described in any basis by the conditions

$$(145) = [23] = 0 \rightarrow u = 0\,,$$
$$(215) = [43] = 0 \rightarrow v = u\,, \qquad \text{(A16)}$$
$$(213) = [54] = 0 \rightarrow v = 1\,.$$

Note that in a general basis where $(\vec{z}_2, \vec{z}_3, \vec{z}_4, \vec{z}_5)$ are complex, at least one point of the real projective plane shown in Fig. 2 is mapped to a point on the surface at infinity. Finally, we observe that, just as integrating the beta function integrand along the three intervals $[-\infty, 0]$, $[0,1]$, and $[1,\infty]$ in Fig. 1 gives the three permutations of four external lines, integrating the B_5 integrand over the twelve triangles of Fig. 2 gives the 12 distinct permutations of the five-point function's external lines.[9]

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PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Infinite-Momentum Helicity States*

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(Received 5 January 1972)

We discuss and generalize to arbitrary spin the kind of single-particle spin states which have appeared naturally in field theories in the infinite-momentum frame. These states transform simply under the Galilean symmetry group which is important in the infinite-momentum frame, rather than under the rotation group. We also find that the spinors $U(P,\lambda)$ representing these states are very simple.

I. INTRODUCTION

The states of a single particle with mass M, spin s are generally represented by a state vector $|P,\lambda\rangle$, where P is the momentum of the particle and λ labels its spin state. Many definitions of spin state are available – the most popular being the Jacob and Wick helicity states.[1]

The presently common kinds of spin states transform simply under rotations. They are thus particularly useful for the description of low-energy phenomena, in which rotational symmetry is important (for instance, two-body scattering in the resonance region). In this paper, we will define and discuss a set of spin states which transform simply under the "Galilean" transformations[2,3] which are useful in the description of particles moving in the $+z$ direction with high energy. These spin states have previously been found to emerge naturally in discussions of field theories in the infinite-momentum frame,[3-7] at least for the cases $s=\frac{1}{2}$ and $s=1$.

We begin with a brief review of the infinite-momentum coordinate system,

$$\tau = 2^{-1/2}(t+z), \quad \mathfrak{z} = 2^{-1/2}(t-z),$$

paying particular attention to the Galilean subgroup of the Poincaré group, which leaves the planes τ = constant invariant. We use this Galilean structure to define a convenient "spin" or "internal angular momentum" operator. It is then a simple matter to construct single-particle eigenstates of this operator. We also show that these "infinite-momentum helicity" states look like ordinary Jacob and Wick helicity states when viewed from a reference frame moving in the $-z$ direction with the speed of light.

Finally, we construct the spinors $U(P,\lambda)$ – the higher-spin analogs of the familiar Dirac spinors for spin $\frac{1}{2}$ – representing the infinite-momentum helicity states $|P,\lambda\rangle$. These spinors are found to have a remarkably simple form.

II. REVIEW OF THE INFINITE-MOMENTUM COORDINATES

In order to keep this paper reasonably self-contained, let us recall the change of variables that defines the infinite-momentum frame. Our notation and philosophy follows that of Ref. 3, where a more complete discussion may be found. The components a^μ of a four-vector in the infinite-momentum coordinate system are related to the components $\hat{a}^\mu$ of the same four-vector in the usual coordinate system by the transformation

$$a^0=2^{-1/2}(\hat{a}^0+\hat{a}^3),\quad a^1=\hat{a}^1,\quad a^2=\hat{a}^2,\quad a^3=2^{-1/2}(\hat{a}^0-\hat{a}^3).$$

In particular, the components of a position vector are $x^\mu=(\tau,x^1,x^2,\mathfrak{z})$, where $\tau=2^{-1/2}(t+z)$ and $\mathfrak{z}=2^{-1/2}(t-z)$. In the infinite-momentum system, the coordinate τ plays the role of "time."

We will be particularly interested in the generators of the Poincaré group in the new coordinate system. The generators of translations are $P^\mu=(\eta,P^1,P^2,H)$, where $\eta=2^{-1/2}(E+P_z)$ and $H=2^{-1/2}\times(E-P_z)$. Since $P_\mu x^\mu=H\tau+\eta z-\vec{P}\cdot\vec{x}$, we see that H generates τ translations and thus plays the role of a Hamiltonian.[8] The six generators of Lorentz transformations are conveniently taken to be $K,J,\vec{B},\vec{S}$, which are related to the usual three-component angular momentum and Lorentz boost operators, (J_1,J_2,J_3) and (K_1,K_2,K_3), by

$$\begin{aligned} &K=K_3, && J=J_3,\\ &B^1=(K_1+J_2)/\sqrt{2}, && S^1=(K_1-J_2)/\sqrt{2},\\ &B^2=(K_2-J_1)/\sqrt{2}, && S^2=(K_2+J_1)/\sqrt{2}\,. \end{aligned} \tag{2.1}$$

We recall[3] that the subgroup of the Poincaré group generated by $\eta,\vec{P},H,J,\vec{B}$ is isomorphic to the symmetry group of nonrelativistic quantum mechanics in two dimensions, with

$$\begin{aligned} &H\to \text{Hamiltonian},\\ &\vec{P}\to \text{momentum},\\ &\eta\to \text{mass},\\ &J\to \text{angular momentum},\\ &\vec{B}\to \text{generators of Galilean boosts}, \end{aligned} \tag{2.2}$$

(i.e., $e^{i\vec{v}\cdot\vec{B}}\vec{P}e^{-i\vec{v}\cdot\vec{B}}=\vec{P}+\eta\vec{v}$). The action of the Lorentz boost operator K is also very simple – K serves merely to rescale the other Poincaré generators:

$$\begin{aligned} &e^{i\omega K}\eta e^{-i\omega K}=e^{\omega}\eta,\\ &e^{i\omega K}\vec{P}e^{-i\omega K}=\vec{P},\\ &e^{i\omega K}He^{-i\omega K}=e^{-\omega}H,\\ &e^{i\omega K}Je^{-i\omega K}=J,\\ &e^{i\omega K}\vec{B}e^{-i\omega K}=e^{\omega}\vec{B},\\ &e^{i\omega K}\vec{S}e^{-i\omega K}=e^{-\omega}\vec{S}. \end{aligned} \tag{2.3}$$

III. INFINITE-MOMENTUM HELICITY STATES

A. The Spin Operator j

In order to use the infinite-momentum frame to discuss the dynamics of particles with nonzero spin, one needs a description of the possible states of a single spinning particle which is adapted to that frame. This means that the single-particle states should transform simply under the Poincaré generators η, $\vec{P}$, J, $\vec{B}$, and K, which leave the plane $\tau=0$ invariant.

It is easy to find such states for zero spin. We simply label the state $|\eta,\vec{p}\rangle$ by its "mass" η and transverse momentum $\vec{p}$. [The "energy" of the state is then given by the free-particle Hamiltonian, $h=(\vec{p}^2+M^2)/2\eta$, where M is the covariant mass of the particle.]

It will come as no surprise that the states of a massive particle with spin S can be labeled by the momentum $(\eta,\vec{p})$ of the particle and some $(2S+1)$-valued spin index λ. The states $|\eta,\vec{p},\lambda\rangle$ will transform simply under $\eta,\vec{P},J,\vec{B},K$ if we make use of the isomorphism with two-dimensional nonrelativistic quantum mechanics and let λ label the (two-dimensional, nonrelativistic) spin of the system.

To do this, recall that in nonrelativistic quantum mechanics, the Galilean boost operator is $\underline{B}=-m\underline{R}$. This relation serves to define a transverse-particle position operator for us[9]:

$$\vec{R}\equiv-\frac{1}{\eta}\vec{B}. \tag{3.1}$$

Using $\vec{R}$, we can define an "orbital angular momentum" operator

$$\vec{R}\times\vec{P}=R^1P^2-R^2P^1.$$

Finally, the difference between the total angular momentum J and $\vec{R}\times\vec{P}$ defines the "internal angular momentum" or "spin" operator j:

$$j=J-\vec{R}\times\vec{P}. \tag{3.2}$$

We now choose to let the spin index λ label the eigenvalue of j; thus

$$j|\eta,\vec{p},\lambda\rangle=\lambda|\eta,\vec{p},\lambda\rangle. \tag{3.3}$$

B. The Wigner Construction

We can make this definition of the states $|\eta,\vec{p},\lambda\rangle$ precise (and specify some until now arbitrary phase factors) by making use of an informal version of the famous Wigner construction.[10] We start with the states $|M/\sqrt{2},\vec{0},\lambda\rangle$ of a massive particle at rest. Since the particle is to have spin S, we require that these states transform under rotations according to the spin-S representation of SU(2):

$$U(A)|M/\sqrt{2},\vec{0},\lambda\rangle=\sum_{\lambda'} D^{(s)}(A)_{\lambda'\lambda}|M/\sqrt{2},\vec{0},\lambda'\rangle \tag{3.4}$$

for $A\in SU(2)$. The states $|\eta,\vec{p},\lambda\rangle$ of a particle with some other momentum are defined by applying a certain Lorentz transformation to the rest states $|M/\sqrt{2},\vec{0},\lambda\rangle$:

$$|\eta,\vec{p},\lambda\rangle=e^{-i\vec{v}\cdot\vec{B}}e^{-i\omega K}|M/\sqrt{2},\vec{0},\lambda\rangle, \tag{3.5}$$

where

$$\vec{v}=\vec{p}/\eta,\quad e^{\omega}=\sqrt{2}\,\eta/M.$$

Now since

$$j|M/\sqrt{2},\vec{0},\lambda\rangle=J|M/\sqrt{2},\vec{0},\lambda\rangle=\lambda|M/\sqrt{2},\vec{0},\lambda\rangle$$

and j commutes with $\vec{B}$ and K, we have

$$j|\eta,\vec{p},\lambda\rangle=\lambda|\eta,\vec{p},\lambda\rangle,$$

as desired. It is also easy to verify that the states $|\eta,\vec{p},\lambda\rangle$ transform simply under rotations, "Galilean boosts," and Lorentz z boosts:

$$\begin{aligned}
&e^{-i\phi J}|\eta,p^1,p^2,\lambda\rangle\\
&\quad=e^{-i\phi\lambda}|\eta,\cos\phi p^1-\sin\phi p^2,\sin\phi p^1+\cos\phi p^2,\lambda\rangle,\\
&e^{-i\vec{v}\cdot\vec{B}}|\eta,\vec{p},\lambda\rangle=|\eta,\vec{p}+\eta\vec{v},\lambda\rangle,\\
&e^{-i\omega K}|\eta,\vec{p},\lambda\rangle=|e^{\omega}\eta,\vec{p},\lambda\rangle.
\end{aligned} \tag{3.6}$$

Finally, we note that the states $|\eta,\vec{p},\lambda\rangle$ must be covariantly normalized if the operators $\exp(-i\vec{v}\cdot\vec{B})$ and $\exp(-i\omega K)$ are to be unitary. Thus, we take

$$\langle\eta',\vec{p}',\lambda'|\eta,\vec{p},\lambda\rangle=\delta_{\lambda'\lambda}(2\pi)^3 2\eta\,\delta(\eta'-\eta)\delta^2(\vec{p}'-\vec{p}). \tag{3.7}$$

Note that this gives the covariant momentum-space integral over the mass shell,

$$\begin{aligned}
\langle\Phi|\Psi\rangle&=(2\pi)^{-3}\int_0^\infty\frac{d\eta}{2\eta}\int d\vec{p}\;\sum_\lambda\langle\Phi|\eta,\vec{p},\lambda\rangle\langle\eta,\vec{p},\lambda|\Psi\rangle\\
&=(2\pi)^{-3}\int\frac{d\underline{P}}{2(\underline{P}^2+M^2)^{1/2}}\;\sum_\lambda\langle\Phi|P,\lambda\rangle\langle P,\lambda|\Psi\rangle.
\end{aligned}$$

Similar states $|\eta,\vec{p},\lambda\rangle$ can be easily defined for massless particles; indeed, the infinite-momentum frame is particularly well adapted for the discussion of the spin of massless particles. The interested reader will find a short discussion of the massless case in Appendix A. In addition, the states transform simply under certain Lorentz transformations which play the role of parity and time reversal. These transformation properties are discussed in Appendix B. The transformation properties of the states $|\eta,\vec{p},\lambda\rangle$ under the $\vec{S}$ operators defined in (2.1) are not particularly simple or illuminating (just as the transformation properties of Jacob and Wick helicity states under Lorentz boosts are not particularly simple).[11]

The states $|P,\lambda\rangle$ described here, while well defined for any momentum P^μ, are presumably most useful for the description of particles which are moving with high velocity in the $+z$ direction (as in Ref. 4). States adapted to the description of high-energy particles moving in the $-z$ direction can be obtained in a similar fashion simply by interchanging $\tau\leftrightarrow z$, $\eta\leftrightarrow H$, and $\vec{B}\leftrightarrow\vec{S}$.

C. Relation with Ordinary Helicity States

The infinite-momentum helicity states discussed here have arisen naturally in discussions of field theories in the infinite-momentum frame[3-7] for spins $\frac{1}{2}$ and 1. In Ref. 3 it was shown that, for spin $\frac{1}{2}$, the states $|\eta,\vec{p},\lambda\rangle$ are eigenstates of the ordinary helicity operator, but referred to a reference frame that is moving in the $-z$ direction with (almost) the speed of light. The statement can be established in the general case by noting that the helicity operator $\underline{J}\cdot\underline{P}/|\underline{P}|$, when given an infinite Lorentz boost in the $-z$ direction, becomes just the "Galilean spin" operator j:

$$\begin{aligned}
\lim_{\omega\to\infty}e^{i\omega K_3}\frac{\underline{J}\cdot\underline{P}}{|\underline{P}|}e^{-i\omega K_3}&=\lim_{\omega\to\infty}e^{i\omega K_3}\,\frac{2^{-1/2}J_3(\eta-H)+2^{-1/2}\sum_{k,l=1}^{2}(B^l-S^l)\epsilon_{lk}P^k}{[\frac{1}{2}(\eta-H)^2+\vec{P}^2]^{1/2}}\,e^{-i\omega K_3}\\
&=J+\frac{1}{\eta}\vec{B}\times\vec{P}=j.
\end{aligned} \tag{3.8}$$

IV. WAVE FUNCTIONS OF THE HELICITY STATES

In Ref. 3, it was found that the Dirac spinors representing infinite-momentum helicity states of a spin-$\frac{1}{2}$ particle have a simple form. In this section we will see that the corresponding spinors for higher-spin particles are also simple.

We will discuss here the spinors representing the states of a massive spin-S particle in the $(S,0)$ and $(0,S)$ representations of the Lorentz group – or, more properly, of the covering group SL(2, C). In order to be clear as to what these objects are, let us digress for a moment. Consider a one-particle state $|\psi\rangle$ and the amplitude

$$\psi_\beta(P)=\sum_\lambda \mathfrak{D}^{(S,0)}(\alpha(P))_{\beta\lambda}\langle P,\lambda|\psi\rangle. \tag{4.1}$$

Here $\mathfrak{D}^{(S,0)}$ is the $(S,0)$ representation[12] of SL(2, C) and $\alpha(P)\in \mathrm{SL}(2,C)$ is the standard transformation which relates $|M/\sqrt{2},\vec{0},\lambda\rangle$ to $|\eta,\vec{p},\lambda\rangle$ in the definition (3.5) of infinite-momentum helicity states:

$$\alpha(P)=e^{-i\vec{v}\cdot\vec{B}}e^{-i\omega K}, \tag{4.2}$$

where

$$\vec{v}=\vec{p}/\eta \quad\text{and}\quad e^{\omega}=\sqrt{2}\eta/M.$$

This amplitude can be regarded as a momentum-space wave function of the state $|\psi\rangle$. If the state undergoes a Lorentz transformation, $|\psi\rangle\to|\psi'\rangle = U(A)|\psi\rangle$, the wave function ψ transforms according to the $(S,0)$ representation of SL(2, C):

$$\psi'_\beta(P)=\mathfrak{D}^{(S,0)}(A)_{\beta\gamma}\psi_\gamma(\Lambda(A)^{-1}P).$$

The objects we want to discuss are obtained by setting $|\psi\rangle=|\eta,\vec{p},\lambda\rangle$ in (4.1) and factoring out the resulting δ functions and some factors of 2π and M. Thus we define the spinor $U_\beta(P,\lambda)$ representing the state $|\eta,\vec{p},\lambda\rangle$ and transforming according to the $(S,0)$ representation of SL(2, C):

$$U_\beta(P,\lambda)=M^S\mathfrak{D}^{(S,0)}(\alpha(P))_{\beta\lambda}. \tag{4.3}$$

We will also consider the corresponding $(0,S)$ spinor

$$\begin{aligned}U'_\beta(P,\lambda)&=M^S\mathfrak{D}^{(0,S)}(\alpha(P))_{\beta\lambda}\\&=M^S\mathfrak{D}^{(S,0)}(\alpha(P)^{\dagger-1})_{\beta\lambda}.\end{aligned} \tag{4.4}$$

These spinors are the objects which appear in the momentum-space expansion of the free fields which transform according to the $(S,0)$ and $(0,S)$ representations of SL(2, C). They can also appear in the Feynman rules for perturbation theory[13] and in expressions for S-matrix elements between infinite-momentum helicity states in terms of invariant amplitudes.[14] (For spin $\frac{1}{2}$, the two 2-component objects U_β and U'_β make up the usual 4-component Dirac spinor.)

Our task here is to evaluate the spinors $U_\beta(P,\lambda)$ and $U'_\beta(P,\lambda)$ explicitly. We begin by constructing the standard SL(2, C) transformation $\alpha(P)$ defined in Eq. (4.2). We recall that the generators of rotations in SL(2, C) are the Pauli spin matrices, $\underline{J}=\frac{1}{2}\underline{\tau}$, and the generators of Lorentz boosts are $\underline{K}=\frac{1}{2}i\underline{\tau}$. Using (2.1), we calculate

$$\alpha(P)=\left(\frac{M}{\sqrt{2}}\eta\right)^{-1/2}\begin{pmatrix}\eta & 0\\ p_+ & \dfrac{M}{\sqrt{2}}\end{pmatrix}, \tag{4.5}$$

where we define $p_\pm=2^{-1/2}(p^1\pm ip^2)$. We also need the expression for the matrix elements of $\mathfrak{D}^{(S,0)}(A)$ in terms of the matrix elements of A:

$$\begin{aligned}\mathfrak{D}^{(S,0)}(A)_{\lambda'\lambda}&=[(S+\lambda')!(S-\lambda')!(S+\lambda)!(S-\lambda)!]^{1/2}\\&\quad\times\sum_{a,b,c,d}{}'(a!b!c!d!)^{-1}\\&\quad\times(A_{++})^a(A_{+-})^b(A_{-+})^c(A_{--})^d,\end{aligned} \tag{4.6}$$

where the sum includes all those values of a,b,c,d in the range $0,1,\dots,2S$ which satisfy

$$\begin{aligned}a+b+c+d&=2S,\\a+b-c-d&=2\lambda',\\a-b+c-d&=2\lambda.\end{aligned} \tag{4.7}$$

This result is due to Wigner,[15] at least for the special case in which A is unitary; for the reader's convenience, a short proof is given in Appendix C.

Now we are ready to evaluate $\mathfrak{D}^{(S,0)}(\alpha(P))_{\lambda'\lambda}$, where $\alpha(P)$ is given by (4.5). Since the component $\alpha(P)_{+-}$ is zero, the only nonzero terms in (4.6) are those with $b=0$; but there is only one solution of (4.7) with $b=0$, namely, $a=S+\lambda'$, $b=0$, $c=\lambda-\lambda'$, and $d=S-\lambda$. Since the sum in (4.6) includes only non-negative values of the exponents a,b,c,d, this solution leads to a nonzero matrix element $\mathfrak{D}(\alpha)_{\lambda'\lambda}$ only if $c=\lambda-\lambda'\geq 0$. Thus we obtain for the spinor $U_\alpha(P,\lambda)=M^S\mathfrak{D}^{(S,0)}(\alpha(P))_{\alpha\lambda}$

$$\begin{aligned}U_\alpha(P,\lambda)&=\left(\frac{(S-\alpha)!(S+\lambda)!}{(S+\alpha)!(S-\lambda)!}\right)^{1/2}\frac{2^{\lambda/2}}{(\lambda-\alpha)!}\Theta(\alpha\leq\lambda)\\&\quad\times\eta^S\left(\frac{M}{\eta}\right)^{S-\lambda}\left(\frac{p_+}{\eta}\right)^{\lambda-\alpha},\end{aligned} \tag{4.8}$$

where $\Theta(\alpha\leq\lambda)$ is 1 for $\alpha\leq\lambda$, zero if $\alpha>\lambda$. We find in a similar fashion that the spinor $U'(P,\lambda) = M^S\mathfrak{D}^{(S,0)}(\alpha(P)^{\dagger-1})_{\alpha\lambda}$ is

$$\begin{aligned}U'_\alpha(P,\lambda)&=\left(\frac{(S+\alpha)!(S-\lambda)!}{(S-\alpha)!(S+\lambda)!}\right)^{1/2}\frac{2^{-\lambda/2}}{(\alpha-\lambda)!}\Theta(\lambda\leq\alpha)\\&\quad\times\eta^S\left(\frac{M}{\eta}\right)^{S+\lambda}\left(\frac{-p_-}{\eta}\right)^{\alpha-\lambda}.\end{aligned} \tag{4.9}$$

It is remarkable that these infinite-momentum helicity spinors are so simple. The spinors for Jacob and Wick helicity states,[1] by way of contrast, have a form

$$U_\alpha(P,\lambda)\sim e^{\omega\lambda}e^{-i\phi(\alpha-\lambda)}(\sin\theta)^S \times\sum_{N=-S}^{S}C(N)_{\alpha\lambda}(\tan\tfrac{1}{2}\theta)^N,$$

where (θ,ϕ) are the polar angles of $\underline{P}$ and $\cosh\omega = (P^2+M^2)^{1/2}/M$.

It is also interesting to note that these spinors have a simple limit as $M\to 0$ – the spinor $U(P,\lambda)$ vanishes unless $\lambda = S$ and the spinor $U'(P,\lambda)$ vanishes unless $\lambda = -S$.

ACKNOWLEDGMENTS

This work was part of my dissertation research done under the direction of J. D. Bjorken at Stanford University. My debt to him, and to J. B. Kogut, with whom we collaborated, is very great. In addition, I would like to thank J. B. Kogut, D. J. Gross, and M. Creutz for their helpful criticisms of the manuscript.

APPENDIX A: INFINITE-MOMENTUM HELICITY STATES FOR MASSLESS PARTICLES

The states of a massless particle may also be easily discussed. We start with the states $|\eta_0,\vec{0},\lambda\rangle$ of a particle with a fixed momentum $p_0^\mu=(\eta_0,0,0,0)$. We ask how these states can transform under the little group of p_0^μ – the group of Lorentz transformations that leaves p_0^μ fixed, which is the group generated by J and $\vec{S}$. Since $[J,S^k]=i\epsilon_{kl}S^l$ and $[S^k,S^l]=0$, we see that the only finite-dimensional irreducible representations of the little group are the one-dimensional representations consisting of a single vector $|\eta_0,\vec{0},\lambda\rangle$ with

$$J|\eta_0,\vec{0},\lambda\rangle=\lambda|\eta_0,\vec{0},\lambda\rangle,\qquad \vec{S}|\eta_0,\vec{0},\lambda\rangle=0. \tag{A1}$$

The states $|\eta,p,\lambda\rangle$ for other momenta are now defined by

$$|\eta,\vec{p},\lambda\rangle=e^{-i\vec{v}\cdot\vec{B}}e^{-i\omega K}|\eta_0,\vec{0},\lambda\rangle, \tag{A2}$$

where

$$\vec{v}=\vec{p}/\eta,\quad e^\omega=\eta/\eta_0.$$

As in the massive case, we find that these infinite-momentum helicity states are eigenstates of the Galilean spin operator j and transform simply under $\vec{B}$ and K:

$$j|\eta,\vec{p},\lambda\rangle=\lambda|\eta,\vec{p},\lambda\rangle,\qquad e^{-i\vec{v}\cdot\vec{B}}|\eta,\vec{p},\lambda\rangle=|\eta,\vec{p}+\eta\vec{v},\lambda\rangle,\qquad e^{-i\omega K}|\eta,\vec{p},\lambda\rangle=|e^\omega\eta,\vec{p},\lambda\rangle. \tag{A3}$$

APPENDIX B: TRANSFORMATION PROPERTIES UNDER PARITY AND TIME REVERSAL

It is sometimes necessary to know how the states $|\eta,\vec{p},\lambda\rangle$ transform under such Poincaré transformations as parity and time reversal. Consider first the parity operator.

The conventional parity operator U_P carries z into $-z$, and hence interchanges τ and $\mathfrak{z}$. Thus this operator may be useful for relating the infinite-momentum helicity states $|\eta,\vec{p},\lambda\rangle$ to "reversed" states $|h,\vec{p},\mu\rangle$ defined using an infinite-momentum frame with the roles of τ and $\mathfrak{z}$ interchanged. (If the states $|\eta,\vec{p},\lambda\rangle$ are most useful for describing systems moving with high velocities in the $+z$ direction, then the "reversed" states $|h,\vec{p},\mu\rangle$ are useful for describing systems moving with high velocities in the $-z$ direction.)

If, on the other hand, we are more interested in the internal dynamics of systems moving in the $+z$ direction, then the "mirror-reflection" operator

$$U_R\equiv e^{-i\pi J_x}U_P \tag{B1}$$

is more useful. This operator is exactly analogous to the mirror-reflection operator in two-dimensional nonrelativistic quantum mechanics. The operators η and H are scalars under reflection (assuming, of course, that parity is a symmetry of the theory). The rotation operator J is a pseudoscalar and changes sign under reflection. The operators $\vec{P}$ and $\vec{B}$ are "vectors" and transform according to $(V^1,V^2)\to(-V^1,V^2)$. In addition "pseudovectors" like $F^k=\epsilon_{kl}V^l$ sometimes occur; these transform according to $(F^1,F^2)\to(F^1,-F^2)$. This information is summarized in Table I.

It is quite easy to find how the states $|\eta,\vec{p},\lambda\rangle$ of a massive particle transform under U_R. Assuming that the states of a particle at rest in the lab frame transform under the conventional parity operator according to $U_P|M/\sqrt{2},\vec{0},\lambda\rangle=C_P|M/\sqrt{2},\vec{0},\lambda\rangle$, we find by using the definition (3.5) and Table I that

TABLE I. Behavior of the Poincaré generators under mirror-reflection and τ-reversal transformations.

O	$U_R^{-1}OU_R$	$U_\tau^{-1}OU_\tau$
η	η	η
H	H	H
$\vec{P}$	$(-P^1,P^2)$	$-\vec{P}$
$\vec{B}$	$(-B^1,B^2)$	$\vec{B}$
J	$-J$	$-J$
K	K	$-K$
$\vec{S}$	$(-S^1,S^2)$	$\vec{S}$

$$U_R|\eta,\vec{p},\lambda\rangle = C_R|\eta, -p^1, p^2, -\lambda\rangle . \qquad \text{(B2)}$$

Here C_R is a phase factor equal to $(-i)^{2S}C_P$. Thus, the effect of reflection on the particle states $|\eta,\vec{p},\lambda\rangle$ is very simple: The x component of the momentum is reversed and the spin is flipped.

We come now to time reversal. The conventional time-reversal operator U_T does not seem to be very useful in the infinite-momentum frame. A much more natural operator is

$$U_\tau \equiv e^{-i\pi J_z} U_P U_T , \qquad \text{(B3)}$$

which we might call the "τ-reversal" operator. Assuming that PT is a symmetry of the theory, simple computation shows that U_τ acts just like the time-reversal operator in nonrelativistic quantum mechanics. The mass η and energy H of a system are unchanged under τ reversal; the momentum $\vec{P}$ is reversed; and the boost operator $\vec{B}$ (and hence the position operator $\vec{R} = -\vec{B}/\eta$) is unchanged (cf. Table I).

How do the states $|\eta,\vec{p},\lambda\rangle$ transform under τ reversal? We begin with the transformation law for a particle at rest under the conventional PT operator:

$$U_P U_T|M/\sqrt{2},\vec{0},\lambda\rangle = C_{PT}(-1)^{S-\lambda}|M/\sqrt{2},\vec{0},-\lambda\rangle .$$

Then we find, using the definition (3.5) and Table I, that

$$U_\tau|\eta,\vec{p},\lambda\rangle = C_\tau|\eta,-\vec{p},-\lambda\rangle , \qquad \text{(B4)}$$

where $C_\tau = (i)^{2S}C_{PT}$ is a phase factor. This is just what we would expect for a nonrelativistic time-reversal operator – U_τ simply reverses the particle momentum and flips its spin.

APPENDIX C

We wish to derive an expression for the matrix elements of $\mathfrak{D}^{(S,0)}(A)$ in terms of the matrix elements of the matrix $A \in SL(2,C)$. To obtain such an expression we consider the reducible representation $\mathfrak{D}^{(1/2,0)}\times\mathfrak{D}^{(1/2,0)}\times\cdots\times\mathfrak{D}^{(1/2,0)}$ with $2S$ factors of $\mathfrak{D}^{(1/2,0)}$. This representation acts on the space of spinors $\xi_{\alpha_1\cdots\alpha_{2S}}$, where each index α takes the values $\pm\frac{1}{2}$, according to the rule $\xi\to\xi' = \mathfrak{D}(A)\xi$:

$$\xi'_{\alpha_1\cdots\alpha_{2S}} = \sum_{(\beta)} A_{\alpha_1\beta_1}\cdots A_{\alpha_{2S}\beta_{2S}}\xi_{\beta_1\cdots\beta_{2S}} . \qquad \text{(C1)}$$

It is not difficult to see that the space of totally symmetric spinors is left invariant under these transformations and that the representation of $SL(2,C)$ defined in the symmetric subspace by (C1) is $\mathfrak{D}^{(S,0)}$. A suitable[16] orthonormal basis for the symmetric subspace consists of the $2S+1$ vectors $\xi(\lambda)$, $\lambda = -S,\ldots,S$, defined by

$$\xi(\lambda)_{\alpha_1\cdots\alpha_{2S}} = [(2S)!(S+\lambda)!(S-\lambda)!]^{-1/2}\sum_{\text{permutations}}\delta_{\alpha_1,1/2}\cdots\delta_{\alpha_{S+\lambda},1/2}\delta_{\alpha_{S+\lambda+1},-1/2}\cdots\delta_{\alpha_{2S},-1/2} . \qquad \text{(C2)}$$

Here there are $S+\lambda$ factors of $\delta_{\alpha,1/2}$ and $S-\lambda$ factors of $\delta_{\alpha,-1/2}$. The desired matrix elements of $\mathfrak{D}^{(S,0)}(A)$ are simply

$$\mathfrak{D}^{(S,0)}(A)_{\lambda'\lambda} = \sum_{(\alpha)(\beta)}\xi(\lambda')^*_{\alpha_1\cdots\alpha_{2S}}A_{\alpha_1\beta_1}\cdots A_{\alpha_{2S}\beta_{2S}}\xi(\lambda)_{\beta_1\cdots\beta_{2S}} . \qquad \text{(C3)}$$

Thus the matrix elements of $\mathfrak{D}^{(S,0)}(A)_{\lambda'\lambda}$ are polynomials in the matrix elements $A_{++}, A_{+-}, A_{-+}, A_{--}$ of A.

It is not difficult to compute the coefficient of the general term $(A_{++})^a(A_{+-})^b(A_{-+})^c(A_{--})^d$ in this polynomial by a straightforward counting argument. Imagine inserting the expression (C2) for $\xi(x)^*$ and $\xi(\lambda)$ into (C3). We see that the coefficient of $(A_{++})^a(A_{+-})^b(A_{-+})^c(A_{--})^d$ is

$$[(2S)!(S+\lambda')!(S-\lambda')!]^{-1/2}[(2S)!(S+\lambda)!(S-\lambda)!]^{-1/2}$$

times the number N of permutations of the indices in $\xi(\lambda')^*$ and $\xi(\lambda)$ which contribute a term $(A_{++})^a(A_{+-})^b \times (A_{-+})^c(A_{--})^d$ to the sum. That is, N is $(2S)!$ times the number of ways $S+\lambda'$ white balls and $S-\lambda'$ black balls can be put into a white box with $S+\lambda$ slots and a black box with $S-\lambda$ slots so that a white balls are in the white box, b white balls are in the black box, c black balls are in the white box, and d black balls are in the black box. Clearly, N is zero unless

$$\begin{aligned} a+b+c+d &= 2S,\\ a+b-c-d &= 2\lambda',\\ a-b+c-d &= 2\lambda. \end{aligned} \qquad \text{(C4)}$$

If this condition is satisfied, then

$$N = (2S)!\frac{(S+\lambda')!(S-\lambda')!(S+\lambda)!(S-\lambda)!}{a!b!c!d!} .$$

Thus

$$\mathfrak{D}^{(S,0)}(A)_{\lambda'\lambda}=[(S+\lambda')!(S-\lambda')!(S+\lambda)!(S-\lambda)!]^{1/2}\sum_{a,b,c,d}{}'(a!b!c!d!)^{-1}(A_{++})^a(A_{+-})^b(A_{-+})^c(A_{--})^d, \quad \text{(C5)}$$

where the sum includes all those values of a, b, c, d in the range $0, 1, \ldots, 2S$ which satisfy (C4).

*Work supported by the National Science Foundation under Grant No. GP-30738X.

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[9]Note that $[R^k, P^l]=i\delta_{kl}$ and $\vec{\dot{R}}\equiv i[H,\vec{R}]=\vec{P}/\eta$. For many-particle states the operator $\vec{R}$ defined by (3.1) corresponds to the position of the center-of-"mass" system. This $\vec{R}$ is closely related to the impact parameter which emerges in discussions of high-energy scattering.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Mixing Angle in Renormalizable Theories of Weak and Electromagnetic Interactions*

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It is suggested that the weak and electromagnetic interactions be incorporated into a theory based on an SU(3)⊗SU(3) gauge-invariant and parity-conserving Lagrangian, in which the lepton fields form a Konopinski-Mahmoud triplet μ^+, ν, e^-. The unobserved effects which would be produced by 10 of the 12 charged vector bosons in this theory are suppressed if the spontaneous breaking of SU(3)⊗SU(3) down to SU(2)⊗U(1) is much stronger than the spontaneous breaking of SU(2)⊗U(1) down to electromagnetic gauge invariance. The resulting theory is for most purposes equivalent to the previous SU(2)⊗U(1) model, but with mixing angle now fixed at 30°. In consequence, the mass of the charged vector boson which mediates the known weak interactions is now predicted to be 74.6 GeV. This model also provides a natural mechanism for producing an electron mass of order αm_μ.

Several years ago it was suggested[1] that a renormalizable theory of the weak and electromagnetic interactions might be constructed from a gauge-invariant Lagrangian by allowing a spontaneous breakdown of the gauge symmetry. This proposal has now been revived by a number of theoretical studies,[2] which tend to confirm the renormalizability of models of this general class.

There are many possible presumably renormalizable models, based on different underlying symmetries and different patterns of symmetry breaking. However, particular attention has been given to a simple model[1] of the weak and electromagnetic interactions, based on a previously suggested[3] SU(2)⊗U(1) gauge group, under which the left-handed leptons transform as two independent doublets $(\nu_\mu, \mu^-)_L$ and $(\nu_e, e^-)_L$, while the right-handed leptons transform as two independent sing-