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Consistency of Spin-One Theory*

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The energy eigenvalues of a generalized equation for the motion of the spin-1 particle in a homogeneous magnetic field are solved by using a very simple method. From the explicit expression of the eigenvalues, it is found that the spin-1 theory is consistent when the anomalous magnetic moment $\kappa(H^2)$ is a nonpolynomial function of the magnetic field strength $|H|$ that obeys certain conditions. Hence, the usual way of writing the spin-1 equation, with constant anomalous magnetic moment, is inconsistent. A particular form for $\kappa(H^2)$ which gives consistency for spin-1 theory is discussed. Some physical implications are also presented.

I. INTRODUCTION

The question of the consistency of theories of charged particles moving in an external electromagnetic field has been with us for a long time.[1] The present status of the question is that, for charged particles with spin less than or equal to one, theories with minimal electromagnetic couplings are consistent, while for spin higher than one they are believed to be inconsistent.[1,2] Another related subject is the energy eigenvalue problem which started as early as when Dirac first proposed his spin-$\frac{1}{2}$ theory.[3] Only recently have the exact eigenvalues of the spin-$\frac{1}{2}$ and spin-1 equations with anomalous-magnetic-moment couplings been obtained by using the conventional method of solving the differential equation[4,5] and by a new method proposed by Tsai and Yildiz.[6] From the eigenvalues obtained, it is found that,[6] for the spin-1 case, the energy eigenvalues can become pure imaginary for sufficiently high magnetic field strength H. The occurrence of complex eigenvalues for spin-0 charged particles moving in an electrostatic potential has also been known for a long time.[7] The spin-1 case discussed in Ref. 6 furnished another example of complex energy eigenvalues for particles moving in a static field. Since the energy eigenvalue for a system is a physical quantity and should be real, the appearance of the complex energy eigenvalues implies that both the spin-0 and the spin-1 theories are inconsistent. The physical interpretation of these inconsistencies remains unknown.

The purposes of this paper are (1) to introduce an improvement of the method of Ref. 6 for calculating energy eigenvalues, and (2) to show that spin-1 theory becomes consistent when the anomalous magnetic moment $\kappa(H^2)$ is a nonpolynomial function of the magnetic field strength H that obeys certain conditions. In Sec. II, we present an improved version of the calculation method introduced in Ref. 6. By introducing a new identity, we show that instead of working on the quartic characteristic equation [Eq. (24) of Ref. 6], we need only consider a quadratic characteristic equation [Eq. (11) of this paper], which greatly simplifies the algebra of calculation. The eigenvalues of the spin-1 vector theory with some additional (nonminimal) gauge-invariant coupling terms are solved by this improved method. With the resulting eigenvalues, we show in Sec. III how the inconsistency arises in the eigenvalues of Ref. 6 and how to construct a consistent spin-1 theory. A particular case which gives a consistent theory is also discussed.

II. ENERGY EIGENVALUES

In this section, we will solve the energy eigenvalues for a more general spin-1 equation by using a method improved over that of Ref. 6.

The usual way of writing the spin-1 equation with anomalous-magnetic-moment coupling[8] is

$$(m^2+\pi^\nu\pi_\nu)\phi_\mu-\pi_\mu(\pi^\nu\phi_\nu)+ieq(1+\kappa)F_{\mu\nu}\phi^\nu=0\,, \quad (1)$$

where $\pi^\nu=(1/i)\partial^\nu-eqA^\nu$, A^μ is the electromagnetic potential, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and κ is the (constant) anomalous magnetic moment of the spin-1 particle.

However, there are other nonminimal gauge-invariant coupling terms of higher order in the electromagnetic field of the form

$$aF_{\mu\nu}F^{\nu\lambda}\phi_\lambda + bF_{\mu\nu}F^{\nu\lambda}F_{\lambda\sigma}\phi^\sigma + \cdots \tag{2}$$

which take into account the magnetic polarizability of the spin-1 particle and are expected to contribute in the strong magnetic field. Therefore, for the motion of a charged particle in a homogeneous magnetic field, a more general form of the equation can be written as

$$(m^2 + \pi^\nu\pi_\nu)\phi_\mu - \pi_\mu(\pi^\nu\phi_\nu) + ieq[1+\kappa(H^2)]F_{\mu\nu}\phi^\nu + \frac{e^2}{m^2}\chi(H^2)F_{\mu\nu}F^{\nu\lambda}\phi_\lambda = 0\,, \tag{3}$$

where we choose $\vec{H}$ to be in the z direction and have used the identities

$$\begin{aligned} &F_{ik}F_{kj} = H_iH_j - \delta_{ij}\vec{H}^2\,,\\ &H_iF_{ij} = 0\,,\\ &F_{0\mu}=0, \quad F_{12} = H\,. \end{aligned} \tag{4}$$

In Eq. (3), $\kappa(H^2)$ and $\chi(H^2)$ are possibly infinite polynomials that result from using Eq. (4) in the form (2), which has been added to the left-hand side of Eq. (1). We can unify the cases of infinite and finite polynomials by letting $\kappa(H^2)$ and $\chi(H^2)$ be as yet arbitrary functions, and by considering the infinite-polynomial case to be the result of a formal series expansion of these functions about the point $H^2 = 0$.

Using Eq. (3) instead of Eq. (1), we start the calculation paralleling Ref. 6. Equation (3) implies the subsidiary condition

$$m^2\pi^\mu\phi_\mu - \pi^\mu[1-\kappa(H^2)]ieqF_{\mu\nu}\phi^\nu + \frac{e^2}{m^2}\chi(H^2)\pi^\mu F_{\mu\nu}F^{\nu\lambda}\phi_\lambda = 0\,. \tag{5}$$

In the following, we omit the arguments of $\kappa(H^2)$ and $\chi(H^2)$ for notational convenience and consider the case when $\pi_3 = p_3 = 0$ for simplicity. Then, with the help of Eqs. (4) and (5), the spatial component of Eq. (3) becomes

$$(p^0)^2\phi_i = (m^2+\vec{\pi}^2)\phi_i - \frac{1}{m^2}\pi_i\left[ieq(1-\kappa)\pi_kF_{kj}\phi^j + \frac{e^2}{m^2}H^2\chi\pi_j\phi^j\right] + ieq(1+\kappa)F_{ik}\phi^k + \frac{e^2}{m^2}\chi F_{ik}F^{kj}\phi_j\,, \tag{6}$$

which can be rewritten in the matrix form as

$$\begin{aligned}(p^0)^2\phi = \Big(&m^2 + \vec{\pi}^2 - eq(1+\kappa)\vec{S}\cdot\vec{H} + eq\frac{(1-\kappa)}{m^2}(\vec{\pi}^2 - eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H}\\ &- \frac{eq}{m^2}(1-\kappa)(\vec{S}\cdot\vec{\pi})^2\vec{S}\cdot\vec{H} - \frac{e^2}{m^4}\chi\vec{S}\cdot\vec{H}(\vec{S}\cdot\vec{\pi})^2\vec{S}\cdot\vec{H} - \frac{e^2}{m^2}\chi\vec{S}\cdot\vec{H}\Big)\phi\,,\end{aligned} \tag{7}$$

where ϕ is a three-component column vector and S is the spin matrix for the spin-1 system in the representation $(S_j)_{ik} = i\epsilon_{ijk}$, so that we have

$$\pi_i\pi_j = [\vec{\pi}^2\delta_{ij} - eq(\vec{S}\cdot\vec{H})_{ij}] - [(\vec{S}\cdot\vec{\pi})^2]_{ij}\,, \quad F_{ij} = i(\vec{S}\cdot\vec{H})_{ij}\,,$$

$$S_iS_jS_k + S_kS_jS_i = \delta_{ij}S_k + \delta_{kj}S_i\,, \quad \{\vec{S}\cdot\vec{H},\ (\vec{S}\cdot\vec{\pi})^2\} = (\vec{\pi}^2 - eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H}\,,$$

which were used to arrive at Eq. (7).

With the help of the identities (recall $\pi_3 = 0$)

$$(\vec{S}\cdot\vec{\pi})^2(\vec{S}\cdot\vec{H}) = \tfrac{1}{2}(\vec{\pi}^2 - eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H} + \tfrac{1}{4}HN_-\,,$$

$$N_\pm = S_-^{\,2}\pi_+^{\,2} \pm S_+^{\,2}\pi_-^{\,2}\,, \quad S_\pm = S_1 \pm iS_2\,, \quad \pi_\pm = \pi_1 \pm i\pi_2\,,$$

Eq. (7) can be further simplified to the form

$$\begin{aligned}(p^0)^2\phi = \Big(&m^2 + \vec{\pi}^2 - eq(1+\kappa)\vec{S}\cdot\vec{H} + \frac{eq}{2m^2}(1-\kappa)(\vec{\pi}^2 - eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H}\\ &- \frac{e^2}{m^4}\chi[m^2 + \tfrac{1}{2}(\vec{\pi}^2 - eq\vec{S}\cdot\vec{H})](\vec{S}\cdot\vec{H})^2 - (\alpha + \beta S_3)N_-\Big)\phi\,,\end{aligned} \tag{8}$$

where

$$\alpha = \frac{eq(1-\kappa)H}{4m^2}\,, \quad \beta = \frac{e^2\chi}{4m^4}H^2\,,$$

and we have used the identity

$$N_\mp = -S_3N_\pm\,. \tag{9}$$

Equation (8) is the equation from which the energy eigenvalues are to be solved. Note that Eq. (18) of Ref. 6 can be obtained from Eq. (8) by letting $\chi=0$.

To obtain the energy eigenvalues, we proceed as follows. We first rewrite Eq. (8) in the form

$$y=-(\alpha+\beta S_3)N_-\phi\,, \tag{10}$$

with

$$y=(p^0)^2-\Big(m^2+\vec{\pi}^2-eq(1+\kappa)\vec{S}\cdot\vec{H}+\frac{eq}{2m^2}(1-\kappa)(\vec{\pi}^2-eq\vec{S}\cdot\vec{H})\vec{S}\cdot\vec{H}-\frac{e^2}{m^4}\chi[m^2+\tfrac{1}{2}(\vec{\pi}^2-eq\vec{S}\cdot\vec{H})](\vec{S}\cdot\vec{H})^2\Big).$$

Multiplying Eq. (8) from the left by $-(\alpha+\beta S_3)N_-$ and making use of Eqs. (9) and (10) and the relations

$$[N_-,y]=\lambda N_-\,,$$

$$\lambda=2(1-\kappa)eq\vec{S}\cdot\vec{H}\Big(1+\frac{1}{2m^2}(\vec{\pi}^2-2eq\vec{S}\cdot\vec{H})\Big)-\frac{e^2H^2}{m^4}\chi eq\vec{S}\cdot\vec{H}\,,$$

$$[S_3{}^2,N_\pm]=0\,,$$

$$[\vec{\pi}^2-2eq\vec{S}\cdot\vec{H},N_\pm]=0\,,$$

we obtain

$$y^2\phi+\lambda[-(\alpha+\beta S_3)N_-]\phi-(\alpha^2-\beta^2S_3{}^2)N_-{}^2\phi=0$$

or

$$[y^2+\lambda y+\gamma]=0\,, \tag{11}$$

where we have also used

$$\{S_3,N_\pm\}=0\,,$$

$$\gamma=-(\alpha^2-\beta^2S_3{}^2)N_-{}^2=4(\alpha^2-\beta^2S_3{}^2)[S_3{}^2(\vec{\pi}^2)^2+3(eq\vec{S}\cdot\vec{H})^2-4\vec{\pi}^2eq\vec{S}\cdot\vec{H}]\,.$$

It is the use of the identity, Eq. (9), that provides the improvement over the calculation in Ref. 6; there the equation analogous to our Eq. (11) [Eq. (21) in Ref. 6] has a term proportional to N_+ on the right-hand side, and the procedure that produces Eq. (11) from Eq. (10) is iterated to eliminate $N_\pm$, resulting in a quartic characteristic equation. In view of Eq. (9), this is unnecessary; the quadratic equation [Eq. (11)] contains only the commuting operators S_3 and $\vec{\pi}^2$ (see Ref. 9) and may be solved algebraically for the eigenvalues.

The eigenvalues can be obtained by solving Eq. (11) with the weak-field physical boundary condition

$$(p^0)^2\sim m^2+\vec{\pi}^2-eq(1+\kappa)S_3H+O(H^2)\quad\text{as}\quad H\to0\,,$$

which can be obtained directly from Eq. (8). Explicitly, the eigenvalues are, for $S_3=0$,

$$(p^0)^2=m^2[1+(2n+1)\xi]\,, \tag{12}$$

and for $S_3=\pm1$,

$$(p^0)^2=m^2\Big[1+\eta+\tfrac{1}{2}(1-\kappa)\xi^2-\chi\xi^2(1+\tfrac{1}{2}\eta)+\tfrac{1}{2}qS_3(1-\kappa)\Big(2+\eta-\frac{\chi\xi^2}{1-\kappa}\Big)\xi\times\Big(1-\frac{(\eta^2-\xi^2)[1-\chi^2\xi^2/(1-\kappa)^2]}{[2+\eta-\chi\xi^2/(1-\kappa)]^2}\Big)^{1/2}\Big]\,, \tag{13}$$

where

$$\xi=\frac{eH}{m^2}\,,\quad \eta=(2n+1-2qS_3)\,.$$

Note that $(p^0)^2$ is real, since

$$\eta^2\geq\xi^2\,,$$

$$\Big(2+\eta-\frac{\chi\xi^2}{1-\kappa}\Big)^2-(\eta^2-\xi^2)\Big(1-\frac{\chi^2\xi^2}{(1-\kappa)^2}\Big)=\Big(\frac{\xi}{\eta}(2+\eta)-\frac{\chi\eta\xi}{1-\kappa}\Big)^2+\frac{4}{\eta^2}(\eta^2-\xi^2)(1+\eta)\geq0\,.$$

III. CONSISTENT SPIN-ONE EQUATION

Equations (12) and (13) are the energy eigenvalues of the more general spin-1 equation, Eq. (3). The consistency of the theory requires that $(p^0)^2\geq0$. However, as we will show below, not any choice of $\kappa(\xi^2)$ and $\chi(\xi^2)$ satisfies this requirement. For example, the results of Ref. 6 may be reproduced by setting $\chi(\xi^2)=0$, and $\kappa(\xi^2)=\kappa=$ const in Eq. (13). In that case and for $n=0$, we have

$$(p^0)^2/m^2=1-\kappa\xi\quad\text{for}\quad qS_3=+1\,,$$

$$(p^0)^2/m^2\sim3\kappa\xi\quad\text{as}\quad\xi\to\infty\quad\text{for}\quad qS_3=-1\,,$$

which implies that the condition $(p^0)^2\geq0$ holds only when $\kappa=0$. A similar result clearly obtains if $\kappa(\xi^2)$ is a (finite) polynomial function of ξ^2. This is unlike the spin-$\frac{1}{2}$ case, where the theory is consistent for any value of the anomalous moment. We have encountered many attempts to explain this difference.[10]

It has been suggested that the inconsistency of the spin-1 theory with constant-anomalous-magnetic-moment couplings is related to the instability of the vacuum in such a theory, or to the constancy of the magnetic field over an infinite period of time, or to the omission of radiative corrections in the calculation. These "explanations" beg the question, for if they are accepted then we must also explain why the spin-$\frac{1}{2}$ calculation is immune to these problems. There is also the notion that perhaps spin-1 particles are necessarily composite in some fundamental sense, and decompose in a strong magnetic field. This last is not satisfactory either, but provides a clue – we might naively expect the particle to become distorted first and this raises the question of magnetic polarizability.

In the following, we propose a way to avoid this inconsistency and to construct a consistent spin-1 theory. One notes that the usual way of writing the spin-1 equation, Eq. (1), is valid only in the weak-magnetic-field case. For the general case, especially in a very strong magnetic field, we cannot neglect the magnetic polarizability of the particle.[11] As will be shown below, the inclusion of the latter is necessary and gives rise to a consistent spin-1 theory.

We ask ourselves: For the energy eigenvalues of Eq. (3), is there any choice of $\kappa(\xi^2) \neq 0$ and $\chi(\xi^2)$ such that $(p^0)^2$ is positive definite? The answer is yes, and we find it is sufficient to discuss the case when $\chi(\xi^2)=0$. In this case, the conditions for Eq. (13) to be positive definite are the following:

(i) For the ground state with $n=0$ and $qS_3=+1$, we have

$$(p^0)^2 = m^2(1-\kappa\xi),$$

which implies that

$$\kappa(\xi^2) \leq 1/\xi. \tag{14}$$

No finite polynomial (in H) of nonminimal gauge-invariant terms in Eq. (3) can satisfy Eq. (14).

(ii) For the other states, since $\eta \geq \xi \geq 0$, we have the conditions

$$1+\eta+\tfrac{1}{2}(1-\kappa)\xi^2 \geq 0,$$

$$1+\eta+\tfrac{1}{2}(1-\kappa)\xi^2 \geq \tfrac{1}{4}(1-\kappa)^2(4+4\eta+\xi^2),$$

which, in view of Eq. (14), are equivalent to

$$1+\eta \geq \kappa(\kappa-1)\xi^2. \tag{15}$$

Another condition for a particle with anomalous magnetic moment is

$$\kappa(\xi^2) \to \kappa' = \text{const} < \infty \quad \text{as} \quad \xi \to 0. \tag{16}$$

From the above discussion, we see that any choice of $\kappa(\xi^2)$ which satisfies conditions (14), (15), and (16) guarantees that $(p^0)^2 \geq 0$, and hence yields a consistent theory.[12] From Eqs. (14) and (15), it is easy to see that

$$|\kappa(\xi^2)| \leq 1/\xi. \tag{17}$$

This implies that if the nonminimal couplings are added as a power series in ξ, they must form an infinite series that sums to a nonpolynomial function that satisfies Eq. (17). The particular choice

$$\kappa(\xi^2) = \kappa'/(1+\tfrac{1}{4}\kappa'^2\xi^2), \tag{18}$$

where κ' is an arbitrary number, satisfies conditions (14) and (15). With this choice, the ground-state energy is found to be

$$(p^0)^2 = m^2(1-\tfrac{1}{2}\kappa'\xi)^2(1+\tfrac{1}{4}\kappa'^2\xi^2)^{-1},$$

which is similar to the spin-$\frac{1}{2}$ case.

In summary, a simplified method to calculate the energy eigenvalues for a general spin-1 equation has been presented. It has also been shown that the complex energy eigenvalues for the motion of spin-1 particles in a homogeneous magnetic field, discussed in Ref. 6, are artificial and are due to the exclusion of magnetic-polarizability effects; explicitly, they are due to the fact that the anomalous magnetic moment did not vanish in very strong magnetic fields in the case of constant κ. The conditions for a consistent spin-1 theory have been stated and a simple example satisfying those conditions has been presented. We hope that a similar argument can be used to avoid the inconsistency of the electrostatic potential-well case of Ref. 7 and the inconsistency of the spin-$\frac{3}{2}$ theory discovered by Johnson and Sudarshan ten years ago.[1] We also hope that our discussion here may shed some light on the phenomenological theory of strong interactions; i.e., in order to extend the known low-energy phenomena to high energy, we suspect from this that we cannot use polynomial interactions but should use nonpolynomial interactions.

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[9]We use the well-known result that for a particle moving in a homogeneous magnetic field, $\vec{\pi}^2=(2n+1)eH$ $(n=0,1,2,\ldots)$.

[10]We thank Professor S. Coleman for bringing some of these notions to our attention.

[11]This does not affect the spin-$\frac{1}{2}$ case, for there $\kappa\rightarrow\kappa(H^2)$ has no effect whatsoever on the calculation or conclusions. See, e.g., Ref. 6.

[12]One of us has shown that equations of the form of Eq. (13) with real and positive eigenvalues satisfy unitarity and hence give consistent theories. See Wu-yang Tsai, Harvard University thesis, 1971 (unpublished).

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Quantum-Mechanical Treatment of an Electron Undergoing Synchrotron Radiation*

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The problem of an electron moving perpendicular to an intense magnetic field is approached from the framework of quantum mechanics. A numerical solution to the related rate equations describing the probabilities of occupation of the electron's energy states is put forth along with the expected errors involved. The quantum-mechanical approach is found to predict a significant amount of energy broadening with time for an initially monoenergetic electron beam entering a region of an intense magnetic field as long as the product of initial energy and magnetic field is of order 50 MG BeV or larger.

We consider an electron with momentum $\vec{P}$ moving perpendicular to a constant homogeneous magnetic field of magnitude H. Where H_0 is the Dirac-electron–magnetic-field interaction Hamiltonian, the solution Ψ of the Dirac equation $[i\hbar(d/dt)-H_0]\Psi=0$ yields stationary-state wave functions with energy eigenvalues[1]

$$E_n=mc^2(1+2nH/H_0)^{1/2},\quad n=0,1,2,3,\ldots$$

where

$m\equiv$ rest mass of an electron,

$c\equiv$ speed of light in vacuum,

$H_0\equiv 4.414\times10^{13}$ G.

The charged particle's interaction with the local photon field provides the mechanism for spontaneous transitions from state n to lower or upper states. We shall be concerned with the downward transitions only because we expect the effect of upward transitions on the energy of the particle at time t to be small.

Regarding the electron-photon field interaction Hamiltonian[2]

$$U^*=\frac{e}{L^{3/2}}\sum_{\vec{\chi}}\left(\frac{2\pi c\hbar}{\chi}\right)^{1/2}(\vec{\alpha}\cdot\vec{a}^*)\,e^{i(c\chi t-\vec{\chi}\cdot\vec{r})}$$

as a small perturbation, where

$e\equiv$ charge of an electron in absolute value,

$L\equiv$ side of Born periodicity cube,

$\vec{\chi}\equiv$ photon wave vector,

$\hbar\equiv$ Planck's constant,

$\vec{\alpha}\equiv$ Dirac velocity matrix,

$\vec{a}^*\equiv$ photon creation operator,

$t\equiv$ elapsed time,

$\vec{r}\equiv$ position vector of electron,

we may find the transition rates $\lambda_{nn'}$ characterizing a transition from state n to state n' according to[3]

$$\lambda_{nn'}\cong\frac{d}{dt}\left|(-i\hbar^{-1})\int_0^t d\tau\left[e^{-iE_{n'}(t-\tau)/\hbar}\langle\Psi_{n'}|U^*(\tau)|\Psi_n\rangle\,e^{-iE_n\tau/\hbar}\right]\right|^2.$$

Here $\Psi_{n'}$ represents the system state vector after one-photon emission, and Ψ_n represents the system