{ll} remains finite while a_{ul} becomes infinite, or when both become infinite. In the latter case we will suppose that the scattering lengths remain proportional to each other, leaving to the more dedicated reader the enumeration of the various permutations which are possible if the scattering lengths are related by more complicated growth laws [such as $a_{ll}=k(a_{ul})^{\lambda}$, or $a_{ll}=\exp(a_{ul})$, etc.]. Since the growth of $\mathrm{tr}(\tilde{K}_J{}^n)$ with scattering length is logarithmic, the constant of proportionality a_{ll}/a_{ul} is irrelevant.

There are also two particularly interesting values of γ, namely, 0 and ∞. The first corresponds to the Born-Oppenheimer[15] molecule of two heavy identical masses which interact with a very light one, whereas the limit $\gamma\to\infty$ corresponds to two identical particles in a fixed potential. Suppose a_{ll} remains finite while $a_{ul}\to\infty$. Then for $\gamma>0$

$$\mathrm{tr}(\tilde{K}_J)\to\frac{(1+\gamma)^2}{\pi[\gamma(2+\gamma)]^{1/2}}(-1)^J Q_J(1+\gamma)\times\ln\left[\Lambda a_{ul}\left(\frac{\gamma(2+\gamma)}{(1+\gamma)^2}\right)^{1/2}\right] \tag{31}$$

and

$$\mathrm{tr}(\tilde{K}_J{}^2)\to\frac{(1+\gamma)^4}{\pi^2\gamma(2+\gamma)}\ln\left[\Lambda a_{ul}\left(\frac{\gamma(2+\gamma)}{(1+\gamma)^2}\right)^{1/2}\right]\times\int_0^{\pi}\frac{d\theta}{\sin\theta}Q_J{}^2\left(\frac{1+\gamma}{\sin\theta}\right), \tag{32}$$

so that

$$\lim_{a_{ul}\to\infty}\frac{\mathrm{tr}(\tilde{K}_J{}^2)}{\mathrm{tr}(\tilde{K}_J)}=(-1)^J\frac{(1+\gamma)^2}{\pi[\gamma(2+\gamma)]^{1/2}}\frac{\int_0^{\pi}\frac{d\theta}{\sin\theta}Q_J{}^2\left(\frac{1+\gamma}{\sin\theta}\right)}{Q_J(1+\gamma)}. \tag{33}$$

It is clear from Eq. (33) that for each even J there is a value of γ sufficiently small that the right-hand side will exceed unity. That is, in contrast to the identical-particle case, the Born-Oppenheimer case will have infinite numbers of bound states in arbitrarily high (even) partial waves, even when the like pair effectively do not interact (i.e., when $a_{ll}<\infty$, the second term in $\tilde{K}_J$ remains nonsingular). This qualitative result does not change even when we consider the case where both a_{ul} and $a_{ll}\to\infty$, since the second term of $\tilde{K}_J$ becomes small compared with the first when $\gamma\to 0$. That is, the possibility (suggested by the fact that the second term of $\tilde{K}_J$ is always a positive operator independent of J) that the l-l interaction with $a_{ll}\to\infty$ might produce infinitely many three-body bound states in odd partial waves also does not actually occur. It may at first seem surprising that only the even waves are attractive, but in fact the choice of signs in Eq. (30) implies that the like pair are bosons.

Examining the limiting case $\gamma\to\infty$, we find that when $a_{ll}<\infty$,

$$\lim_{a_{ul}\to\infty}\frac{\mathrm{tr}(\tilde{K}_J{}^2)}{\mathrm{tr}(\tilde{K}_J)}=(-1)^J\frac{1+\gamma}{\pi}\frac{\int_0^{\pi}\frac{d\theta}{\sin\theta}Q_J{}^2\left(\frac{1+\gamma}{\sin\theta}\right)}{Q_J(1+\gamma)}, \tag{34}$$

which for large γ approaches $2/\pi$ when $J=0$ and vanishes like γ^{-J} when $J>0$. An upper bound to $\|\tilde{K}_J\|$ which also depends on γ like γ^{-J} is easily obtained. Thus, there definitely cannot occur infinitely many bound states with $J>0$. But since $2/\pi<1$, we have for $J=0$ an inconclusive result, since $2/\pi$ is only a lower bound on the accumulation point of the eigenvalues of $\tilde{K}_0$. However, we already know that when $a_{ll}<\infty$, and $\gamma\to\infty$, the number of bound states is finite, since the wave function of a noninteracting pair of identical particles in an external potential is a product of single-particle (sp) wave functions and the binding energy is the sum of the sp energies. But these are zero if $a_{ul}=\infty$. The static limit can be singular, therefore, only if a_{ul} and a_{ll} both become infinite. The integrals involved in evaluating the traces for $J=0$, $\gamma=\infty$ are rather complicated, so we evaluated them numerically rather than relying on crude bounds. We find (to within about 1% accuracy)

$$\lim_{a_{ul}\to\infty,a_{ll}\to\infty}\frac{\mathrm{tr}(\tilde{K}_0{}^2)}{\mathrm{tr}(\tilde{K}_0)}=\frac{2/\pi^2+50.2/\pi^3+488/\pi^4}{1/\pi+10.6/\pi^2}=4.91\pm0.05. \tag{35}$$

When $J>0$, $\gamma\to\infty$, $a_{ul}\propto a_{ll}\to\infty$, the situation is more complicated. We have, for $J\geqslant 1$, $\gamma\to\infty$,

$$\|\tilde{K}_J\|\leqslant\frac{8}{\pi^2}4^{J+1}[Q_J(\sqrt{2})]^2\frac{\Gamma^4(\frac{1}{2}+\frac{1}{2}J)}{\Gamma^2(J+1)} \tag{36}$$

(see Appendix D), which for $J=1$ is a number roughly equal to 0.84, and which decreases rapidly with J thereafter. We conclude that the number of bound states with $J>1$ is finite in the static limit when $a_{ul}\propto a_{ll}\to\infty$, as it is in the identical-boson case.

IV. DISCUSSION

We have shown that three-body systems interacting *via* short-range forces have a logarithmically growing number of bound states near zero energy as the two-body scattering lengths become infinite, just as first discovered by Efimov. We have shown that this occurs not only for three identical particles, but also for two identical and one different. In the latter situation we have studied the two interesting limits of two very heavy and one light ($2H$-$1L$) and of one heavy and two light ($1H$-$2L$). In the $2H$-$1L$ case we found that the heavy-heavy interaction did not matter: There is an infinite number of states so long as the light-heavy scattering length becomes infinite. We can see physically why this should be so since if the masses of the heavy particles become infinite, they become static sources and there is an infinite number of ways of placing them relative to each other – each corresponding to a three-body state if the range of their interaction is effectively infinite. The infinite variety of possible orientations is of course the reason why the number of states can be infinite with $J>0$. In the $1H$-$2L$ case, all scattering lengths have to become infinite for there to be an infinite number of bound states. This too is easily understood: If one particle is static and the other two do not interact, the eigenstates are product wave functions and there is no infinity of energy eigenvalues.

One of the most surprising aspects of the Efimov effect is that the number of three-body bound states is not monotonic in the potential strength. The number of states diverges only for coupling strength corresponding to the first zero-energy two-body bound state. For *stronger* or *weaker* coupling, the number of three-body bound states is finite. On the other hand, one can easily show using Feynman's theorem[16] that at least for purely attractive potentials all two- and three-body bound states get more tightly bound with increasing coupling. The way out of these seemingly contradictory facts is to note that three-body states must occur below the lowest scattering threshold. In the absence of two-body bound states, this threshold is $E=0$, but if there are two-body bound states, the lowest scattering threshold in the three-body system is $E=-b$, where b is the binding energy of the most tightly bound two-body state. Hence the scattering threshold itself is a function of coupling strength. As the coupling is increased, b increases more rapidly than the binding energy of the nth three-body state (where $n<\infty$) and hence the threshold overruns, and devours infinitely many three-body bound states, forcing them onto unphysical sheets of the scattering amplitude. What happens to the Efimov states after they are eliminated? What sheet do they go on to? Do they show up as resonances? These are all questions we are presently studying. This nonmonotonicity of the number of bound states is, of course, not special to the Efimov states and has been previously noted by Simon.[3]

Although we have shown that the Efimov mechanism can produce true three-body bound states only when the *first* two-body bound state has zero energy, it is interesting to investigate in more detail what happens when the nth two-body bound state is at $E=0$. Near any two-body bound state, the two-body t matrix has the approximate form (5) or (7) with $a\to\infty$, and the trace of the appropriate three-body kernel will again diverge as $\ln(\Lambda a)$. However, the kernel will now gain an imaginary part from the two-body scattering cut which runs from $-b$ to $+\infty$, and this will give the trace an imaginary part if evaluated for $E>-b$. It is easy to see that the trace of the imaginary part of the kernel remains finite at $E=0$ and $a=\infty$. This means that whenever there is a two-body bound state with zero binding energy, there are an infinte set of solutions of the Schrödinger equation with E complex and very near zero. What sheet these eigenvalues are on, and their functional dependence of the two-body coupling are all interesting questions. It is, of course, particularly interesting to see if any of these "states" will be observable resonances. We are presently investigating these matters. In addition to our analytic attack on the problem, we have made a numerical model.[17] So far we have found the Efimov bound states numerically and we plan to investigate the sheet structure numerically as well.

Aspects of Efimov's discovery are of interest for other problems in physics. The emergence or disappearance of an infinite number of bound states as the coupling is increased gives some indication of the terrible nature of the singularity in the three-body scattering amplitude at zero energy and $g=g_0$. This singularity, and also the voracious behavior of moving thresholds, must have significance for a wide range of problems, particularly in particle physics. The existence and nature of the infinite number of weakly bound states may also be significant in a number of statistical-mechanics problems, since this effect may occur for any $(n+1)$-body system when the n-body system first develops a bound state. We are investigating ways of verifying this for the $(n+1)$-body system.

ACKNOWLEDGMENTS

We are grateful to V. Efimov and to C. Goebel for helpful correspondence, and to D. F. Freeman for once again lending us his constructive skepticism.

APPENDIX A

In this appendix we establish the monotonicity conditions on the eigenvalues given in Eq. (18). The eigenvalues of $K_J(E,g)$ [defined in Eq. (17)] are defined by

$$\tilde{K}_J\tilde{\phi}=\tilde{\eta}_J\tilde{\phi}, \quad (A1)$$

so that (we suppose that $\tilde{K}_J$ has been symmetrized)

$$\frac{1}{\tilde{\eta}_J}\frac{\partial\tilde{\eta}_J}{\partial E}=\left(\tilde{\phi},\frac{\partial\tilde{K}_J}{\partial E}\tilde{\phi}\right)\Big/(\tilde{\phi},\tilde{K}_J\tilde{\phi}) \quad (A2)$$

and

$$\frac{1}{\tilde{\eta}_J}\frac{\partial\tilde{\eta}_J}{\partial g}=\left(\tilde{\phi},\frac{\partial\tilde{K}_J}{\partial g}\tilde{\phi}\right)\Big/(\tilde{\phi},\tilde{K}_J\tilde{\phi}). \quad (A3)$$

[Since the scattering length a is a monotonically increasing continuous function of g for $g<g_0$, we can replace $\partial\tilde{K}_J/\partial g$ by $(\partial\tilde{K}_J/\partial a)(da/dg)$ in (A3).] Now we use the fact that[18]

$$\frac{1}{2pp'}Q_J\left(\frac{p^2+p'^2-E}{pp'}\right)=\int_0^\infty dt\, te^{-(p^2+p'^2-2E)t}j_J(pt)j_J(p't) \quad (A4)$$

to write (we let $\psi(p)=[1/a+(\frac{3}{4}p^2-E)^{1/2}]^{-1/2}\tilde{\phi}(p)$)

$$\frac{1}{\tilde{\eta}_J}\frac{\partial\tilde{\eta}_J}{\partial E}=\frac{(\tilde{\phi},[B\tilde{K}_J+\tilde{K}_JB]\tilde{\phi})}{(\tilde{\phi},\tilde{K}_J\tilde{\phi})}+\left[2\int_0^\infty dt\, t^2e^{2Et}\left|\int_0^\Lambda dp p^2\psi(p)e^{-p^2t}j_J(pt)\right|^2\right]\left[\int_0^\infty dt\, te^{2Et}\left|\int_0^\Lambda dp p^2\psi(p)e^{-p^2t}j_J(pt)\right|^2\right]^{-1}, \quad (A5)$$

where the positive definite operator B is given in the momentum representation by

$$B(p,p')=\delta(p-p')\tfrac{1}{4}(\tfrac{3}{4}p^2-E)^{-1/2}[1/a+(\tfrac{3}{4}p^2-E)^{1/2}]^{-1}. \quad (A6)$$

The second term of (A5) is clearly positive, and the first is just $2(\tilde{\phi},B\tilde{\phi})/(\tilde{\phi},\tilde{\phi})$, so that $(1/\tilde{\eta}_J)(\partial\tilde{\eta}_J/\partial E)\geq 0$ for $E\leq 0$, $a<\infty$, as advertised. In a similar fashion we find

$$\frac{1}{\tilde{\eta}_J}\frac{\partial\tilde{\eta}_J}{\partial a}=\frac{(\tilde{\phi},[C\tilde{K}_J+\tilde{K}_JC]\tilde{\phi})}{(\tilde{\phi},\tilde{K}_J\tilde{\phi})}, \quad (A7)$$

where

$$C(p,p')=\delta(p-p')\frac{1}{2a^2}[1/a+(\tfrac{3}{4}p^2-E)^{1/2}]^{-1} \quad (A8)$$

is obviously a positive operator. But the right-hand side of (A7) is just $2(\tilde{\phi},C\tilde{\phi})/(\tilde{\phi},\tilde{\phi})\geq 0$, and since $da/dg>0$, we have

$$\frac{1}{\tilde{\eta}_J}\frac{\partial\tilde{\eta}_J}{\partial g}=\frac{da}{dg}\frac{2(\tilde{\phi},C\tilde{\phi})}{(\tilde{\phi},\tilde{\phi})}\geq 0.$$

Finally, from (A4) we see that $(-1)^JK_J$ is a positive operator since for any vector ϕ

$$(\phi,(-1)^J\tilde{K}_J\phi)=\frac{8}{\pi}\int_0^\infty dt\, te^{2Et}\left|\int_0^\Lambda dp p^2\frac{\phi(p)e^{-p^2t}j_J(pt)}{[1/a+(\frac{3}{4}p^2-E)^{1/2}]^{1/2}}\right|^2\geq 0. \quad (A9)$$

APPENDIX B

We show that the number of times a "peculiar" eigenvalue of $K(E,g)$ can equal unity is finite. We note that for E real and less than $-b(g)$, $K(E,g)$ is a compact operator which by similarity transformation can be made Hermitian. By virtue of Eq. (14) all solutions of the equation

$$\eta(E,g)=1 \quad (B1)$$

must satisfy

$$-B(g)\leq E\leq -b(g). \quad (B2)$$

Hence if there are an infinite number of roots of (B1), they must possess at least one accumulation point in this interval. Hence it must be possible to find two values of E arbitrarily close to each other within the interval for which (B1) is satisfied. However, solutions of the equation

$$K(E,g)\phi(E,g)=\phi(E,g), \quad (B3)$$

suitably symmetrized, are bound-state solutions of the Schrödinger equation. But such solutions corresponding to different energy eigenvalues are orthogonal, i.e.,

$$(S\phi(E_1,g),S\phi(E_2,g))=0,\quad E_1\neq E_2 \quad (B4)$$

where S is a symmetrization operator (which is a projection operator and therefore has unit norm). But as we shall see the eigenvalues of

$$K(E, g)\phi(E, g) = \eta(E, g)\phi(E, g) \tag{B5}$$

are continuous functions of E and therefore cannot satisfy (B4) for two arbitrarily close values E_1 and E_2 of E corresponding to the same η. To show this we write (suppressing g)

$$|(S\phi(E_1), S\phi(E_2))| \geq \big|(S\phi(E_1), S\phi(E_1)) - |(S\phi(E_1), S\phi(E_2) - S\phi(E_1))|\big| . \tag{B6}$$

The right-hand side of (B6) can be written $(1-\gamma)$, where

$$\gamma = |(S\phi(E_1), S\phi(E_2) - S\phi(E_1))| \leq \|\phi(E_2) - \phi(E_1)\| \tag{B7}$$

by Schwarz's inequality. If the ϕ's are continuous, $\gamma \to 0$ as $E_1 \to E_2$ and hence γ cannot cancel the one and preserve the orthogonality of the left-hand side of (B6). To prove this continuity, consider the quantity

$$\begin{aligned} &\eta_1\phi_1 - \eta_2\phi_2 = K_1\phi_1 - K_2\phi_2 , \\ &\eta_1(\phi_1 - \phi_2) + \phi_2(\eta_1 - \eta_2) = (K_1 - K_2)\phi_1 + K_2(\phi_2 - \phi_1), \end{aligned} \tag{B8}$$

where we have used (B5), suppressed the g label, and written $f(E_1) = f_1$, $F(E_2) = f_2$. Take matrix elements of (B8) on the left with an arbitrary eigenfunction ψ_2 of K_2 corresponding to eigenvalues λ_2 ($\lambda_2 \neq \eta_2$). Using $(\phi_2, \psi_2) = 0$ from the Hermiticity of K_2, we obtain

$$(\eta_1 - \lambda_2)(\psi_2, (\phi_1 - \phi_2)) = (\psi_2, (K_1 - K_2)\phi_1) . \tag{B9}$$

Using the operator continuity of $K(E)$ in the real interval (B2), the right-hand side of (B9) vanishes as $E_1 \to E_2$. $\eta_1 - \lambda_2$ does not vanish in that limit, hence $(\psi_2, \phi_1 - \phi_2)$ must vanish. Consider the set of all ψ_2 plus the vector $\chi = \phi_2 + \phi_1$. This set is complete, but $(\chi, \phi_1 - \phi_2) = 0$. (We assume we are working in a real basis.) Hence as E_1 approaches E_2, the projections of $\phi_1 - \phi_2$ on a complete set vanish and thus the norm of $\phi_1 - \phi_2$ must vanish, which proves the result. If the roots of (B5) are degenerate so that more than one eigenvalue equals unity at some point, orthogonal combinations must be formed to make the proof go through. Of course an infinte number of roots cannot be unity if the kernel is compact.

We note that if $(1/\eta)(d\eta/dg) > 0$ for all real $E < -b(g)$, as will be the case for a large class of potentials including most separable potentials, the proof given here can easily be extended to show that each η can equal unity only once (for fixed g) as a function of E.

APPENDIX C

We want to show that if $w_n(x)$ are positive functions and $\{s_n\}_{n=1}^{\infty}$ a sequence with limit $l = \lim_{n\to\infty} s_n$, the function

$$f(x) = \sum_{n=1}^{\infty} w_n(x)s_n \Big/ \sum_{n=1}^{\infty} w_n(x)$$

approaches l as $x \to \infty$ if and only if for each $m > 0$,

$$\lim_{x\to\infty} \frac{\sum_{n=1}^{m} w_n(x)}{\sum_{n=1}^{\infty} w_n(x)} = 0 .$$

The method used here is that of Ford.[12] It is clear that the condition is necessary, since we can construct an example which does not satsify the hypothesis: Simply choose $w_n(x) = x/(xn^2+1)$, $n = 1, 2, \ldots,$ and $s_n = 1 - n^{-2}$. Then $l = 1$ and

$$\lim_{x\to\infty} \frac{\sum_{n=1}^{m} x/(xn^2+1)}{\sum_{n=1}^{\infty} x/(xn^2+1)} \geq \lim_{x\to\infty} \frac{(1+1/x)^{-1}}{\sum_{n=1}^{\infty} 1/n^2} = \frac{6}{\pi^2} > 0,$$

which violates the hypothesis. We see that as a consequence,

$$\lim_{x\to\infty} \frac{\sum_{n=1}^{\infty} [x/(xn^2+1)](1 - 1/n^2)}{\sum_{n=1}^{\infty} x/(xn^2+1)} = 1 - \frac{\pi^2}{15} \neq 1 .$$

To show that the condition is sufficient, we look at

$$|l - f(x)| = \frac{\left|\sum_{n=1}^{\infty} w_n(x)(l - s_n)\right|}{\sum_{n=1}^{\infty} w_n(x)} \leq \frac{\sum_{n=1}^{\infty} w_n(x)|l - s_n|}{\sum_{n=1}^{\infty} w_n(x)} .$$

Now $|l - s_n|$ is bounded for all n by, say, $\lambda > 0$. On the other hand, given $\epsilon > 0$, there is an integer $m > 0$ such that $n \geq m$ implies $|s_n - l| \leq \epsilon/2$. Thus

$$|l - f(x)| \leq \lambda \frac{\sum_{n=1}^{m} w_n(x)}{\sum_{n=1}^{\infty} w_n(x)} + \frac{\epsilon}{2} \frac{\sum_{n=m+1}^{\infty} w_n(x)}{\sum_{n=1}^{\infty} w_n(x)} .$$

We now need merely take x sufficiently large that

$$\lambda \frac{\sum_{n=1}^{m} w_n(x)}{\sum_{n=1}^{\infty} w_n(x)} \leq \frac{\epsilon}{2} .$$

APPENDIX D

We now use a theorem given by Tiktopoulos[19] which is useful in finding bounds on operators represented by integral kernels to bound the kernels in our problem. Consider $K(x, y)$ where x and y may be vector variables, $d(x)$ may be a volume element in multidimensional space, and the domain of integration is some set, D, of x space. Then we have

$$\|K\| = \sup_{\|\psi\|=1,\,\|\varphi\|=1} |(\psi, K\varphi)| = \sup_{\|\psi\|=1,\,\|\varphi\|=1} \left| \int d(x) \int d(y)\, \psi^*(x) K(x,y) \varphi(y) \right|,$$

and if $\sigma(x)$ is a positive function, we have

$$\|K\| \leq \sup_{\|\varphi\|=1,\,\|\psi\|=1} \int d(x) \int d(y) |\psi(x)| |K(x,y)| |\varphi(y)| \frac{\sigma^{1/2}(x)\sigma^{1/2}(y)}{\sigma^{1/2}(y)\sigma^{1/2}(x)}.$$

Using the Cauchy-Schwarz inequality we now find

$$\|K\| \leq \sup_{\|\varphi\|,\,\|\psi\|=1} \left\{ \left[\int d(x) \int d(y) |\psi(x)|^2 |K(x,y)| \frac{\sigma(y)}{\sigma(x)} \right]^{1/2} \left[\int d(x) \int d(y) \frac{\sigma(x)}{\sigma(y)} |K(x,y)| |\psi(y)|^2 \right]^{1/2} \right\}.$$

Since $\|\varphi\|^2 = \int d(x) |\varphi(x)|^2 = 1$ we have

$$\|K\| \leq \left[\sup_{x\in D} \frac{1}{\sigma(x)} \int d(y) |K(x,y)| \sigma(y) \right]^{1/2} \left[\sup_{y\in D} \frac{1}{\sigma(y)} \int d(x)\, \sigma(x) |K(x,y)| \right]^{1/2}.$$

Since we are interested in a symmetric kernel, $|K(x,y)| = |K(y,x)|$, we have

$$\|K\| \leq \sup_{x\in D} \frac{1}{\sigma(x)} \int d(y) |K(x,y)| \sigma(y), \tag{D1}$$

where $\sigma(x)$ is *any* positive function. Of course, choices of σ which make the right side of (D1) infinite are not particularly useful.

To apply (D1) to the symmetrized form of our limiting kernel $\tilde{K}_J(0, g_0)$ of Eq. (17), we note that for $Z > 1$, $Z^{J+1} Q_J(Z)$ is monotonically decreasing in Z, and so for $Z > Z_0 > 1$,

$$Q_J(Z) \leq Q_J(Z_0) \left(\frac{Z_0}{Z} \right)^{J+1}. \tag{D2}$$

Thus

$$\|\tilde{K}_J^{\text{sym}}\| \leq \frac{4}{\pi} \left(\frac{4}{3} \right)^{1/2} \sup_{0\leq p\leq\Lambda} \frac{1}{\sigma(p) p^{3/2}} \int_0^\Lambda dp'\, p'^{1/2} \sigma(p') Q_J\left(\frac{p^2 + p'^2}{pp'} \right). \tag{D3}$$

Let $\sigma(p) = p^{-3/2}$, which is surely >0. Then

$$\|\tilde{K}_J^{\text{sym}}\| \leq \frac{8}{\pi\sqrt{3}} \sup_{0\leq p\leq\Lambda} \int_0^{\Lambda/p} \frac{dx}{x} Q_J\left(\frac{1+x^2}{x} \right) = \frac{8}{\pi\sqrt{3}} \int_0^\infty \frac{dx}{x} Q_J\left(\frac{1+x^2}{x} \right).$$

Now employing (D2) and noting that $(1+x^2)/x \geq 2$ for $x \geq 0$, we have

$$\|\tilde{K}_J^{\text{sym}}\| \leq \frac{8}{\pi\sqrt{3}} 2^{J+1} Q_J(2) \int_0^\infty \frac{dx\, x^J}{(1+x^2)^{J+1}} = \frac{8}{\pi\sqrt{3}} 2^J Q_J(2) \frac{\Gamma^2(\frac{1}{2}+\frac{1}{2}J)}{\Gamma(J+1)}. \tag{D4}$$

In obtaining the bound (36), we ignore the first term of (30) for $\gamma \to \infty$, $J > 1$, since it is zero. The second term (when symmetrized) is ($\gamma \to \infty$, $a_{ul} = a_{ll} = \infty$)

$$\tilde{K}_J'(p,p') = \frac{8}{\pi^2} \frac{\theta(\Lambda - p)\theta(\Lambda - p')}{p^{3/2} p'^{3/2}} \int_0^\Lambda \frac{dp''}{p''} Q_J\left(\frac{p^2 + \frac{1}{2}p''^2}{pp''} \right) Q_J\left(\frac{p'^2 + \frac{1}{2}p''^2}{p'p''} \right). \tag{D5}$$

Again choosing $\sigma(p) = p^{-3/2}$, we have

$$\|\tilde{K}_J'\| \leq \frac{8}{\pi^2} \sup_{0\leq p\leq\Lambda} \int_0^{\Lambda/p} \frac{dx}{x} \int_0^{\Lambda/p} \frac{dy}{y} Q_J\left(\frac{1+\frac{1}{2}x^2}{x} \right) Q_J\left(\frac{y^2 + \frac{1}{2}x^2}{xy} \right) \leq \frac{8}{\pi^2} \int_0^\infty \frac{dx}{x} Q_J\left(\frac{1+\frac{1}{2}x^2}{x} \right) \int_0^\infty \frac{dy}{y} Q_J\left(\frac{2y^2+1}{2y} \right). \tag{D6}$$

Noting that $(1+\frac{1}{2}x^2)/x \geq \sqrt{2}$, $(1+2y^2)/2y \geq \sqrt{2}$, we apply (D2) to get

$$\|\tilde{K}_J'\| \leq \frac{8}{\pi^2} 4^{J+1} [Q_J(\sqrt{2})]^2 \frac{\Gamma^4(\frac{1}{2}+\frac{1}{2}J)}{\Gamma^2(J+1)}. \tag{D7}$$

*Work supported in part by the National Science Foundation.

†Work supported in part by the U. S. Atomic Energy Commission.

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[8]See, e.g., M. L. Goldberger and K. M. Watson, *Collision Theory* (Wiley, New York, 1964), p. 286.

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[10]These eigenvalues are, by definition, the limits of the (discrete) eigenvalues of the compact kernels $K(E,g)$. The question of whether the bounded but noncompact limiting kernel $K(0,g_0)$ also possesses a continuous spectrum does not enter into our proof.

[11]Reference 6, p. 367.

[12]W. R. Ford, *Studies in Divergent Series and Summability* (Chelsea, New York, 1960), p. 79; G. H. Hardy, *Divergent Series* (Oxford Univ. Press, Oxford, 1949), p. 49.

[13]We have tacitly assumed the sequence $\{(-1)^J\tilde{S}_\nu(J)\}_{\nu=1}^{\infty}$ has a definite limit. That this is indeed the case follows from the fact that it is bounded and nonincreasing, $(-1)^J\tilde{S}_\nu(J) > (-1)^J\tilde{S}_{\nu+1}(J)$. Cf. T. Apostol, *Mathematical Analysis* (Addison-Wesley, Reading, Mass., 1957), pp. 43, 355.

[14]We have used the relation [cf. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge Univ. Press, Cambridge, 1962), p. 317]

$$Q_J(Z) = \pi^{1/2}\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})}(2Z)^{-J-1}\,{}_2F_1(\tfrac{1}{2}J+1,\ \tfrac{1}{2}J+\tfrac{1}{2};\ J+\tfrac{3}{2};\ Z^{-2})$$

valid for $|Z|>1$, together with the fact that for Z real and >1 the hypergeometric function ${}_2F_1(\frac{1}{2}J+1,\ \frac{1}{2}J+\frac{1}{2};\ J+\frac{3}{2};\ Z^{-2})$ exceeds unity and is real, to derive the inequality

$$Q_J(Z) \le \pi^{1/2}\frac{\Gamma(J+1)}{\Gamma(J+\frac{3}{2})}(2Z)^{-J-1}, \quad Z>1.$$

Then

$$\int_0^\pi \frac{d\theta}{\sin\theta}Q_J{}^2(2/\sin\theta) \le \frac{\pi\,\Gamma^2(J+1)}{\Gamma^2(J+\frac{3}{2})\,16^{J+1}}\int_0^\pi d\theta\,(\sin\theta)^{2J+1}$$

and since $\int_0^\pi d\theta\,(\sin\theta)^{2J+1} = \Gamma(J+1)/\Gamma(J+\frac{3}{2})$, we have the inequality which led to Eq. (26).

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PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Relativistic Eikonal Dynamics*

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(Received 27 December 1971)

The eikonal formula for the scattering due to neutral-vector-meson exchange at very high energy in the generalized ladder approximation is obtained as the exact solution of a dynamical equation. This equation is obtained by applying high-energy approximations to an equation of the quasipotential type. The latter equation is known to have the correct low-energy limit and it therefore interpolates between high- and low-energy approximations to the generalized ladder approximation.

I. INTRODUCTION

Calculations of scattering amplitudes in the high-energy limit have led to formulas of the eikonal type. Guided by the derivation of the eikonal formula from nonrelativistic theory, Lévy and Sucher[1] applied similar approximations to the generalized ladder diagrams[2] of a field theory of neutral-meson exchange to obtain the familiar exponential form of the scattering amplitude. The same results were also obtained in the relativistic theory by means of functional techniques.[3,4] In these approaches no dynamical equation was used, the dynamics being furnished by relativistic quantum