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PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Loop Graph in the Dual-Tube Model*

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The one-loop graph in the dual-tube model is constructed. The conditions for no divergences or new singularities are exactly those found by Lovelace for factorization of the "Pomeranchukon" in the strip model. Loops correspond to electrostatics on multiple tori only if spurious particles are permitted to circulate in the loops.

The usual dual-resonance model can be understood in terms of an electrostatic analog in which the ether is a two-dimensional strip.[1] Resonances correspond to long, thin strips and loop diagrams to annuli. External particles correspond to charges on the edges of the strip. Singularities in the scattering amplitude are associated either with an accumulation of charges corresponding to external particles, or to singularities in the shape of the ether surface.[2] For example, the one-loop diagram corresponds to an annulus of ether with either all charges on one boundary (planar loop) or some on each (nonplanar). When the hole shrinks to zero, one gets a divergence[3] (planar case) or the "Pomeranchukon" singularity[4] (nonplanar).

Almost a year ago I proposed a model[5] with a different kind of duality than that of the strip model. In the strip model only planar channels are dual to each other, while in the new model all channels are dual. The ether is a closed two-dimensional surface (a sphere for the tree diagrams) and resonances correspond to tubes instead of strips, with external particles entering as charges anywhere on the surface. I conjectured that higher-order diagrams would correspond to electrostatics on multiple tori (spheres with n handles). It was further conjectured that the difficulties in the strip model coming from shrinking the hole in the annulus might disappear in this model.

In this article I calculate the one-loop diagram in the tube model. We shall see that the electrostatic analog applies only if one permits spurious states to circulate in the loop, but that, whether or not they are permitted, there exist choices of dimensions and assumed Virasoro-type gauges for which there are no divergences or new singularities. For the case where these spurious states are projected out, the dimensionalities are exactly those Lovelace found for the factorization of the "Pomeranchukon" in the strip model.

Sewing of the Loop

The n-particle loop amplitude may be obtained either from the amplitude for an excited state $\lambda \to \lambda + n$ scalars, analogously to Ref. 3, or from n propagators and vertices, in analogy to Ref. 6. The factorized form had been found by Yoshimura[7] and by Del Giudice and Di Vecchia,[8] though we use a slightly modified form

$$V(q) = 2\pi \exp[q\textstyle\sum(2r)^{-1/2}(a_r + b_r)] \times \exp[q\textstyle\sum(2r)^{-1/2}(a_r^\dagger + b_r^\dagger)] \quad (1)$$

and

$$D(\Pi) = (2\pi)^{-1}\int d^2w\, |w|^{-4-\Pi^2/2} w^{R_a} w^{*R_b}. \quad (2)$$

Here a_r and b_r are two infinite sets of (four-vector) harmonic-oscillator lowering operators, $R_a = \sum r a_r^\dagger a_r$, and Π is the four-momentum of the resonance. We notice that the propagator automatically projects out states with $R_a \neq R_b$, which cannot couple to the scalars. We shall let D be the number of dimensions of each of the oscillators and E be the number of oscillator dimensions made spurious by sets of Virasoro identities[9] (again for *each* of the oscillators). We then find, in the usual way,[6]

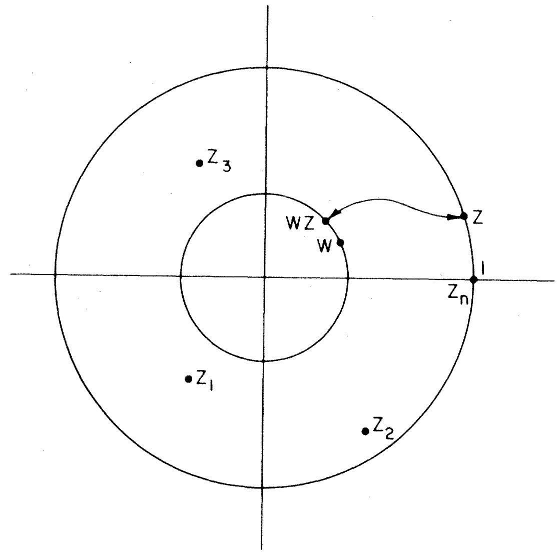

FIG. 1. The toroidal ether for the one-loop graph, pictured as an annulus with points z on the outside boundary identified with wz on the inside.

$$L = \int d^Dk\, \mathrm{Tr}\{V(q_1)D(k+q_1)\cdots D(k-q_n)V(q_n)D(k)\}$$

$$= \int d^Dk \int \prod_{i=1}^{n-1} d^2z_i \int d^2w |w|^{-4-k^2/2} \prod_{i=1}^{n} |z_i|^{-q_i\cdot k} \prod_{r=1}^{\infty} |1-w^r|^{-2(D-E)} \prod_{i<j} |z_j - z_i|^{-q_i\cdot q_j} \prod_{i,j}\prod_{r=1}^{\infty} |1 - w^r z_i/z_j|^{-q_i\cdot q_j}, \quad (3)$$

where $z_n = 1$. This form may be visualized in terms of solving electrostatics on an annulus as in Fig. 1, with the point z on the outside boundary identified with the point wz on the inside. Each of the terms in the last two products corresponds to the exponential of the interaction energy $-q_i q_j \ln|u_i - u_j|$, where u_i is the position of the ith charge or one of its images under $z \to w^n z$. This is equivalent to sewing a tube of length $-\ln|w|$ to make a torus, but first giving it a twist through an angle of $\arg w$. Thus it is not exactly electrostatics on a torus.

From this picture it is possible to see several symmetries of the integrand. The first is a periodicity in each $\ln z_i$ with period $\ln w$ coupled with $k \to k + q_i$, which just corresponds to taking the charge around the torus back to where it was. This is similar to the periodicity[10] in nonplanar loops in the strip model when all of the charges on a given boundary are multiplied by w. Another is under all $z_i \to 1/z_i$, $k \to -k$, $w \to w$. These may be shown explicitly. There is also an apparent symmetry of the tube under $w \to w^{-1}$, but I have been unable to show this explicitly. The integrand is, of course, miserably defined for $|w| > 1$.

Even if, as I shall assume, this w integration may be confined to $|w| < 1$, one may be horrified[11] by the exponential blowup of the last product at the infinite number of points $w = e^{2\pi i\theta}$, θ rational. To investigate this, let us evaluate the k integral by the usual blind application of a Wick rotation, to get[12]

$$L = \int d^2w\, |w|^{-4}(2\pi/-\ln|w|)^{D/2} \prod_{r=1}^{\infty} |1-w^r|^{-2(D-E)} \times 2^{-2m}\int \prod_{i=1}^{n-1} |z_i|^{-2} d^2z_i \exp[(\mathrm{Re}\textstyle\sum q_i \ln z_i)^2/2\ln|w|] \displaystyle\prod_{i<j} \left| \frac{\vartheta_1((\ln z_i - \ln z_j)/2i,\, w^{1/2})}{\vartheta_1'(0,\, w^{1/2})} \right|^{-q_i\cdot q_j} \quad (4)$$

The integrand is a single-valued function of w and may thus be considered a periodic function of $\tau = (2\pi i)^{-1}\ln w$. I define new variables

$$\tau' = -1/\tau, \quad w' = e^{2\pi i\tau'}, \quad \xi_i = \exp(\tau' \ln z_i) \quad (5)$$

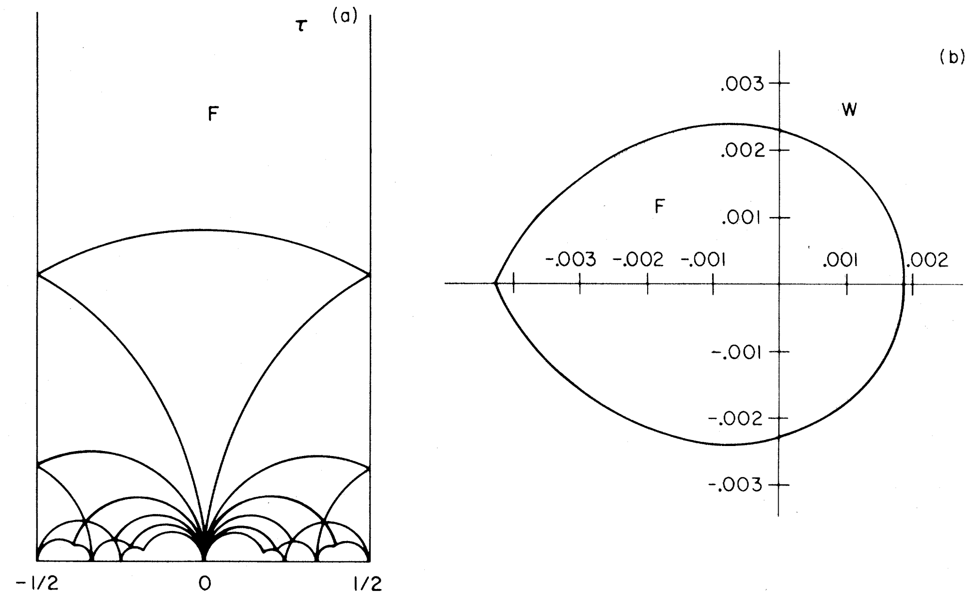

FIG. 2. (a) Fundamental regions of the modular group. The region marked F is the one I choose to integrate over. (b) The region F in the w plane $[w=\exp(2\pi i\tau)]$.

and use the Jacobi imaginary transformation to show that the form[13] of the integrand remains unchanged if

$$D=26, \quad E=2\,, \tag{6}$$

which are exactly the conditions required by Lovelace for factorization of the Pomeranchukon in the strip model. I assume these peculiar dimensions and note that after the z integrations, the integrand becomes invariant under the transformations $\tau\to\tau+1$ and $\tau\to-1/\tau$ which generate the modular group.[14] In attempting to integrate over $|w|<1$, i.e., $|\mathrm{Re}\tau|<\frac{1}{2}$, $\mathrm{Im}\tau>0$, we are integrating over an infinite number of copies of the fundamental region $|\mathrm{Re}\tau|<\frac{1}{2}$, $\mathrm{Im}\tau>0$, $|\tau|>1$. See Fig. 2. We handle this new periodicity exactly as the others, restricting ourselves to one fundamental region. The fundamental region has only the singularity at $w=0$, and all the singularities on the natural boundary $|w|=1$ are excluded, being simply images of $w=0$.

Thus for the peculiar case $D=26$, $E=2$, the loop diagram has no divergences[15] and no new singularities. The form for the loop amplitude is

$$\int_F d^2w\, 2^{-2m}|w|^{-4}(2\pi/-\ln|w|)^{13}\prod_{r=1}^{\infty}|1-w^r|^{-48}$$
$$\times\int_{|w|<|z|<1}\prod_{i=1}^{n-1}|z_i|^{-2}d^2z_i\exp[(\mathrm{Re}\textstyle\sum q_i\ln z_i)^2/2\ln|w|]\displaystyle\prod_{i<j}\left|\frac{\vartheta_1((\ln z_i-\ln z_j)/2i,\, w^{1/2})}{\vartheta_1'(0,\, w^{1/2})}\right|^{-q_i\cdot q_j}. \tag{7}$$

The tube model in a 26-dimensional space is clearly not to be confused with the world we live in. In addition to the absurd dimensionality it has no apparent place for isospin with suppressed exotics. Nonetheless, the elimination of some of the difficulties is encouraging, especially in light of the possibility that alterations in the Born term may change the details of dimensionality. To the extent that the Born term is the Pomeranchukon tree diagram of the usual model (it is similar but not identical) it gives an explicit verification that no new troubles arise when sewing Pomeranchukon loops.

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[10]M. Kaku and C. B. Thorn, Phys. Rev. D 1, 2869 (1970); Gross *et al.* (Ref. 4). Note that, in the strip model, all the particles on a given boundary must be simultaneously shifted.

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$$D=(2\pi)^{-1}\int_0^\infty d^2w\, w^{-3-p^2/2-R_a-R_b}$$

one finds that $R_a - R_b$ is conserved in $\lambda \to \lambda' + n$ scalars, but is not necessarily zero. Then all w's in Eq. (3) become $|w|$, the angular integration is trivial, and we wind up integrating $\int_0^1 dw$. Thus in this case we have electrostatics on the torus. One can use the Jacobi transformation to show that the $w=1$ singularity is just duplicating the $w=0$ one if $D=24$, $E=0$. Note that the line (w real) = (τ pure imaginary) only passes through two copies of the fundamental region.

[12]A. Neveu and J. Scherk, Phys. Rev. D 1, 2355 (1970). The Jacobi theta functions are described in E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge Univ. Press, New York, 1952).

[13]The range of integration of the ξ_i's apparently comes out wrong. However the integrand is periodic in $\ln\xi_i$ with periods $\ln w'$ and $2\pi i$. It can then be shown that the new integration area is a unit cell, as was the old.

[14]*Higher Transcendental Functions* (Bateman Manuscript Project), edited by A. Erdélyi (McGraw-Hill, New York, 1953), Vol. III.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Dual N-Point Functions in PGL($N-2,C$)-Invariant Formalism

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We derive a new expression for the dual N-point function integrand which is invariant under the action of the projective general linear group PGL($N-2$, C). The $(N-1)(N-3)$ free complex parameters of the group are used to make the integrand independent of the values of $(N-1)$ points of complex dimension $(N-3)$ which appear in the integrand. These points uniquely specify the location of all $\frac{1}{2}(N-1)(N-2)$ hyperplanes which appear as branch singularities of the integrand when it is viewed as a function on $(N-3)$-dimensional complex projective space. In contrast to the Koba-Nielsen formalism, the PGL($N-2$, C)-invariant form of the N-point integrand allows transformations which mix the $(N-3)$ integration variables and permits greater freedom in the placement of the branch singularities while preserving a simple hyperplane structure for the singularities.

I. INTRODUCTION

A large portion of the literature dealing with the dual N-point functions[1] has made use of the appealing Koba-Nielsen description[2] of the N-point function integrands. The purpose of this paper is to introduce a generalization of the Koba-Nielsen formalism in which the N-point integrands become invariant under the projective general linear group PGL($N-2, C$). When the dual N-point integrands are written in PGL($N-2, C$)-invariant form, we may move the branch singularities of the integrand wherever we please in $(N-3)$-dimensional complex projective space. We therefore view PGL($N-2, C$) as a natural singularity-