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Algebraic Factorization of Scattering Amplitudes at Physical Landau Singularities*

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The behavior of the scattering amplitude in the vicinity of a physical Landau singularity is considered. It is shown that its singular part may be written as an algebraic product of the scattering amplitudes for each vertex of the corresponding Landau graph times a certain explicitly determined singularity factor which depends only on the type of singularity (triangle graph, square graph, etc.) and on the masses and spins of the internal particles. Thus the well-known result for single-particle-exchange poles is generalized to arbitrary physical Landau singularities. Also, it is shown that for any Landau singularity there exists a finite polynomial in the scalar products of the external four-momenta whose vanishing gives the Landau singularity curve. A general, purely algebraic, method is given for constructing this polynomial.

I. INTRODUCTION

Among the various properties of scattering amplitudes, one of the best known and most useful is the behavior in the neighborhood of single-particle-exchange poles. It is known that if the n external legs of a given amplitude be divided into two disjoint subsets $(1, \ldots, r)$ and $(r+1, \ldots, n)$, and if the total quantum numbers of each subset allow for the exchange of a known physical particle of mass M, and if we let

$$\sum_{i=1}^{r} P_\mu^i = -\sum_{i=r+1}^{n} P_\mu^i = P_\mu ,$$

then we find:

1. The amplitude has a pole at points satisfying $P^2 = M^2$.
2. The four-momentum of the corresponding exchanged particle is P_μ.
3. In the immediate vicinity of the pole, if the exchanged particle has spin 0, the amplitude may be written in the form

$$A_{\alpha_1 \cdots \alpha_n}(P_1, \ldots, P_n) = A^{(1)}_{\alpha_1 \cdots \alpha_r}(P_1, \ldots, P_r, -P) \frac{i}{P^2 - M^2 + i\epsilon} A^{(2)}_{\alpha_{r+1} \cdots \alpha_n}(P, P_{r+1}, \ldots, P_n)$$
$$+ \text{other nonsingular terms},$$

where $A^{(1)}$ and $A^{(2)}$ are the amplitudes for the two independent scattering processes connected by the exchanged particle. If the exchanged particle has spin $\frac{1}{2}$, then we have

$$A_{\alpha_1 \cdots \alpha_n}(P_1, \ldots, P_n) = \sum_{\beta, \beta'} A^{(1)}_{\alpha_1 \cdots \alpha_r \beta}(P_1, \ldots, P_r, -P) \frac{i(\not{P} + M)_{\beta\beta'}}{P^2 - M^2 + i\epsilon} A^{(2)}_{\beta' \alpha_{r+1} \cdots \alpha_n}(P, P_{r+1}, \ldots, P_n)$$
$$+ \text{other nonsingular terms}.$$

In space-time this represents the possibility of the over-all scattering taking place in two far separated clusters, one in the forward light cone of the other, with a physical particle on its mass shell being emitted by the earlier cluster and absorbed by the later cluster.[1]

It is well known that scattering amplitudes also possess singularities corresponding to more complicated types of particle exchanges. These are called Landau singularities and their location is given by the Landau equations. Each Landau singularity (i.e., each type of particle exchange) may be represented by a graph in a standard way.[2]

We regard a reduced graph of a given graph as a distinct graph, representing a distinct type of particle exchange. We shall be interested only in physical-region Landau singularities, i.e., only Landau singularities which correspond to real physical space-time processes.[3] One may then ask the question: How much of properties 1 through 3 for single-particle exchanges may be generalized to arbitrary physical-region Landau singularities? As we shall prove in what follows, all of it may be generalized, more precisely, we shall consider an arbitrary scattering amplitude

$$A_{\alpha_1 \cdots \alpha_n}(P_1, \ldots, P_n)$$

and an arbitrary physical Landau singularity of that amplitude with its corresponding graph. The graph has n external lines. Let it have m internal lines, l independent loops, and p vertices. Let r_j external lines and s_j internal lines meet at the jth vertex. Then

$$\sum_{j=1}^{p} r_j = n .$$

Let

$$P_{(i,j)}, \quad i=1,\ldots,p; \quad j=1,\ldots,r_i$$

be the four-momentum of the jth external line which enters at the ith vertex. Let R_j, $j=1,\ldots,l$, be the jth independent loop momentum. Finally, let Q_i, $i=1,\ldots,m$, be the four-momentum of the ith internal line and let

$$K_i = \sum_{j=1}^{r_i} P_{(i,j)}$$

be the total external four-momentum entering at the ith vertex.

In what follows we shall prove that:

1. There always exists a finite polynomial in the scalar products of the K_i such that $\mathcal{R}(K_i \cdot K_j)=0$ gives the location of the Landau singularity.

2. For a given set of values $(K_1,\ldots,K_p)$ which lie on the given physical Landau singularity [which implies $\mathcal{R}(K_i\cdot K_j)=0$] there exists only one unique set of values for the internal-line four-momenta, $Q_i(K_1,\ldots,K_p)$, which satisfy Landau's equations. In other words, if four-momentum is conserved at each vertex, and if $Q_i^2=M_i^2$, $i=1,\ldots,m$, where M_i is the mass of the ith internal line, and if $\sum\alpha_i Q_{i\mu}=0$ around each independent loop of the graph, with $\alpha_i>0$, $i=1,\ldots,m$, where α_i are the Feynman parameters for each internal line, then each Q_i is uniquely determined.

3. For $P_1,\ldots,P_n$ in the immediate vicinity of the given Landau singularity, the scattering amplitude may be written in the form

$$\begin{aligned} A_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n) = \sum_{\beta(1,1)\cdots\beta(p,sp)} & A^{(1)}_{\alpha(1,1)\cdots\alpha(1,r_1)\beta(1,1)\cdots\beta(1,s_1)}(P_{(1,1)},\ldots,P_{(1,r_1)},Q_{(1,1)},\ldots,Q_{(1,s_1)}) \\ & \times A^{(2)}_{\alpha(2,1)\cdots\alpha(2,r_2)\beta(2,1)\cdots\beta(2,s_2)}(P_{(2,1)},\ldots,P_{(2,r_2)},Q_{(2,1)},\ldots,Q_{(2,s_2)})\times\cdots \\ & \times A^{(p)}_{\alpha(p,1)\cdots\alpha(p,r_p)\beta(p,1)\cdots\beta(p,s_p)}(P_{(p,1)},\ldots,P_{(p,r_p)},Q_{(p,1)},\ldots,Q_{(p,s_p)}) \\ & \times \phi_{\beta(1,1)\cdots\beta(1,s_1)\cdots\beta(p,1)\cdots\beta(p,s_p)}(K_1,\ldots,K_p) + \text{other nonsingular terms,} \end{aligned} \tag{1}$$

where $A^{(1)}$ through $A^{(p)}$ are the scattering amplitudes for the p independent scattering processes taking place at each vertex, and ϕ is determined only by the type of Landau singularity (triangle, square, acnode, etc.) and the type (mass and spin) of each internal particle. ϕ is analytic in the immediate vicinity of the given Landau singularity, and singular on it. The $Q_{(i,j)}$ are the uniquely determined internal momenta given by 2.

In order that 1 and 2 may hold only one restriction need be observed: At least one internal line must have nonzero mass. If property 3 is to hold we must further require that the given Landau singularity point does not also lie on the Landau curve of another graph, of which the given graph is a contraction. This excludes mainly normal threshold singularities and isolated points where a given Landau singularity curve touches one of its reduced-graph singularity curves.[4]

Property 2 is really a theorem in classical mechanics. It says, for example, that if we fire three relativistic billiard balls at each other, and the lightest of the three balls bounces back and forth 13 times between the two heavier ones and goes off, then given only the initial and final momenta of the three balls, we can uniquely reconstruct the momenta of each of the three balls at any time during the scattering. It is assumed the distance between collisions is large compared to the radius of the billiard balls.

Given property 2, it is not surprising that property 3 holds. The total amplitude for a given process with specified initial and final states, is a sum over the amplitudes for the possible intermediate states. If the intermediate state is uniquely determined, then the sum collapses into one term which is a product of the cluster amplitudes. What is a little surprising is that the functions ϕ are, in simple cases such as the triangle graph, square graph, etc., very simple expressions as we shall show by explicit calculation.

The results have a number of applications. Theoretically the cluster-decomposition program of analyzing the momentum-space behavior of fieldtheory amplitudes (or S-matrix amplitudes) implied by their behavior at large space-time separations has never gotten beyond the case of noninteracting clusters and the one-particle-exchange case. The results obtained here enable one to extend the cluster-decomposition program to include multiparticle exchanges. Experimentally, there has been interest for many years in trying to use the triangle singularity to explain certain enhancements in strongly interacting three-particle final states.[5] This effort was hampered by the fact that one had to do lengthy dispersion calculations in order to estimate the intensity and width of any given enhancement.[6] The results given below enable one to make the same estimates by substituting the appropriate masses (some of which may be unstable complex masses) and real four momenta into the simple formulas obtained below.

II. DEMONSTRATIONS

Proof of Property 1

Our notation will be that given in the Introduction. It is well known that there exists a function $D((K_i \cdot K_j), M_i^2, \alpha_i)$ called the discriminant, which is a homogeneous polynomial in the α_i, and whose coefficients are linear in the $(K_i \cdot K_j)$ and M_i^2, such that the equations $\partial D/\partial \alpha_i = 0$, $i = 1, \ldots, m$ are equivalent to the usual Landau conditions for the existence of a Landau singularity.[7] There is a theorem in algebra which states that given any m homogeneous polynomials, $\mathcal{P}_i(\alpha_i, \ldots, \alpha_n)$, in m unknowns, there exists a unique minimal homogeneous polynomial in the coefficients, $\mathcal{R}((K_i \cdot K_j), M_i^2)$, called the resultant, such that $\mathcal{R} = 0$ is a necessary and sufficient condition for the existence of a solution to the system of equations $\mathcal{P}_1 = 0, \ldots, \mathcal{P}_m = 0$ distinct from the trivial solution $(\alpha_1 = 0, \ldots, \alpha_m = 0)$.[8] Further, this polynomial may be explicitly written down in the general case as the quotient of two determinants.[9] Note that in particular when we have m homogeneous *linear* polynomials in m unknowns, the resultant becomes just the usual determinant. Clearly $\mathcal{R}((K_i \cdot K_j), M_i^2)$ is the desired polynomial in the external momenta whose vanishing gives a necessary and sufficient condition for a Landau singularity. Note that $\mathcal{R} = 0$ gives unphysical as well as physical Landau singularities. This establishes property 1. This method offers an algebraic alternative to the usual geometric dual-diagram method, initiated by Landau, for finding the location of singularities.

Proof of Property 2

By conservation of four-momentum at the vertices, the m internal-line conditions may be written as $Q_i^2 = M_i^2$, where

$$Q_{i\mu} = \left(\sum_{j=1}^{p} \epsilon_{ij}^1 K_{j\mu} + \sum_{j=1}^{l} \epsilon_{ij}^2 R_{j\mu} \right) \tag{2}$$

and

$$\epsilon_{ij} = -1, 0, +1 .$$

Consider the problem of locating an extremum of Q_1^2, satisfying subsidiary conditions

$$Q_i^2 = M_i^2, \quad i = 2, 3, \ldots, m \tag{3}$$

in the $4l$-dimensional space of the $R_{j\mu}$. The $K_{j\mu}$ are regarded as fixed. In order for a point in the $R_{j\mu}$ space to be an extremum it must be a simultaneous solution of the equations

$$Q_i^2 = M_i^2, \quad i = 2, 3, \ldots, m$$

$$\frac{\partial}{\partial R_{j\mu}} \left(Q_1^2 + \sum_{i=2}^{m} \beta_i (Q_i^2 - M_i^2) \right) = 0, \quad j = 1, \ldots, l \tag{4}$$

where the β_i are the Lagrange multipliers. If an extremum happens to satisfy, in addition to Eqs. (4), the equation

$$Q_1^2 = M_1^2 , \tag{5}$$

then clearly we have a solution to the Landau equations, and the external momenta $K_{i\mu}$ must lie on a Landau singularity. Thus any set of internal loop momenta, $R_{j\mu}$, which satisfies the Landau equations gives an extremum of Q_1^2 for fixed $Q_i^2 = M_i^2$, $i = 2, 3, \ldots, m$. In fact, if the given Landau singularity is a physical one, $\Rightarrow \beta_i > 0, i = 2, 3, \ldots, m$, then the $R_{j\mu}$ which solve the Landau equations give a maximum.

Proof. Let η_σ, $\sigma = 1, 2, \ldots, (4l - m + 1)$, parametrize the surface in the $R_{j\mu}$ space determined by the $(m-1)$ subsidiary conditions in the neighborhood of a given solution of Eqs. (4). Then for points on this surface we have

$$\frac{\partial Q_i^2}{\partial \eta_\sigma} = 2 Q_{i\mu} \frac{\partial Q_{i\mu}}{\partial \eta_\sigma} = 0, \quad i = 2, \ldots, m \tag{6}$$

and

$$\frac{\partial^2 Q_i^2}{\partial \eta_\sigma \partial \eta_\rho} = 2 \left(\frac{\partial Q_{i\mu}}{\partial \eta_\sigma} \frac{\partial Q_{i\mu}}{\partial \eta_\rho} + Q_{i\mu} \frac{\partial^2 Q_{i\mu}}{\partial \eta_\sigma \partial \eta_\rho} \right) = 0, \quad i = 2, \ldots, m . \tag{7}$$

At the extremum point we have further from (4)

$$\frac{\partial Q_1^2}{\partial R_{j\mu}} = -\sum_{i=2}^{m} \beta_i \frac{\partial Q_i^2}{\partial R_{j\mu}} . \tag{8}$$

Also since $\partial Q_{i\mu}/\partial R_{j\nu} = \epsilon_{ij}^2 \delta_{\mu\nu} = \text{const}$, anywhere on the surface we have

$$\frac{\partial^2 Q_{i\mu}}{\partial \eta_\sigma \partial \eta_\rho} = \sum_{j,\nu} \frac{\partial Q_{i\mu}}{\partial R_{j\nu}} \left(\frac{\partial^2 R_{j\nu}}{\partial \eta_\sigma \partial \eta_\rho} \right) . \tag{9}$$

From (7), (8), and (9) it follows that at the extremum

$$2Q_{1\mu}\frac{\partial^2 Q_{1\mu}}{\partial\eta_\sigma\partial\eta_\rho}=2Q_{1\mu}\sum_{j,\nu}\frac{\partial Q_{1\mu}}{\partial R_{j\nu}}\frac{\partial^2 R_{j\nu}}{\partial\eta_\sigma\partial\eta_\rho}$$

$$=\sum_{j,\nu}\frac{\partial Q_1^{\,2}}{\partial R_{j\nu}}\frac{\partial^2 R_{j\nu}}{\partial\eta_\sigma\partial\eta_\rho}$$

$$=-\sum_{i=2}^{m}\sum_{j,\nu}\beta_i\frac{\partial Q_i^{\,2}}{\partial R_{j\nu}}\frac{\partial^2 R_{j\nu}}{\partial\eta_\sigma\partial\eta_\rho}$$

$$=-2\sum_{i=2}^{m}\beta_i Q_{i\mu}\frac{\partial^2 Q_{i\mu}}{\partial\eta_\sigma\partial\eta_\rho}$$

$$=2\sum_{i=2}^{m}\beta_i\frac{\partial Q_{i\mu}}{\partial\eta_\sigma}\frac{\partial Q_{i\mu}}{\partial\eta_\rho}. \qquad (10)$$

Using (10) we find that at the extremum

$$\frac{\partial^2 Q_1^{\,2}}{\partial\eta_\sigma\partial\eta_\rho}=2\left(\frac{\partial Q_{1\mu}}{\partial\eta_\sigma}\frac{\partial Q_{1\mu}}{\partial\eta_\rho}+Q_{1\mu}\frac{\partial^2 Q_{1\mu}}{\partial\eta_\sigma\partial\eta_\rho}\right)$$

$$=2\left(\frac{\partial Q_{1\mu}}{\partial\eta_\sigma}\frac{\partial Q_{1\mu}}{\partial\eta_\rho}+\sum_{i=2}^{m}\beta_i\frac{\partial Q_{i\mu}}{\partial\eta_\sigma}\frac{\partial Q_{i\mu}}{\partial\eta_\rho}\right). \qquad (11)$$

Since $Q_i^{\,2}=M_i^{\,2}>0$, $Q_{i\mu}$ is timelike and so by (6), for an arbitrary displacement $d\eta_\sigma$, we find that

$$\Delta Q_{i\mu}=\sum_\sigma\frac{\partial Q_{i\mu}}{\partial\eta_\sigma}d\eta_\sigma$$

is spacelike. Using this result we see from (11) that for arbitrary $d\eta_\sigma$,

$$\sum_{\sigma,\rho}\frac{\partial^2 Q_1^{\,2}}{\partial\eta_\sigma\partial\eta_\rho}d\eta_\sigma\,d\eta_\rho=2\left((\Delta Q_1)^2+\sum_{i=2}^{m}\beta_i(\Delta Q_i)^2\right)<0, \qquad (12)$$

since $\beta_i>0$ at a physical Landau singularity. Q.E.D.

This establishes property 2 except for the possibility that there may be a finite number (>1) of maxima. That this is not the case may be seen as follows. Assume there were two distinct positive α solutions to Landau's equation for a given Landau graph, and given external four-momenta. Then $\sum\alpha_i Q_i=\sum\overline{\alpha}_i\overline{Q}_i=0$ around each closed loop, and $Q_i^{\,2}=\overline{Q}_i^{\,2}=M_i^{\,2}$ with [as in Eq. (2)]

$$Q_i=\sum_j\epsilon_{ij}^1K_j+\sum_j\epsilon_{ij}^2R_j,$$

$$\overline{Q}_i=\sum_j\epsilon_{ij}^1K_j+\sum_j\epsilon_{ij}^2\overline{R}_j,$$

for each internal line. This implies $(Q_i-\overline{Q}_i)_\mu=\sum_j\epsilon_{ij}^2(R_{j\mu}-\overline{R}_{j\mu})$. Multiplying on the left by α_iQ_i and summing over i and μ, we find

$$\sum_{i=1}^{m}\alpha_i[Q_i^{\,2}-(Q_i\cdot\overline{Q}_i)]=0,$$

since $\sum_i\alpha_iQ_i\epsilon_{ij}^2=\sum\alpha_iQ_i$ around the jth closed loop $=0$. However $Q_i^{\,2}=\overline{Q}_i^{\,2}=M_i^{\,2}$, and if we assume $Q_{i0}\overline{Q}_{i0}>0$, then $Q_i^{\,2}-(Q_i\overline{Q}_i)\leq 0$ for all i, and since $\alpha_i>0$ for all i, $Q_i^{\,2}=Q_i\overline{Q}_i\Rightarrow Q_i=\overline{Q}_i$.[10] Thus there is only one maximum. We assumed that $Q_i^0\overline{Q}_i^0>0$. We regard solutions for which the time directions of some of the internal lines have been reversed, as corresponding to distinct *types* of processes. We may accordingly redefine the notion of "a distinct Landau graph" to include a specification of the time direction of each internal line.

Proof of Property 3

Consider for general values of the external momenta, the $(4l-m)$-dimensional surface in the $R_{j\mu}$ space satisfying $Q_i^{\,2}=M_i^{\,2}$, $i=1,\ldots,m$. It is the intersection of the $(4l-m+1)$-dimensional surface given by $Q_i^{\,2}=M_i^{\,2}$, $i\geq 2$, and the surface $Q_1^{\,2}=M_1^{\,2}$. When the external momenta move onto a given point on the physical Landau curve, $M_1^{\,2}$ becomes the maximum value of $Q_1^{\,2}$ for fixed $Q_i^{\,2}=M_i^{\,2}$, $i\geq 2$, so clearly for these particular values of the external momenta, the $(4l-m)$-dimensional surface (or the branch of it in the vicinity of the maximum) satisfying $Q_i^{\,2}=M_i^{\,2}$, $i=1,\ldots,m$, degenerates into a single point. Let $\overline{K}_{j\mu}$ be the given set of external momenta on the physical Landau curve, and let $\overline{R}_{j\mu}$ be the point which maximizes $Q_1^{\,2}$. Clearly by continuity arguments, for any $\epsilon>0$, there exists a $\delta>0$:

$$\sum_{j,\mu}|K_{j\mu}-\overline{K}_{j\mu}|^2<\delta$$

implies that if $R_{j\mu}$ is any point in the $(4l-m)$-dimensional surface determined by the equations $Q_i^{\,2}(K_{j\mu},R_{j\mu})=M_i^{\,2}$, $i=1,\ldots,m$, then

$$\sum_{j,\mu}|R_{j\mu}-\overline{R}_{j\mu}|^2<\epsilon.$$

Now consider any perturbation-theory contribution to the given n-point amplitude,

$$A_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n)=\int\prod_{i=1}^{l'}d^4R_i\,\frac{B_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n,R_1,\ldots,R_{l'})}{\prod_{j=1}^{m'}(Q_j^{\,2}-M_j^{\,2}+i\epsilon)},$$

where the given Feynman graph has m' internal lines and l' independent loops.[11] If this Feynman graph has no Landau singularity at the given point $\overline{K}_{j\mu}$, then we include its contribution among "other nonsingular terms." If it does have a Landau singularity at the given point, we may write, without loss of generality,

$$A_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n)=\int\prod_{i=1}^{l}d^4R_i\prod_{j=1}^{m}\left(\frac{1}{Q_j^2-M_j^2+i\epsilon}\right)\int\prod_{i=l+1}^{l'}d^4R_i\,\frac{B_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n,R_1,\ldots,R_{l'})}{\prod_{j=m+1}^{m'}(Q_j^2-M_j^2+i\epsilon)},\qquad(13)$$

where the Q_i, $i=1,\ldots,m$ are the uncontracted lines in the Landau diagram (the internal lines of the reduced graph) and Q_i, $i=m+1,\ldots,m'$ are the contracted lines. Further, the R_i are chosen so that Q_i for $i\le m$ depends only on $R_1,\ldots,R_l$ and not on $R_{l+1},\ldots,R_{l'}$. This is always possible.[12] R_1 through R_l are the l independent loop momenta of the reduced graph. It follows from the analysis of Eden *et al.*[13] that the part of the amplitude which is singular at the given point arises solely from the coincidence of the m poles in the integrand $1/(Q_i^2-M_i^2+i\epsilon)$, $i=1,\ldots,m$. Therefore, if we only wish to look at the singular part of A, we need not integrate each of the $R_{j\mu}$, $j=1,\ldots,l$ from $-\infty$ to $+\infty$. We may write instead

$$A_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n)=\int_V\prod_{i=1}^{l}d^4R_i\prod_{j=1}^{m}\left(\frac{i}{Q_j^2-M_j^2+i\epsilon}\right)\int_\infty\prod_{i=l+1}^{l'}d^4R_i\frac{B_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n,R_1,\ldots,R_{l'})}{\prod_{j=m+1}^{m'}(Q_j^2-M_j^2+i\epsilon)}$$
$$+\text{other nonsingular terms},\qquad(14)$$

where V is a $4l$-dimensional volume containing all points $(R_{1\mu},\ldots,R_{l\mu})$ within δ ($\delta>0$ and arbitrarily small) of some point on the $(4l-m)$-dimensional surface given by $[Q_i(K_1,\ldots,K_p,R_1,\ldots,R_l)]^2=M_i^2$, $i=1,\ldots,m$. By the above, as the external momenta $(K_1,\ldots,K_p)$ approach the given physical Landau singularity, V shrinks to a sphere of radius δ centered at $\overline{R}_{j\mu}$. If we require that two distinct Landau singularities (having distinct graphs) not coincide at the given point in the space of the external momenta, then the second factor in (14) will be analytic in a neighborhood of $\overline{R}_{j\mu}$, and by choosing δ sufficiently small we may regard the second factor as a constant as $R_{j\mu}$ varies over V. Thus we obtain

$$A_{\alpha_1\cdots\alpha_n}(P_1,\ldots,P_n)=\int_\infty\prod_{i=l+1}^{l'}d^4R_i\frac{B(P_1,\ldots,P_n,\overline{R}_1,\ldots,\overline{R}_l,R_{l+1},\ldots,R_{l'})}{\prod_{j=m+1}^{m'}(Q_j^2-M_j^2+i\epsilon)}\int_V\prod_{i=1}^{l}d^4R_i\prod_{j=1}^{m}\left(\frac{i}{Q_j^2-M_j^2+i\epsilon}\right)$$
$$+\text{other nonsingular terms}.\qquad(15)$$

Here,

$$Q_j=Q_j(K_1,\ldots,K_p,R_1,\ldots,R_l)\quad\text{for } j=1,\ldots,m,$$

and

$$Q_j=Q_j(P_1,\ldots,P_n,\overline{R}_1,\ldots,\overline{R}_l,R_{l+1},\ldots,R_{l'})\quad\text{for } j=m+1,\ldots,m'.$$

Also, let

$$\overline{Q}_j=Q_j(K_1,\ldots,K_p,\overline{R}_1,\ldots,\overline{R}_l),\quad j=1,\ldots,m.$$

Clearly the first factor on the right-hand side of (15) is just equal to the algebraic product of the independent Feynman graph contributions to the cluster amplitudes at each vertex, times a factor of $(\not{Q}_i+M_i)$ for each intermediate spin-$\frac{1}{2}$ particle in the Landau graph. If we put $\eta_{\beta(1,1)\cdots\beta(p,s_p)}$ equal to this product of intermediate spin factors, and if we set

$$\xi(K_1,\ldots,K_p)=\int_V\prod_{i=1}^{l}d^4R_i\prod_{j=1}^{m}\left(\frac{i}{Q_j^2-M_j^2+i\epsilon}\right)\qquad(16)$$

and then put

$$\phi_{\beta(1,1)\cdots\beta(p,s_p)}=\eta\xi,\qquad(17)$$

we have explicitly constructed the function ϕ which appears in Eq. (1) and if we then sum over all Feynman graph contributions to the given amplitude we find that we have established property 3. It should be noted that the function ξ is not uniquely defined. Clearly we can add any function, analytic in the neighborhood of the singularity, to ξ without invalidating property 3. Thus, for example, we could just as well take

$$\xi=\int_\infty\prod_{i=1}^{l}d^4R_i\prod_{j=1}^{m}\left(\frac{i}{Q_j^2-M_j^2+i\epsilon}\right),\qquad(18)$$

or even better we can use the analysis of Cutkosky[14] and make a change of variables in (18) writing

$$\xi=\int_{a_1}^{b_1}dQ_1^2\cdots\int_{a_m}^{b_m}dQ_m^2\int\frac{\prod d\rho_i}{J}\prod_{j=1}^{m}\left(\frac{i}{Q_j^2-M_j^2+i\epsilon}\right),\qquad(19)$$

where $\int\prod d\rho$ is the integral over the remaining angle variables, and J is the Jacobian of the transformation. Then the analysis of Cutkosky shows that the singular part of the right-hand side of (19) is

$$(2\pi)^{(m-1)}\int_{a_m}^{b_m} dQ_m{}^2 \int \frac{\prod d\rho_i}{J}\left(\frac{i}{Q_m{}^2 - M_m{}^2 + i\epsilon}\right), \quad (20)$$

and we may take ξ equal to expression (20).

III. EXAMPLES

The Triangle Landau Graph

Our notation is fixed in Fig. 1.

1. Let $y_{12} = (K_3{}^2 - M_1{}^2 - M_2{}^2)/2M_1M_2$, $y_{11} = -1$ and cyclic permutations of the above. Then as is easily obtained by the method described in Sec. II,

$$\mathcal{R}(K_1{}^2, K_2{}^2, K_3{}^2) = \mathrm{Det}(y_{ij}) = 0 \quad (21)$$

gives the Landau singularity curve.

2. The physical Landau curve has six branches in the $K_{i\mu}$ space. Consider the specific branch where $K_1{}^2 > (M_2+M_3)^2$, $K_3{}^2 > (M_1+M_2)^2$, $K_2{}^2 < (M_1 - M_3)^2$, $K_{10} > 0$, $K_{30} < 0$.

Let us go to the rest frame of $K_{1\mu}$ and then rotate so that $\vec{K}_3 = -p_3\hat{z}$, where $p_3 > 0$. Then

$$K_1 = (E_1, 0, 0, 0),$$

$$K_2 = (E_3 - E_1, 0, 0, p_3),$$

$$K_3 = -(E_3, 0, 0, p_3), \quad E_3 > 0.$$

Let the loop momentum $R_\mu = Q_{2\mu}$; this implies

$$Q_{1\mu} = R_\mu + K_{3\mu},$$

$$Q_{2\mu} = R_\mu,$$

$$Q_{3\mu} = R_\mu - K_{1\mu}.$$

Then solving for the unique intermediate-state momentum as described in Sec. II, we find

$$\begin{aligned}\overline{Q}_{2\mu} &= \overline{R}_\mu \\ &= \frac{1}{2E_1}\left((M_2{}^2 - M_3{}^2 + K_1{}^2), [\rho(K_1{}^2, M_2{}^2, M_3{}^2)]^{1/2}\hat{z}\right), \\ \overline{Q}_{1\mu} &= \overline{R}_\mu + K_{3\mu}, \\ \overline{Q}_{3\mu} &= \overline{R}_\mu - K_{1\mu},\end{aligned} \quad (22)$$

where $\rho(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$. The other branches are obtained by cyclic permutation and by the over-all reflection $K_{i\mu} \to -K_{i\mu}$.

3. Finally evaluating expression (20) for the case at hand we find for all six branches

$$\xi = \frac{+i}{8\pi[\rho(K_1{}^2, K_2{}^2, K_3{}^2)]^{1/2}} \ln[\mathcal{R}(K_1{}^2, K_2{}^2, K_3{}^2)], \quad (23)$$

where if $\mathcal{R} > 0$, $\ln\mathcal{R}$ is real and if $\mathcal{R} < 0$, $\ln\mathcal{R} = \ln(-\mathcal{R}) - i\pi$. If all three internal lines are spinless, then $\phi = \xi$. If, for example, the Q_1 line were a spin-$\frac{1}{2}$ particle, and other lines spin 0, then

$$\phi_{\beta\beta'}(K_1, K_2, K_3) = (\bar{\not{R}}(K_1, K_2, K_3) + \not{K}_3 + M_1)_{\beta\beta'} \times \xi(K_1{}^2, K_2{}^2, K_3{}^2). \quad (24)$$

The Square Landau Graph

Our notation is fixed in Fig. 2.

1. Let

$$y_{12} = \frac{M_1{}^2 + M_2{}^2 - K_2{}^2}{2M_1M_2} \text{ and cyclic permutations},$$

$$y_{11} = 1 \text{ and cyclic permutations},$$

$$y_{24} = \frac{M_2{}^2 + M_4{}^2 - s}{2M_2M_4},$$

$$y_{13} = \frac{M_1{}^2 + M_3{}^2 - t}{2M_1M_3}.$$

Then one easily finds

$$\mathcal{R}(K_1{}^2, K_2{}^2, K_3{}^2, K_4{}^2, s, t) = \mathrm{Det}(y_{ij}) = 0 \quad (25)$$

gives the Landau singularity curve.

2. The physical Landau curve has 14 branches in the $K_{i\mu}$ space.

(a) Four branches are in the regions $K_1{}^2 > (M_1 + M_4)^2$, $K_3{}^2 > (M_2 + M_3)^2$, $K_2{}^2 < (M_1 - M_2)^2$, $K_4{}^2 < (M_3 - M_4)^2$, $K_{10} > 0$, $K_{30} < 0$ and its distinct cyclic permutations and over-all reflections, $K_{i\mu} \to -K_{i\mu}$, $i = 1, 2, 3, 4$.

(b) Eight branches are in the regions $K_1{}^2 > (M_1 + M_4)^2$, $K_2{}^2 > (M_1 + M_2)^2$, $K_3{}^2 < (M_2 - M_3)^2$, $K_4{}^2 < (M_3 - M_4)^2$, $K_{10} > 0$, $K_{20} < 0$ and its distinct cyclic permutations

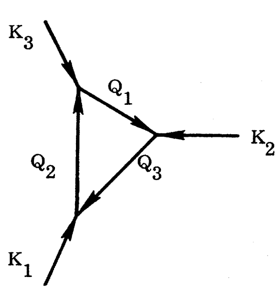

FIG. 1. The triangle Landau graph.

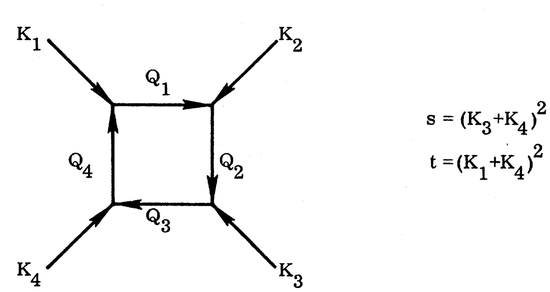

FIG. 2. The square Landau graph.

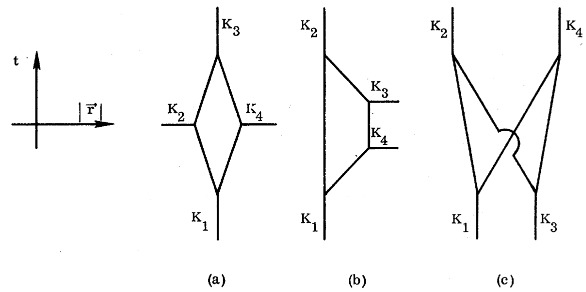

FIG. 3. The three distinct possible types of space-time processes.

and over-all reflections.

(c) Two branches are in the regions $K_1^2 > (M_1+M_4)^2$, $K_3^2 > (M_2+M_3)^2$, $K_2^2 > (M_1+M_2)^2$, $K_4^2 > (M_3+M_4)^2$, $K_{10} > 0$, $K_{30} > 0$, $K_{20} < 0$, $K_{40} < 0$ and its distinct cyclic permutations and over-all reflections.

Typical space-time diagrams for the three types of processes are shown in Fig. 3.

Let

$$\begin{aligned} Q_{1\mu} &= R_\mu , \\ Q_{2\mu} &= R_\mu + K_{2\mu} , \\ Q_{3\mu} &= R_\mu + K_{2\mu} + K_{3\mu} , \\ Q_{4\mu} &= R_\mu - K_{1\mu} . \end{aligned} \tag{26}$$

For the three cases pictured above we can always choose a Lorentz frame, where

$$\begin{aligned} K_{1\mu} &= (E_1, 0, 0, 0) , \\ K_{2\mu} &= (E_2, 0, 0, p_2), \quad p_2 > 0 , \\ K_{3\mu} &= (E_3, p_{3x}, 0, p_{3z}), \quad p_{3x} > 0 , \\ K_{4\mu} &= -(E_1 + E_2 + E_3, p_{3x}, 0, p_2 + p_{3z}) . \end{aligned}$$

Then a simple calculation shows that for each of the three cases pictured above, the unique intermediate momenta are given by (26) and

$$\overline{R}_\mu = (R_0, r\hat{u}) ,$$

where

$$\begin{aligned} R_0 &= \frac{M_1^2 - M_4^2 + K_1^2}{2E_1} , \\ r &= \frac{[\rho(K_1^2, M_1^2, M_4^2)]^{1/2}}{2E_1} , \\ u_x &= -[1 - (u_z)^2]^{1/2} , \\ u_y &= 0 , \\ u_z &= \left(\frac{1}{2p_2 r}\right)(M_1^2 - M_2^2 + K_2^2 + 2E_2 R_0) . \end{aligned} \tag{27}$$

3. Evaluating expression (20) we find for all branches

$$\xi = \frac{i}{16 M_1 M_2 M_3 M_4 [\mathcal{R}(K_1^2, K_2^2, K_3^2, K_4^2, s, t)]^{1/2}} , \tag{28}$$

where if $\mathcal{R} < 0$, then $\sqrt{\mathcal{R}} = -i\sqrt{-\mathcal{R}}$.

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[14]Eden *et al.*, Ref. 7, p. 111.