PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Motion of Particles in Einstein's Relativistic Field Theory. V. Gravitation and the Lorentz-Covariant Procedure

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(Received 27 August 1971)

In several earlier papers the author developed an approximation method for finding the Lorentz-covariant equations of structure and motion of interacting particles (represented by singularities) in Einstein's theory of the nonsymmetric field. In these earlier papers two specific procedures were suggested for investigating a particular form of partial differential equation which appears when carrying out the approximation method. One of these two specific procedures was Lorentz-covariant at each stage of the analysis and one was not. In both cases the final result was Lorentz-covariant. In the earlier papers, when applying the approximation method to specific particles, only the noncovariant procedure was used. In this paper we illustrate the use of the Lorentz-covariant procedure by applying this procedure up to fourth order to the case of interacting simple neutral particles – particles which interact only through the gravitational field.

I. INTRODUCTION

In several earlier papers[1] we developed a general method for finding step by step to any order of approximation desired the Lorentz-covariant equations of structure and motion of particles represented by singularities in Einstein's theory of the nonsymmetric field. We called these particles ideal particles. In one of the earlier papers (paper II) we applied the above-mentioned method to the case of certain simple ideal particles located in a perfectly isolated region of the continuum[2] and found, in the second and fourth orders of approximation, respectively, an electromagnetic and a gravitational interaction between the particles.

In the first of the earlier papers (paper I) two specific procedures were suggested for investigating the solutions to a certain form of partial differential equation which always appears when carrying out the general method for finding equations of structure and motion. One of these two specific procedures was Lorentz-covariant at each stage of the analysis. The other procedure was not. In both cases the final result was Lorentz-covariant.

In paper II we made use of the noncovariant procedure to find the Lorentz-covariant equations of structure and motion of various types of interacting simple ideal particles. In this paper we shall make use of the Lorentz-covariant procedure to find the fourth-order Lorentz-covariant equations of structure and motion satisfied by interacting simple ideal particles which are neutral and spinless to second order. We are reinvestigating this problem – which was investigated in paper II using the noncovariant procedure – in order to illustrate the Lorentz-covariant procedure and in order to prepare ourselves for later investigations of more complex interactions.

A Comparison with the Havas-Goldberg Method. Before we proceed with the investigation the author would like to compare his method for finding the Lorentz-covariant equations of structure and motion of particles with the method of Havas and Goldberg.[3] Both are methods for finding the equations of structure and motion of particles in Einstein's gravitational theory[4] if the particles are represented by singularities in the fundamental tensor field (metric field) $g_{\mu\nu}$.

The author regards the essential difference between the two methods as arising from the fact that Havas and Goldberg make use of the Dirac δ function to represent certain singular properties of the fundamental tensor field – when particles are represented by singularities – while the author does not make use of such functions.[5] Through the use of δ-function notation involving the introduction of a singular matter tensor (energy-momentum tensor), Havas and Goldberg express integrability conditions on the field $g_{\mu\nu}$ at each order of approximation as conditions on the singular matter tensor and the field $g_{\mu\nu}$. These conditions determine the general form of the singular matter tensor and the equations of structure and motion of the particles (singularities) at each order of approximation.

Unfortunately, using the Havas-Goldberg method certain terms in the equations of structure and motion are not fully determined through the formal use of the method (certain terms are found to be formally infinite if the field $g_{\mu\nu}$ is infinite at the positions of singularities representing particles) and must be determined by some auxiliary procedure. The author finds the particular auxiliary procedures suggested by Havas and Goldberg for evaluating these terms – the Bhabha and Harish-Chandra procedure and the Riesz procedure – artificial and unsatisfactory.[6] Be that as it may, no one using the Havas-Goldberg method, as far as the author knows, has ever been able to obtain the complete equations of structure and motion of particles beyond the lowest order of approximation involving interaction (the fourth order in our notation), and the method has been in use for nine years.

In contrast to Havas and Goldberg the author satisfies the integrability conditions on the field $g_{\mu\nu}$ at each order of approximation by actually solving the field equations to that order in the immediate vicinity of each particle (singularity). This is sufficient to satisfy the integrability conditions as it has been shown by the author[7] that if at each order the field equations are satisfied in the immediate vicinity of each particle in the region under investigation, then the integrability conditions on the field $g_{\mu\nu}$ to that order are satisfied over the entire region.[8] In the process of solving the field equations to a certain order of approximation in the vicinity of each particle, and thus satisfying the integrability conditions on the field $g_{\mu\nu}$, one finds the equations of structure and motion satisfied by the particles (singularities) to that order of approximation. There are no calculational difficulties in solving the field equations in the vicinity of each particle but the method does involve much tedious labor in the higher orders of approximation. The equations of structure and motion of particles are, however, uniquely determined by the process and contain no ambiguous or infinite terms.[9]

In this paper we shall solve the fourth-order field equations in the vicinity of certain simple particles in order to satisfy the integrability conditions on the field $g_{\mu\nu}$ to that order, and in order to determine the equations of structure and motion of the particles to fourth order. All terms in the equations of structure and motion are easily found. No terms are ambiguous or infinite and no auxiliary procedures for evaluating the terms are introduced.

If instead we had used the Havas-Goldberg method to determine the equations of structure and motion of the particles to fourth order we would not have solved the field equations to the fourth order in the vicinity of each particle but only to the second order. We would be applying one of the special procedures suggested by Havas and Goldberg to determine the terms in the equations of structure and motion which are formally undetermined using their method of satisfying integrability conditions on the field $g_{\mu\nu}$. In their paper Havas and Goldberg actually find the equations of structure and motion of particles to the fourth order using the Bhabha and Harish-Chandra procedure although they suggest it would be more satisfactory to use the Riesz procedure. Using their method and the Bhabha and Harish-Chandra procedure Havas and Goldberg obtain the same equations of structure and motion to the fourth order (excepting radiation reaction terms[10]) as we obtain using the author's method.[11]

What then is the advantage of the author's method over that of Havas and Goldberg? First, the author does not have to introduce special procedures to evaluate formally undetermined quantities – procedures which the author finds arbitrary and unjustified.[12] Second, the author's method, in contrast to the Havas-Goldberg method (and other Lorentz-covariant methods of which the author is aware), presents no special computational difficulties (other than tedious labor) in going to the higher orders of approximation.[13]

II. THE FIELD ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$

First we must reinvestigate, this time using the Lorentz-covariant procedure, the solutions[14] ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ to the Eqs. (1.63) of paper II, that is, to the equations[15,16]

$$\Box^2\,{}_{[4]}\gamma^{*I}_{(\mu\nu)} = {}_{[4]}t_{\mu\nu} + O(\kappa^6)\,, \tag{1}$$

where

$$\begin{aligned}{}_{[4]}t_{\mu\nu} = {} & \tfrac{1}{2}\,{}_{[2]}\gamma_{\rho\sigma,\mu}\,{}_{[2]}\gamma^{\rho\sigma}{}_{,\nu} - \tfrac{1}{4}\,{}_{[2]}\gamma_{,\mu}\,{}_{[2]}\gamma_{,\nu} + {}_{[2]}\gamma_{\mu\rho,\sigma}\,{}_{[2]}\gamma_{\nu}{}^{\rho,\sigma} + {}_{[2]}\gamma_{\mu\rho}{}^{,\sigma}\,{}_{[2]}\gamma_{\nu\sigma}{}^{,\rho} - {}_{[2]}\gamma_{\mu\rho,\sigma}\,{}_{[2]}\gamma^{\rho\sigma}{}_{,\nu} - {}_{[2]}\gamma_{\nu\rho,\sigma}\,{}_{[2]}\gamma^{\rho\sigma}{}_{,\mu} \\ & - \tfrac{1}{4}\eta_{\mu\nu}\,{}_{[2]}\gamma_{\rho\sigma,\lambda}\,{}_{[2]}\gamma^{\rho\sigma,\lambda} + \tfrac{1}{2}\eta_{\mu\nu}\,{}_{[2]}\gamma_{\rho\sigma,\lambda}\,{}_{[2]}\gamma^{\rho\lambda,\sigma} + \tfrac{1}{8}\eta_{\mu\nu}\,{}_{[2]}\gamma_{,\rho}\,{}_{[2]}\gamma^{,\rho} - {}_{[2]}\gamma_{\rho\sigma}\,{}_{[2]}\gamma_{\mu\nu}{}^{,\rho\sigma}\,,\end{aligned} \tag{2}$$

with

$$\begin{aligned}{}_{[2]}\gamma_{\mu\nu} &= \sum_{p}{}^{(p)}_{[2]}\gamma_{\mu\nu}\,, \\ {}^{(p)}_{[2]}\gamma_{\mu\nu} &= {}^{(p)}a_{\mathrm{ret}}\,{}^{(p)}[(m^G/c^2)u_\mu u_\nu(ru)^{-1}]_{\mathrm{ret}} + {}^{(p)}a_{\mathrm{adv}}\,{}^{(p)}[(m^G/c^2)u_\mu u_\nu(ru)^{-1}]_{\mathrm{adv}}\,,\end{aligned} \tag{3}$$

and

$${}^{(p)}m^{G}=O(\kappa^{2}),\quad {}^{(p)}\dot{m}^{G}=O(\kappa^{4}), \tag{4}$$

$${}^{(p)}u^{\rho}=O(\kappa^{0}),\quad {}^{(p)}\dot{u}^{\rho}=O(\kappa^{2}). \tag{5}$$

We shall only be interested in those solutions to (1) for which ${}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu}$ takes the form[17]

$${}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu}=\sum_{p}{}^{(p)}a_{\rm ret}\,{}^{(p)}[{}_{[4]}C^{I}_{\mu}(ru)^{-1}]_{\rm ret}+\sum_{p}{}^{(p)}a_{\rm adv}\,{}^{(p)}[{}_{[4]}C^{I}_{\mu}(ru)^{-1}]_{\rm adv}$$
$$+\sum_{p}{}^{(p)}a_{\rm ret}\,{}^{(p)}[{}_{[4]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\rm ret}{}^{,\nu}+\sum_{p}{}^{(p)}a_{\rm adv}\,{}^{(p)}[{}_{[4]}C^{I}_{[\mu\nu]}(ru)^{-1}]_{\rm adv}{}^{,\nu}+O(\kappa^{6}), \tag{6}$$

where ${}^{(p)}_{[4]}C^{I}_{\mu}$ and ${}^{(p)}_{[4]}C^{I}_{[\mu\nu]}$ are understood to be functions of ${}^{(p)}\tau$ only, and

$${}^{(p)}_{[4]}C^{I}_{[\mu\nu]}\,{}^{(p)}u^{\nu}=0. \tag{7}$$

We shall assume that solutions to (1) satisfying (6) always exist.[18]

The field ${}_{[2]}\gamma_{\mu\nu}$ appearing in Eqs. (1) can be broken up in the following manner:

$${}_{[2]}\gamma_{\mu\nu}={}^{(p)}_{[2]}\gamma^{\rm self}_{\mu\nu}+{}^{(p)}_{[2]}\gamma^{\rm ext}_{\mu\nu}, \tag{8}$$

where

$${}^{(p)}_{[2]}\gamma^{\rm self}_{\mu\nu}={}^{(p)}_{[2]}\gamma_{\mu\nu},\quad {}^{(p)}_{[2]}\gamma^{\rm ext}_{\mu\nu}=\sum_{p'\neq p}{}^{(p')}_{[2]}\gamma_{\mu\nu}. \tag{9}$$

The field ${}^{(p)}_{[2]}\gamma^{\rm self}_{\mu\nu}$ in (9) can itself be split into two parts,

$${}^{(p)}_{[2]}\gamma^{\rm self}_{\mu\nu}={}^{(p)}_{[2]}\gamma^{{\rm self}\,s}_{\mu\nu}+{}^{(p)}_{[2]}\gamma^{{\rm self}\,n}_{\mu\nu}, \tag{10}$$

where

$${}^{(p)}_{[2]}\gamma^{{\rm self}\,s}_{\mu\nu}=\tfrac{1}{2}({}^{(p)}_{[2]}\gamma_{\mu\nu\,{\rm adv}}+{}^{(p)}_{[2]}\gamma_{\mu\nu\,{\rm ret}}), \tag{11}$$

$${}^{(p)}_{[2]}\gamma^{{\rm self}\,n}_{\mu\nu}=\tfrac{1}{2}{}^{(p)}a({}^{(p)}_{[2]}\gamma_{\mu\nu\,{\rm adv}}-{}^{(p)}_{[2]}\gamma_{\mu\nu\,{\rm ret}}), \tag{12}$$

and

$${}^{(p)}a={}^{(p)}a_{\rm adv}-{}^{(p)}a_{\rm ret}. \tag{13}$$

We therefore write

$${}_{[2]}\gamma_{\mu\nu}={}^{(p)}_{[2]}\gamma^{s}_{\mu\nu}+{}^{(p)}_{[2]}\gamma^{n}_{\mu\nu}, \tag{14}$$

where

$${}^{(p)}_{[2]}\gamma^{s}_{\mu\nu}={}^{(p)}_{[2]}\gamma^{{\rm self}\,s}_{\mu\nu}, \tag{15}$$

$${}^{(p)}_{[2]}\gamma^{n}_{\mu\nu}={}^{(p)}_{[2]}\gamma^{{\rm self}\,n}_{\mu\nu}+{}^{(p)}_{[2]}\gamma^{\rm ext}_{\mu\nu}. \tag{16}$$

The field ${}^{(p)}_{[2]}\gamma^{s}_{\mu\nu}$ is singular along the world line of the pth particle while the field ${}^{(p)}_{[2]}\gamma^{n}_{\mu\nu}$ is nonsingular along this world line.

We shall assume that over the region of the continuum we are investigating Eqs. (1) have a solution ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ which satisfies (6) and can be expanded in a Lorentz-covariant power series (Laurent series) around the world line ${}^{(p)}\xi^{\rho}$ of an arbitrary particle in the region, that is in a power series in ${}^{(p)}\epsilon$, where

$${}^{(p)}\epsilon={}^{(p)}(-r_{\rho}r^{\rho})^{1/2},\quad {}^{(p)}r^{\rho}=x^{\rho}-{}^{(p)}\xi^{\rho}. \tag{17}$$

The power series expansion in ${}^{(p)}\epsilon$ need only exist in the immediate vicinity of the world line of the pth particle.[19] The vector ${}^{(p)}r^{\rho}$ is understood to be a spacelike vector.

As discussed in paper I the world line of the pth particle is parametrized by the quantity ${}^{(p)}\tau$ defined through the equation

$$d\,{}^{(p)}\tau^{2}=\eta_{\mu\nu}\,d\,{}^{(p)}\xi^{\mu}\,d\,{}^{(p)}\xi^{\nu}. \tag{18}$$

We can extend the domain of definition of the quantity ${}^{(p)}\tau$ to include the neighborhood of the world line of the pth particle through the use of the equation

$${}^{(p)}r_{\rho}\,{}^{(p)}u^{\rho}=0, \tag{19}$$

which is understood as defining surfaces of constant ${}^{(p)}\tau$ in the neighborhood of the pth particle. The quantity ${}^{(p)}\tau$ will then be known as the proper time in the neighborhood of the pth particle. We see from the above definition that surfaces of constant ${}^{(p)}\tau$ in the neighborhood of the pth particle are planes perpendicular to the world line of the pth particle.

If in the neighborhood of the pth particle we differentiate (19) with respect to x^{σ} we find

$${}^{(p)}r_{\rho,\sigma}\,{}^{(p)}u^{\rho}+{}^{(p)}r_{\rho}\,{}^{(p)}u^{\rho}{}_{,\sigma}=0. \tag{20}$$

Making use of

$${}^{(p)}r_{\rho,\sigma}=\eta_{\rho\sigma}-{}^{(p)}u_{\rho}\,{}^{(p)}\tau_{,\sigma},\quad {}^{(p)}u^{\rho}{}_{,\sigma}={}^{(p)}\dot{u}^{\rho}\,{}^{(p)}\tau_{,\sigma}, \tag{21}$$

this means that

$${}^{(p)}\tau_{,\rho}={}^{(p)}u_{\rho}[1-{}^{(p)}\epsilon\,{}^{(p)}(\alpha\dot{u})]^{-1}, \tag{22}$$

where

$${}^{(p)}(\alpha\dot{u})={}^{(p)}\alpha_{\rho}\,{}^{(p)}\dot{u}^{\rho},\quad {}^{(p)}\alpha_{\rho}=\frac{{}^{(p)}r_{\rho}}{{}^{(p)}\epsilon}. \tag{23}$$

All quantities in Eqs. (20)–(23) are understood as evaluated at proper time ${}^{(p)}\tau$ in the neighborhood of the pth particle.

From the above we see that the parameter ${}^{(p)}\tau$ will be a well-defined single-valued function of position in the neighborhood of the world line of the pth particle if that neighborhood is restricted to those points of proper time ${}^{(p)}\tau$ for which

$$ {}^{(p)}\epsilon < {}^{(p)}(\alpha \dot{u})^{-1} . \tag{24} $$

When expanding the various quantities appearing in this paper in a power series in ${}^{(p)}\epsilon$ we shall always consider ourselves working in such a restricted region surrounding the pth particle. The planes of constant ${}^{(p)}\tau$ in such a restricted region will be denoted by ${}^{(p)}\sigma$. From now on whenever the quantity ${}^{(p)}\epsilon$ appears in this paper it will be understood as evaluated on the surface ${}^{(p)}\sigma$, and all expansions in ${}^{(p)}\epsilon$ in the neighborhood of the pth particle will be understood as being over this same surface.

In order to investigate the solutions to Eqs. (1) satisfying the conditions (6) we shall expand ${}_{[4]}t_{\mu\nu}$ in a power series in ${}^{(p)}\epsilon$ in the neighborhood of the pth particle. Making use of (14) and the fact that when expanded in such a power series in ${}^{(p)}\epsilon$

$$ {}^{(p)}_{[2]}\gamma^{s}_{\mu\nu} = {}^{(p)}[(m^G/c^2)u_\mu u_\nu \epsilon^{-1}] + O(\kappa^4), \tag{25} $$

$$ {}^{(p)}_{[2]}\gamma^{s}_{\mu\nu,\rho} = {}^{(p)}[(m^G/c^2)\alpha_\rho u_\mu u_\nu \epsilon^{-2}] + O(\kappa^4), \tag{26} $$

$$ {}^{(p)}_{[2]}\gamma^{s}_{\mu\nu,\rho\sigma} = {}^{(p)}[(m^G/c^2)(3\alpha_\rho \alpha_\sigma u_\mu u_\nu - u_\rho u_\sigma u_\mu u_\nu + \eta_{\rho\sigma} u_\mu u_\nu)\epsilon^{-3}] + O(\kappa^4), \tag{27} $$

we find as the power-series expansion in ${}^{(p)}\epsilon$ of ${}_{[4]}t_{\mu\nu}$

$$ \begin{aligned} {}_{[4]}t_{\mu\nu} = {}^{(p)}\Big[&(m^G/c^2)^2(\tfrac{1}{4}\alpha_\mu \alpha_\nu - u_\mu u_\nu + \tfrac{1}{8}\eta_{\mu\nu})\epsilon^{-4} - {}_{[2]}\gamma^{s}_{\mu\nu,\rho\sigma}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma} - r_\lambda\,{}_{[2]}\gamma^{s}_{\mu\nu,\rho\sigma}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda} \\ &+ \tfrac{1}{2}\,{}_{[2]}\gamma^{s}_{\rho\sigma,\mu}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}{}_{,\nu} + \tfrac{1}{2}\,{}_{[2]}\gamma^{s}_{\rho\sigma,\nu}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}{}_{,\mu} + {}_{[2]}\gamma^{s}_{\mu\rho,\sigma}\,{}_{[2]}\tilde{\gamma}_{\nu}{}^{\rho,\sigma} + {}_{[2]}\gamma^{s}_{\nu\rho,\sigma}\,{}_{[2]}\tilde{\gamma}_{\mu}{}^{\rho,\sigma} + {}_{[2]}\gamma^{s}_{\mu\rho,\sigma}\,{}_{[2]}\tilde{\gamma}_{\nu}{}^{\sigma,\rho} \\ &+ {}_{[2]}\gamma^{s}_{\nu\rho,\sigma}\,{}_{[2]}\tilde{\gamma}_{\mu}{}^{\sigma,\rho} - {}_{[2]}\gamma^{s}_{\mu\rho,\sigma}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}{}_{,\nu} - {}_{[2]}\gamma^{s}_{\nu\rho,\sigma}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}{}_{,\mu} - {}_{[2]}\gamma^{s}_{\rho\sigma,\nu}\,{}_{[2]}\tilde{\gamma}_{\mu}{}^{\rho,\sigma} - {}_{[2]}\gamma^{s}_{\rho\sigma,\mu}\,{}_{[2]}\tilde{\gamma}_{\nu}{}^{\rho,\sigma} - \tfrac{1}{4}\,{}_{[2]}\gamma^{s}{}_{,\mu}\,{}_{[2]}\tilde{\gamma}_{,\nu} \\ &- \tfrac{1}{4}\,{}_{[2]}\gamma^{s}{}_{,\nu}\,{}_{[2]}\tilde{\gamma}_{,\mu} - \tfrac{1}{2}\eta_{\mu\nu}\,{}_{[2]}\gamma^{s}_{\rho\sigma,\lambda}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda} + \eta_{\mu\nu}\,{}_{[2]}\gamma_{\rho\sigma,\lambda}\,{}_{[2]}\tilde{\gamma}^{\rho\lambda,\sigma} + \tfrac{1}{4}\eta_{\mu\nu}\,{}_{[2]}\gamma^{s}{}_{,\lambda}\,{}_{[2]}\tilde{\gamma}^{,\lambda} + O(\epsilon^{-1})\Big] + O(\kappa^6) . \end{aligned} \tag{28} $$

We are using the notation ${}^{(p)}_{[2]}\tilde{\gamma}_{\mu\nu}$ and ${}^{(p)}_{[2]}\tilde{\gamma}_{\mu\nu,\rho}$ in (28) to denote the fields ${}^{(p)}_{[2]}\gamma^{n}_{\mu\nu}$ and ${}^{(p)}_{[2]}\gamma^{n}_{\mu\nu,\rho}$ evaluated along the world line of the pth particle.

Since the equations

$$ \Box^2\,{}^{(p)}_{[4]}\varphi^{1}_{\mu\nu} = {}^{(p)}[(m^G/c^2)^2(\alpha_\mu \alpha_\nu + \tfrac{1}{2}\eta_{\mu\nu})\epsilon^{-4}], \tag{29} $$

$$ \Box^2\,{}^{(p)}_{[4]}\varphi^{2}_{\mu\nu} = {}^{(p)}[(m^G/c^2)^2(u_\mu u_\nu)\epsilon^{-4}], \tag{30} $$

$$ \Box^2\,{}^{(p)}_{[4]}\varphi^{3}_{\mu\nu} = {}^{(p)}[{}_{[2]}\gamma^{s}_{\mu\nu,\rho\sigma}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}], \tag{31} $$

$$ \Box^2\,{}^{(p)}_{[4]}\varphi^{4}_{\mu\nu} = {}^{(p)}[r_\lambda\,{}_{[2]}\gamma^{s}_{\mu\nu,\rho\sigma}\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}], \tag{32} $$

$$ \Box^2\,{}^{(p)}_{[4]}\varphi_{\mu\nu\kappa\lambda\rho\sigma} = {}^{(p)}[{}_{[2]}\gamma^{s}_{\mu\nu,\rho}\,{}_{[2]}\tilde{\gamma}_{\kappa\lambda,\sigma}] \tag{33} $$

have the solutions[20]

$$ {}^{(p)}_{[4]}\varphi^{1}_{\mu\nu} = {}^{(p)}[(m^G/c^2)^2(\tfrac{1}{4}\alpha_\mu \alpha_\nu - \tfrac{1}{4}u_\mu u_\nu)\epsilon^{-2}] + O(\kappa^6), \tag{34} $$

$$ {}^{(p)}_{[4]}\varphi^{2}_{\mu\nu} = {}^{(p)}[(m^G/c^2)^2(-\tfrac{1}{2}u_\mu u_\nu)\epsilon^{-2}] + O(\kappa^6), \tag{35} $$

$$ {}^{(p)}_{[4]}\varphi^{3}_{\mu\nu} = {}^{(p)}[(m^G/c^2)(\tfrac{1}{2}\alpha_\rho \alpha_\sigma\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}u_\mu u_\nu)\epsilon^{-1} + O(\epsilon)] + O(\kappa^6), \tag{36} $$

$$ \begin{aligned} {}^{(p)}_{[4]}\varphi^{4}_{\mu\nu} = {}^{(p)}\Big[&(m^G/c^2)(\tfrac{1}{4}\alpha_\rho \alpha_\sigma \alpha_\lambda\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\mu u_\nu + \tfrac{1}{2}\alpha_\rho\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\sigma u_\lambda u_\mu u_\nu \\ &- \tfrac{1}{4}\alpha_\lambda\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\rho u_\sigma u_\mu u_\nu + \tfrac{1}{4}\alpha_\rho\,{}_{[2]}\tilde{\gamma}^{,\rho}u_\mu u_\nu) + O(\epsilon)\Big] + O(\kappa^6), \end{aligned} \tag{37} $$

$$ {}^{(p)}_{[4]}\varphi_{\mu\nu\kappa\lambda\rho\sigma} = {}^{(p)}[(m^G/c^2)(\tfrac{1}{2}\alpha_\rho\,{}_{[2]}\tilde{\gamma}_{\kappa\lambda,\sigma}u_\mu u_\nu) + O(\epsilon)] + O(\kappa^6), \tag{38} $$

we see from (28) and (29)–(38) that every solution ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ to (1), expandable in a power series in ${}^{(p)}\epsilon$ in the neighborhood of the pth particle, can be written in the form

$$
{}_{[4]}\gamma^{*I}_{(\mu\nu)} = {}^{(p)}\Big[(m^G/c^2)^2(\tfrac{1}{16}\alpha_\mu\alpha_\nu + \tfrac{7}{16}u_\mu u_\nu)\epsilon^{-2} + (m^G/c^2)(-\tfrac{1}{2}\alpha_\rho\alpha_\sigma\,{}_{[2]}\tilde{\gamma}^{\rho\sigma}u_\mu u_\nu)\epsilon^{-1}
$$

$$
+(m^G/c^2)(-\tfrac{1}{4}\alpha_\rho\alpha_\sigma\alpha_\lambda\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\mu u_\nu + \tfrac{1}{4}\alpha_\lambda\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\rho u_\sigma u_\mu u_\nu
$$

$$
-\tfrac{1}{2}\alpha_\rho\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\sigma u_\lambda u_\mu u_\nu - \tfrac{1}{2}\alpha_\mu\,{}_{[2]}\tilde{\gamma}_{\rho\nu,\sigma}u^\rho u^\sigma - \tfrac{1}{2}\alpha_\nu\,{}_{[2]}\tilde{\gamma}_{\rho\mu,\sigma}u^\rho u^\sigma
$$

$$
+\tfrac{1}{2}\eta_{\mu\nu}\alpha_\rho\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\sigma u_\lambda + \tfrac{1}{4}\alpha_\mu\,{}_{[2]}\tilde{\gamma}_{\rho\sigma,\nu}u^\rho u^\sigma + \tfrac{1}{4}\alpha_\nu\,{}_{[2]}\tilde{\gamma}_{\rho\sigma,\mu}u^\rho u^\sigma
$$

$$
-\tfrac{1}{4}\eta_{\mu\nu}\alpha_\lambda\,{}_{[2]}\tilde{\gamma}^{\rho\sigma,\lambda}u_\rho u_\sigma + \tfrac{1}{2}\alpha_\sigma\,{}_{[2]}\tilde{\gamma}_{\rho\mu}{}^{,\sigma}u^\rho u_\nu + \tfrac{1}{2}\alpha_\sigma\,{}_{[2]}\tilde{\gamma}_{\rho\nu}{}^{,\sigma}u^\rho u_\mu
$$

$$
+\tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde{\gamma}_{\rho\mu,\sigma}u^\sigma u_\nu + \tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde{\gamma}_{\rho\nu,\sigma}u^\sigma u_\mu - \tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde{\gamma}_{\rho\sigma,\mu}u^\sigma u_\nu
$$

$$
-\tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde{\gamma}_{\rho\sigma,\nu}u^\sigma u_\mu - \tfrac{1}{4}\alpha^\rho\,{}_{[2]}\tilde{\gamma}_{,\rho}u_\mu u_\nu - \tfrac{1}{8}\alpha_\mu\,{}_{[2]}\tilde{\gamma}_{,\nu} - \tfrac{1}{8}\alpha_\nu\,{}_{[2]}\tilde{\gamma}_{,\mu}
$$

$$
+\tfrac{1}{8}\eta_{\mu\nu}\alpha^\rho\,{}_{[2]}\tilde{\gamma}_{,\rho}) + O(\epsilon)\Big] + {}_{[4]}\gamma'^{h}_{(\mu\nu)} + O(\kappa^6)\,, \tag{39}
$$

where ${}_{[4]}\gamma'^{h}_{(\mu\nu)}$ is a solution to the homogeneous equations

$$
\Box^2\,{}_{[4]}\gamma'^{h}_{(\mu\nu)} = 0\,. \tag{40}
$$

Note that for such a solution[21]

$$
{}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu} = {}^{(p)}\Big[(m^G/c^2)(-{}_{[2]}\tilde{\gamma}_{\rho\mu,\lambda}u^\rho u^\lambda + \tfrac{1}{2}\,{}_{[2]}\tilde{\gamma}_{\rho\sigma,\mu}u^\rho u^\sigma - \tfrac{1}{2}\,{}_{[2]}\tilde{\gamma}_{\rho\sigma,\lambda}u^\rho u^\sigma u^\lambda u_\mu
$$

$$
+\tfrac{1}{2}\,{}_{[2]}\tilde{\gamma}_{,\rho}u^\rho u_\mu - \tfrac{1}{4}\,{}_{[2]}\tilde{\gamma}_{,\mu})\epsilon^{-1} + O(\epsilon^0)\Big] + {}_{[4]}\gamma'^{h}_{(\mu\nu)}{}^{,\nu} + O(\kappa^6)\,. \tag{41}
$$

As previously mentioned we are only interested in the solutions ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ to (1) for which ${}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu}$ takes the form (6). If we expand the right-hand side of (6) in a power series in ${}^{(p)}\epsilon$ we find

$$
{}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu} = {}^{(p)}\Big[{}_{[4]}C^I_\mu\epsilon^{-1} + \alpha^\nu\,{}_{[4]}C^I_{[\mu\nu]}\epsilon^{-2} + {}_{[4]}\dot{C}^I_{[\mu\nu]}u^\nu\epsilon^{-1} + O(\epsilon^0)\Big] + O(\kappa^6)\,. \tag{42}
$$

Comparing (41) with (42) we see that we shall only be interested in those solutions ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ of the form (39) to Eqs. (1) for which

$$
{}_{[4]}\gamma'^{h}_{(\mu\nu)}{}^{,\nu} = {}^{(p)}\Big[{}_{[4]}C'^{h}_\mu\epsilon^{-1} + \alpha^\nu\,{}_{[4]}C'^{h}_{[\mu\nu]}\epsilon^{-2} + {}_{[4]}\dot{C}'^{h}_{[\mu\nu]}u^\nu\epsilon^{-1} + O(\epsilon^0)\Big] + O(\kappa^6)\,. \tag{43}
$$

The quantities ${}_{[4]}C'^{h}_\mu$ and ${}_{[4]}C'^{h}_{[\mu\nu]}$ in (43) are understood to be functions of ${}^{(p)}\tau$ only. If we choose

$$
{}_{[4]}\gamma'^{h}_{(\mu\nu)} = {}^{(p)}\Big[(m^G/c^2)({}_{[2]}\tilde{\gamma}_{\rho\sigma}u^\rho u^\sigma u_\mu u_\nu - \tfrac{1}{4}\,{}_{[2]}\tilde{\gamma}u_\mu u_\nu)\epsilon^{-1}\Big] + {}_{[4]}\gamma^{h}_{(\mu\nu)} + O(\kappa^6)\,, \tag{44}
$$

where ${}_{[4]}\gamma^{h}_{(\mu\nu)}$ is a solution to the homogeneous equations

$$
\Box^2\,{}_{[4]}\gamma^{h}_{(\mu\nu)} = 0 \tag{45}
$$

and

$$
{}_{[4]}\gamma^{h}_{(\mu\nu)}{}^{,\nu} = \sum_p {}^{(p)}a_{\text{ret}}\,{}^{(p)}\big[{}_{[4]}C^h_\mu(ru)^{-1}\big]_{\text{ret}} + \sum_p {}^{(p)}a_{\text{adv}}\,{}^{(p)}\big[{}_{[4]}C^h_\mu(ru)^{-1}\big]_{\text{adv}}
$$

$$
+\sum_p {}^{(p)}a_{\text{ret}}\,{}^{(p)}\big[{}_{[4]}C^h_{[\mu\nu]}(ru)^{-1}\big]_{\text{ret}}{}^{,\nu} + \sum_p {}^{(p)}a_{\text{adv}}\,{}^{(p)}\big[{}_{[4]}C^h_{[\mu\nu]}(ru)^{-1}\big]_{\text{adv}}{}^{,\nu} + O(\kappa^6)\,, \tag{46}
$$

$$
{}^{(p)}_{[4]}C^h_{[\mu\nu]}\,{}^{(p)}u^\nu = O(\kappa^6) \tag{47}
$$

[the quantities ${}^{(p)}_{[4]}C^h_\mu$ and ${}^{(p)}_{[4]}C^h_{[\mu\nu]}$ in (46) and (47) are understood to be functions of ${}^{(p)}\tau$ only], we have such a set of solutions to Eqs. (1) as with this choice of ${}_{[4]}\gamma'^{h}_{(\mu\nu)}$ both (40) and (43) are satisfied. In fact we find for ${}_{[4]}C'^{h}_\mu$ and ${}_{[4]}C'^{h}_{[\mu\nu]}$,

$$
{}_{[4]}C'^{h}_\mu = {}_{[4]}C^h_\mu + (m^G/c^2)({}_{[2]}\tilde{\gamma}_{\rho\sigma,\lambda}u^\rho u^\sigma u^\lambda u_\mu - \tfrac{1}{4}\,{}_{[2]}\tilde{\gamma}_{,\rho}u^\rho u_\mu)\,, \tag{48}
$$

$$
{}_{[4]}C'^{h}_{[\mu\nu]} = {}_{[4]}C^h_{[\mu\nu]}\,. \tag{49}
$$

This means that ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$, where

$$ {}_{[4]}\gamma^{*I}_{(\mu\nu)} = {}^{(p)}[(m^G/c^2)^2(\tfrac{1}{16}\alpha_\mu\alpha_\nu + \tfrac{7}{16}u_\mu u_\nu)\epsilon^{-2} + (m^G/c^2)(-\tfrac{1}{2}\alpha_\rho\alpha_\sigma\,{}_{[2]}\tilde\gamma^{\rho\sigma}u_\mu u_\nu + {}_{[2]}\tilde\gamma_{\rho\sigma}u^\rho u^\sigma u_\mu u_\nu - \tfrac{1}{4}{}_{[2]}\tilde\gamma u_\mu u_\nu)\epsilon^{-1} $$
$$ + (m^G/c^2)(-\tfrac{1}{4}\alpha_\rho\alpha_\sigma\alpha_\lambda\,{}_{[2]}\tilde\gamma^{\rho\sigma,\lambda}u_\mu u_\nu + \tfrac{1}{4}\alpha_\lambda\,{}_{[2]}\tilde\gamma^{\rho\sigma,\lambda}u_\rho u_\sigma u_\mu u_\nu $$
$$ - \tfrac{1}{2}\alpha_\rho\,{}_{[2]}\tilde\gamma^{\rho\sigma,\lambda}u_\sigma u_\lambda u_\mu u_\nu - \tfrac{1}{2}\alpha_\mu\,{}_{[2]}\tilde\gamma_{\rho\nu,\sigma}u^\rho u^\sigma - \tfrac{1}{2}\alpha_\nu\,{}_{[2]}\tilde\gamma_{\rho\mu,\sigma}u^\rho u^\sigma $$
$$ + \tfrac{1}{2}\eta_{\mu\nu}\alpha_\rho\,{}_{[2]}\tilde\gamma^{\rho\sigma,\lambda}u_\sigma u_\lambda + \tfrac{1}{4}\alpha_\mu\,{}_{[2]}\tilde\gamma_{\rho\sigma,\nu}u^\rho u^\sigma + \tfrac{1}{4}\alpha_\nu\,{}_{[2]}\tilde\gamma_{\rho\sigma,\mu}u^\rho u^\sigma $$
$$ - \tfrac{1}{4}\eta_{\mu\nu}\alpha_\lambda\,{}_{[2]}\tilde\gamma^{\rho\sigma,\lambda}u_\rho u_\sigma + \tfrac{1}{2}\alpha_\sigma\,{}_{[2]}\tilde\gamma_{\rho\mu}{}^{,\sigma}u^\rho u_\nu + \tfrac{1}{2}\alpha_\sigma\,{}_{[2]}\tilde\gamma_{\rho\nu}{}^{,\sigma}u^\rho u_\mu $$
$$ + \tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde\gamma_{\rho\mu,\sigma}u^\sigma u_\nu + \tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde\gamma_{\rho\nu,\sigma}u^\sigma u_\mu - \tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde\gamma_{\rho\sigma,\mu}u^\sigma u_\nu $$
$$ - \tfrac{1}{2}\alpha^\rho\,{}_{[2]}\tilde\gamma_{\rho\sigma,\nu}u^\sigma u_\mu - \tfrac{1}{4}\alpha^\rho\,{}_{[2]}\tilde\gamma_{,\rho}u_\mu u_\nu - \tfrac{1}{8}\alpha_\mu\,{}_{[2]}\tilde\gamma_{,\nu} - \tfrac{1}{8}\alpha_\nu\,{}_{[2]}\tilde\gamma_{,\mu} $$
$$ + \tfrac{1}{8}\eta_{\mu\nu}\alpha^\rho\,{}_{[2]}\tilde\gamma_{,\rho}) + O(\epsilon)] + {}_{[4]}\gamma^{h}_{(\mu\nu)} + O(\kappa^6), \tag{50} $$

and for which one finds

$$ {}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu} = {}^{(p)}[(m^G/c^2)(-\tfrac{1}{2}{}_{[2]}\tilde\gamma_{\rho\mu,\sigma}u^\rho u^\sigma - \tfrac{1}{2}{}_{[2]}\tilde\gamma_{\sigma\mu,\rho}u^\rho u^\sigma + \tfrac{1}{2}{}_{[2]}\tilde\gamma_{\rho\sigma,\mu}u^\rho u^\sigma $$
$$ + \tfrac{1}{2}{}_{[2]}\tilde\gamma_{\rho\sigma,\lambda}u^\rho u^\sigma u^\lambda u_\mu + \tfrac{1}{4}{}_{[2]}\tilde\gamma_{,\rho}u^\rho u_\mu - \tfrac{1}{4}{}_{[2]}\tilde\gamma_{,\mu})\epsilon^{-1} + O(\epsilon^0)] + {}_{[4]}\gamma^{h}_{(\mu\nu)}{}^{,\nu} + O(\kappa^6), \tag{51} $$

is the most general solution to (1) expandable in a power series in ${}^{(p)}\epsilon$ for which ${}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu}$ takes the form (6).[22] We see that with this solution to (1) the ${}^{(p)}_{[4]}C^I_\mu$ and ${}^{(p)}_{[4]}C^I_{[\mu\nu]}$ appearing in (6) have the functional form

$$ {}^{(p)}_{[4]}C^I_\mu = {}^{(p)}[(m^G/c^2)(-\tfrac{1}{2}{}_{[2]}\tilde\gamma_{\sigma\mu,\rho}u^\rho u^\sigma - \tfrac{1}{2}{}_{[2]}\tilde\gamma_{\mu\rho,\sigma}u^\rho u^\sigma + \tfrac{1}{2}{}_{[2]}\tilde\gamma_{\rho\sigma,\mu}u^\rho u^\sigma $$
$$ + \tfrac{1}{2}{}_{[2]}\tilde\gamma_{\rho\sigma,\lambda}u^\rho u^\sigma u^\lambda u_\mu + \tfrac{1}{4}{}_{[2]}\tilde\gamma_{,\rho}u^\rho u_\mu - \tfrac{1}{4}{}_{[2]}\tilde\gamma_{,\mu})] + {}^{(p)}_{[4]}C^h_\mu, \tag{52} $$

$$ {}^{(p)}_{[4]}C^I_{[\mu\nu]} = {}^{(p)}_{[4]}C^h_{[\mu\nu]}\,. \tag{53} $$

Different solutions to the inhomogeneous equations (1) satisfying (6) differ from each other by a solution to the homogeneous equations (45) satisfying (46)–(47). In papers I and III we discussed the fact that these different acceptable solutions to Eqs. (1) reflect only the fact that one can use different coordinate systems in the region under investigation or a different definition of those quantities (in this case the mass and spin) which describe the structure of the particles in the region. There is thus no important loss in generality in choosing

$$ {}^{(p)}_{[4]}C^h_\mu = 0\,, \quad {}^{(p)}_{[4]}C^h_{[\mu\nu]} = 0 \tag{54} $$

in (52) and (53). Also, since along the world line of the pth particle

$$ {}^{(p)}_{[2]}\gamma^{\text{self}\,n}_{\mu\nu,\rho} = O(\kappa^4), \tag{55} $$

we can, making use of (16), replace the quantities ${}^{(p)}_{[2]}\tilde\gamma_{\mu\nu,\rho}$ in (52) by ${}^{(p)}_{[2]}\gamma^{\text{ext}}_{\mu\nu,\rho}$. This means, assuming Eqs. (1) have a solution which satisfies (6) and can be expanded in a power series in ${}^{(p)}\epsilon$, that there is a solution ${}^{(p)}_{[4]}\gamma^{*I}_{(\mu\nu)}$ to (1) such that over a perfectly isolated region containing only simple ideal particles which are neutral and spinless to second order,

$$ {}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu} = \sum_p {}^{(p)}a_{\text{ret}}\,{}^{(p)}[{}_{[4]}C^I_\mu(ru)^{-1}]_{\text{ret}} + \sum_p {}^{(p)}a_{\text{adv}}\,{}^{(p)}[{}_{[4]}C^I_\mu(ru)^{-1}]_{\text{adv}} + O(\kappa^6), \tag{56} $$

where

$$ {}^{(p)}_{[4]}C^I_\mu = {}^{(p)}[(m^G/c^2)(-\tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\sigma\mu,\rho}u^\rho u^\sigma - \tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\mu\rho,\sigma}u^\rho u^\sigma + \tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\rho\sigma,\mu}u^\rho u^\sigma $$
$$ + \tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\rho\sigma,\lambda}u^\rho u^\sigma u^\lambda u_\mu + \tfrac{1}{4}{}_{[2]}\gamma^{\text{ext}}{}_{,\rho}u^\rho u_\mu - \tfrac{1}{4}{}_{[2]}\gamma^{\text{ext}}{}_{,\mu})]\,. \tag{57} $$

The fields ${}_{[2]}\gamma^{\text{ext}}_{\mu\nu}$ in (57) are understood as evaluated at the position of the pth particle.

III. EQUATIONS OF STRUCTURE AND MOTION

The fourth-order equations of mass, motion, and spin associated with interacting simple ideal particles in an inertial system which are neutral and spinless to second order are found from (56) and (57) to be[23]

$$^{(p)}\dot{P}_\mu = {}^{(p)}f_\mu , \tag{58}$$

$$^{(p)}(\dot{S}_{\mu\nu} - \dot{S}_{\mu\rho} u^\rho u_\nu + \dot{S}_{\nu\rho} u^\rho u_\mu) = 0 , \tag{59}$$

where

$$^{(p)}P_\mu = {}^{(p)}(m^G u_\mu + \dot{S}_{\mu\rho} u^\rho) , \tag{60}$$

$$^{(p)}f_\mu = {}^{(p)}[m^G(\tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\sigma\mu,\rho} u^\rho u^\sigma + \tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\mu\rho,\sigma} u^\rho u^\sigma - \tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\rho\sigma,\mu} u^\rho u^\sigma - \tfrac{1}{2}{}_{[2]}\gamma^{\text{ext}}_{\rho\sigma,\lambda} u^\rho u^\sigma u^\lambda u_\mu - \tfrac{1}{4}{}_{[2]}\gamma^{\text{ext}}{}_{,\rho} u^\rho u_\mu + \tfrac{1}{4}{}_{[2]}\gamma^{\text{ext}}{}_{,\mu})] , \tag{61}$$

and

$$^{(p)}S_{\mu\nu}{}^{(p)}u^\nu = 0. \tag{62}$$

The above equations are identical with the equations of mass, motion, and spin found for similar particles in paper II.[24] In that paper, however, we used the noncovariant procedure described in paper I to find them.

APPENDIX A: SOLUTIONS TO CERTAIN EQUATIONS

The fact that (34)–(38) are solutions to Eqs. (29)–(33) in the neighborhood of the pth particle is easy to verify.

From the definition of r^ρ given in (17) we find evaluated on the surface $^{(p)}\sigma$

$$^{(p)}r_{\rho,\sigma} = \eta_{\rho\sigma} - {}^{(p)}\xi_{\rho,\sigma} , \tag{A1}$$

where

$$^{(p)}\xi_{\rho,\sigma} = {}^{(p)}\dot{\xi}\,{}^{(p)}\tau_{,\sigma} = {}^{(p)}u_\rho{}^{(p)}\tau_{,\sigma} . \tag{A2}$$

We are using the notation

$$^{(p)}u_\rho = {}^{(p)}\dot{\xi}_\rho . \tag{A3}$$

From (22) we have, expanding $^{(p)}\tau_{,\sigma}$ in a power series in $^{(p)}\epsilon$ and making use of (5),

$$^{(p)}\tau_{,\sigma} = {}^{(p)}u_\sigma + O(\kappa^2) . \tag{A4}$$

Combining (A1), (A2), and (A4) we find evaluated on the surface $^{(p)}\sigma$

$$^{(p)}r_{\rho,\sigma} = \eta_{\rho\sigma} - {}^{(p)}u_\rho{}^{(p)}u_\sigma + O(\kappa^2) . \tag{A5}$$

From the definition of $^{(p)}\epsilon$ given in (17) we have

$$^{(p)}\epsilon\,{}^{(p)}\epsilon_{,\sigma} = -{}^{(p)}r_\rho{}^{(p)}r^\rho{}_{,\sigma} . \tag{A6}$$

Making use of (A5), (A6), and the definition of $^{(p)}\alpha$ given in (23) we find on the surface $^{(p)}\sigma$

$$^{(p)}\epsilon_{,\sigma} = -{}^{(p)}\alpha_\sigma + O(\kappa^2) . \tag{A7}$$

From the definition of $^{(p)}\alpha_\rho$ given in (23) we have that

$$^{(p)}\alpha_{\rho,\sigma} = {}^{(p)}r_{\rho,\sigma}{}^{(p)}\epsilon^{-1} - {}^{(p)}r_\rho{}^{(p)}\epsilon_{,\sigma}{}^{(p)}\epsilon^{-2} . \tag{A8}$$

Making use of (A5) and (A7) we find from (A8) on the surface $^{(p)}\sigma$

$$^{(p)}\alpha_{\rho,\sigma} = ({}^{(p)}\alpha_\rho{}^{(p)}\alpha_\sigma + \eta_{\rho\sigma} - {}^{(p)}u_\rho{}^{(p)}u_\sigma){}^{(p)}\epsilon^{-1} + O(\kappa^2) . \tag{A9}$$

It also follows from (4), (5), and (A4) that on the surface $^{(p)}\sigma$

$$^{(p)}m^G{}_{,\sigma} = {}^{(p)}\dot{m}^G{}^{(p)}\tau_{,\sigma} = O(\kappa^4) , \tag{A10}$$

$$^{(p)}u_{\rho,\sigma} = {}^{(p)}\dot{u}_\rho{}^{(p)}\tau_{,\sigma} = O(\kappa^2) , \tag{A11}$$

$$({}^{(p)}\tilde{\gamma}_{\mu\nu})_{,\sigma} = {}^{(p)}(\dot{\tilde{\gamma}}_{\mu\nu}\tau_{,\sigma}) = {}^{(p)}(\tilde{\gamma}_{\mu\nu,\rho} u^\rho u_\sigma) + O(\kappa^2) , \tag{A12}$$

$$({}^{(p)}\tilde{\gamma}_{\mu\nu,\lambda\ldots})_{,\sigma} = {}^{(p)}(\dot{\tilde{\gamma}}_{\mu\nu,\lambda\ldots}\tau_{,\sigma}) = {}^{(p)}(\tilde{\gamma}_{\mu\nu,\lambda\ldots\rho} u^\rho u_\sigma) + O(\kappa^2) . \tag{A13}$$

Making use of the relations (A7) and (A9)–(A13) one can easily verify that (34)–(38) are solutions to Eqs. (29)–(33) in the neighborhood of the pth particle. It is also easy to show that in the neighborhood of the pth particle at proper time $^{(p)}\tau$

$$^{(p)}(\alpha_\mu\alpha_\nu\epsilon^{-2})_{,\sigma} = {}^{(p)}[(4\alpha_\mu\alpha_\nu\alpha_\sigma + \alpha_\mu\eta_{\nu\sigma} + \alpha_\nu\eta_{\mu\sigma} - \alpha_\mu u_\nu u_\sigma - \alpha_\nu u_\mu u_\sigma)\epsilon^{-3}] + O(\kappa^2), \quad \text{(A14)}$$

$$^{(p)}(\alpha_\mu\alpha_\nu\epsilon^{-1})_{,\sigma} = {}^{(p)}[(3\alpha_\mu\alpha_\nu\alpha_\sigma + \alpha_\mu\eta_{\nu\sigma} + \alpha_\nu\eta_{\mu\sigma} - \alpha_\mu u_\nu u_\sigma - \alpha_\nu u_\mu u_\sigma)\epsilon^{-2}] + O(\kappa^2), \quad \text{(A15)}$$

$$^{(p)}(\alpha_\mu\alpha_\nu\alpha_\rho)_{,\sigma} = {}^{(p)}[(3\alpha_\mu\alpha_\nu\alpha_\rho\alpha_\sigma + \alpha_\mu\alpha_\nu\eta_{\rho\sigma} + \alpha_\mu\alpha_\rho\eta_{\nu\sigma} + \alpha_\nu\alpha_\rho\eta_{\mu\sigma} - \alpha_\mu\alpha_\nu u_\rho u_\sigma - \alpha_\mu\alpha_\rho u_\nu u_\sigma - \alpha_\nu\alpha_\rho u_\mu u_\sigma)\epsilon^{-1}] + O(\kappa^2), \quad \text{(A16)}$$

$$^{(p)}(\alpha_\mu)_{,\sigma} = {}^{(p)}[(\alpha_\mu\alpha_\sigma + \eta_{\mu\sigma} - u_\mu u_\sigma)\epsilon^{-1}] + O(\kappa^2), \quad \text{(A17)}$$

$$^{(p)}\epsilon^{-2}{}_{,\sigma} = {}^{(p)}[2\alpha_\sigma\epsilon^{-3}] + O(\kappa^2), \quad \text{(A18)}$$

$$^{(p)}\epsilon^{-1}{}_{,\sigma} = {}^{(p)}[\alpha_\sigma\epsilon^{-2}] + O(\kappa^2). \quad \text{(A19)}$$

The relations (A14)–(A19) will prove to be useful in verifying (41), (48), (49), and (51).

APPENDIX B: A COMPARISON WITH THE RESULTS OF THE NONCOVARIANT PROCEDURE

It is easy to show that the field ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ in the neighborhood of the pth particle obtained here by means of the Lorentz-covariant procedure is equivalent to the field ${}_{[4]}\gamma^{I}_{(\mu\nu)}$ of the paper II Appendix B obtained by means of the noncovariant procedure.

On the surface ${}^{(p)}\sigma$ we know that ${}^{(p)}r^\rho$ and ${}^{(p)}u^\rho$ take the form[25]

$$r^s = x^s - (\xi^s)_{t*}, \quad r^4 = ct - ct^*, \quad \text{(B1)}$$

$$u^s = (\xi'^s)_{t*} + O(\xi''^r\xi'^l)_{t*}, \quad u^4 = 1 + O(\xi''^r\xi'^l)_{t*}, \quad \text{(B2)}$$

where we are using the notation

$$t = x^4/c, \quad t^* = \xi^4/c, \quad \text{(B3)}$$

$$\xi'^s = d\xi/cdt, \quad \xi''^s = d^2\xi^s/c^2dt^2. \quad \text{(B4)}$$

The quantities ξ^s and ξ'^s on the right-hand side of the Eqs. (B1) and (B2) are understood as evaluated at time t^*. This is indicated by the subscript t^*. The expression $O(\xi''^r\xi'^l)$ in (B1) and (B2) means a term whose power-series expansion in ξ''^r begins with $\xi''^r\xi'^l$.

From (B1) and (B2) we see that the condition

$$^{(p)}r_\rho\,{}^{(p)}u^\rho = 0 \quad \text{(B5)}$$

on the surface ${}^{(p)}\sigma$ implies

$$c(t-t^*) = (r^s\xi'^s)_{t*} + O(\xi''^r\xi'^l)_{t*}. \quad \text{(B6)}$$

Making use of the Taylor-series expansions

$$(\xi^s)_{t*} = \xi^s - \xi'^s c(t-t^*) + O((t-t^*)^2), \quad \text{(B7)}$$

$$(\xi'^s)_{t*} = \xi'^s - \xi''^s c(t-t^*) + O((t-t^*)^2), \quad \text{(B8)}$$

which are assumed valid in the neighborhood of the world line of the pth particle, it follows from (B6) and

$$\xi''^s = O(\kappa^2) \quad \text{(B9)}$$

that in the neighborhood of the world line of the pth particle

$$c(t-t^*) = r^s\xi'^s + O(\xi''^r\xi'^l) + O(\kappa^2). \quad \text{(B10)}$$

The condition (B9) follows from (5). All quantities on the right-hand side of (B7), (B8), and (B10) are evaluated at time t.

From (B1), (B2), and (B7)–(B10) we find for ${}^{(p)}r^\rho$, ${}^{(p)}u^\rho$, and ${}^{(p)}\epsilon$ in the neighborhood of the world line of the pth particle on ${}^{(p)}\sigma$

$$^{(p)}r^s = x^s - {}^{(p)}\xi^s, \quad {}^{(p)}r^4 = {}^{(p)}r^s\,{}^{(p)}\xi'^s, \quad \text{(B11)}$$

$$^{(p)}u^s = {}^{(p)}\xi'^s, \quad {}^{(p)}u^4 = 1, \quad \text{(B12)}$$

$$^{(p)}\epsilon = ({}^{(p)}r^s\,{}^{(p)}r^s)^{1/2}. \quad \text{(B13)}$$

All quantities on the right-hand side of Eqs. (B11)–(B13) are understood as evaluated at time t. We are neglecting the terms $O({}^{(p)}\xi''^r\,{}^{(p)}\xi'^l)$ and $O(\kappa^2)$ on the right-hand side of Eqs. (B11)–(B13).

Making use of the Taylor-series expansions

$$(\tilde\gamma_{\mu\nu})_{t*} = \tilde\gamma_{\mu\nu} - (\tilde\gamma_{\mu\nu})_{,4}c(t-t^*) + O((t-t^*)^2), \quad \text{(B14)}$$

$$(\tilde\gamma_{\mu\nu,\lambda})_{t*} = \tilde\gamma_{\mu\nu,\lambda} - (\tilde\gamma_{\mu\nu,\lambda})_{,4}c(t-t^*) + O((t-t^*)^2), \quad \text{(B15)}$$

$$(m)_{t*} = m - m_{,4}c(t-t^*) + O((t-t^*)^2), \quad \text{(B16)}$$

we find from (B10) and

$$m_{,4} = O(\kappa^4) \quad \text{(B17)}$$

that in the neighborhood of the pth particle

$$({}^{(p)}\tilde\gamma_{\mu\nu})_{t*} = {}^{(p)}[\tilde\gamma_{\mu\nu} - r^s\tilde\gamma_{\mu\nu,4}\xi'^s], \quad \text{(B18)}$$

$$({}^{(p)}\tilde\gamma_{\mu\nu,\lambda})_{t*} = {}^{(p)}[\tilde\gamma_{\mu\nu,\lambda} - r^s\tilde\gamma_{\mu\nu,\lambda 4}\xi'^s], \quad \text{(B19)}$$

$$({}^{(p)}m)_{t*} = {}^{(p)}m. \quad \text{(B20)}$$

The condition (B17) follows from (4). All quantities on the right-hand side of (B18)–(B20) are evaluated at time t. We are neglecting the terms $O({}^{(p)}\xi^{\prime\prime}\,{}^{(p)}\xi^{\prime l})$ and $O(\kappa^4)$ in (B18)–(B20).

Through the use of (B11)–(B13) and (B18)–(B20) it is easy to show that the field ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ given in (50) and found in this paper by means of the Lorentz-covariant procedure is identical, if we choose ${}_{[4]}\gamma^{h}_{(\mu\nu)}=0$, to the field ${}_{[4]}\gamma^{I}_{(\mu\nu)}$ given in Appendix B of paper II and found by means of the noncovariant procedure.

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[1]C. R. Johnson, Phys. Rev. D 4, 295 (1971); 4, 318 (1971); 4, 3555 (1971); 5, 282 (1972). The papers will be referred to as papers I, II, III, and IV, respectively.

[2]The terms "simple ideal particle" and "perfectly isolated region of the continuum" are defined in paper I. It is assumed that the reader is familiar with the earlier papers I, II, III, and IV. Unless otherwise indicated, the notation used in this paper will be the same as that used in the earlier papers.

[3]P. Havas and J. N. Goldberg, Phys. Rev. 128, 398 (1962). Several methods have been proposed for finding the Lorentz-covariant equations of structure and motion of particles in Einstein's gravitational theory. The method most widely used and discussed at the present time seems to be that of Havas and Goldberg. For a brief discussion of other methods and their weaknesses see the author's papers I and II and the references therein.

[4]In papers I–IV the author's method is actually developed for Einstein's theory of the nonsymmetric field. Since Einstein's gravitational theory is a special case of his theory of the nonsymmetric field the author's method is also applicable to the pure gravitational theory.

[5]Another difference is that Havas and Goldberg treat the acceleration of a particle as of zeroth order in κ while the author treats it as of the same order as the mass of a particle (the second order in our notation). In this regard we agree with the previous work of Kerr [R. P. Kerr, Nuovo Cimento 13, 492 (1959)] and Carmeli [M. Carmeli, *ibid.* 55B, 220 (1968)]. For the point of view of Havas and Goldberg in this controversy see Ref. 3. For the point of view of the author see papers I and II, especially Appendix E of paper II.

It should be pointed out that this controversy does not affect the final form of the equations of motion of particles but only the order at which certain terms appear in the equations of motion. For example Havas and Goldberg obtain radiation reaction terms in the lowest order in which interactions among particles appear while Kerr, Carmeli, and the author insist that such terms should be treated as of higher order and do not have physical meaning in that order of approximation.

[6]Havas and Goldberg would presumably disagree with the author on this assessment. See Ref. 3 for a discussion of the procedures.

Using the Bhabha–Harish-Chandra procedure one introduces infinite renormalization. Using the Riesz procedure one represents particles by singular solutions finite along the world line of a particle. Such singular solutions are, in the author's opinion, artificial constructs. Using the author's method particles are represented in the usual way by singular solutions infinite along the world line of a particle. No infinite renormalization is required using the author's method.

[7]See Sec. V B and Appendix A of paper I.

[8]This is true even though in the immediate vicinity of each particle the solutions to the field equations at each order do not approximate solutions to the full field equations. See Sec. V B of paper I.

[9]There is, of course, always a certain ambiguity in the higher orders of approximation due to the freedom one has in choosing coordinate systems.

[10]See Ref. 5.

[11]The equations of structure and motion we obtain are identical to those obtained earlier by Kerr, Carmeli, and others. If we neglect spin, the equations of motion are just the usual geodesic equations, to second order in κ, for the motion of a test particle in a gravitational field.

[12]See Ref. 6.

[13]A series of papers leading to the equations of structure and motion of neutral particles to the sixth and eighth order of approximation are being written by the author.

[14]In an inertial system the field $\gamma_{(\mu\nu)}$ to fourth order is given by ${}_{[4]}\gamma_{(\mu\nu)}$,

$$ {}_{[4]}\gamma_{(\mu\nu)} = {}_{[4]}\gamma^{*H}_{(\mu\nu)} + {}_{[4]}\gamma^{*I}_{(\mu\nu)} + \delta\,{}_{[4]}\gamma^{*}_{(\mu\nu)}, $$

where

$$ \delta\,{}_{[4]}\gamma^{*}_{(\mu\nu)} = {}_{[4]}\epsilon^{*}_{\mu,\nu} + {}_{[4]}\epsilon^{*}_{\nu,\mu} - \eta_{\mu\nu}\,{}_{[4]}\epsilon^{*,\rho}_{\rho}. $$

The field denoted by ${}_{[4]}\gamma^{*H}_{(\mu\nu)}$ in the above equations is identical to the field ${}_{[4]}\gamma^{H}_{(\mu\nu)}$ of paper II. The field denoted by ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ is identical to the field ${}_{[4]}\gamma^{I}_{(\mu\nu)}$ of that paper. The field $\delta\,{}_{[4]}\gamma^{*}_{(\mu\nu)}$ does not affect the equations of motion of particles to the fourth order and will not be considered further here. For a further discussion of all of this the reader is referred to papers II and III.

[15]There is no loss in generality in assuming terms of order five in κ do not appear in Eqs. (1.63) of paper II. We shall thus replace $O(\kappa^5)$ in those equations by $O(\kappa^6)$.

[16]Indices will always be raised and lowered with the Minkowski metric $\eta_{\mu\nu}$. We shall use the abbreviation $\gamma=\eta^{\rho\sigma}\gamma_{\rho\sigma}$ in this paper.

[17]The reason for this restriction is discussed in Sec. V B of paper I.

[18]Arguments of the existence of such solutions are discussed in Sec. V B and Appendix A of paper I.

[19]Although we shall investigate ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ in the immediate vicinity of the pth particle it is only assumed that ${}_{[4]}\gamma_{(\mu\nu)}$,

$$ {}_{[4]}\gamma_{(\mu\nu)} = {}_{[4]}\gamma^{*H}_{(\mu\nu)} + {}_{[4]}\gamma^{*I}_{(\mu\nu)} + {}_{[4]}\epsilon^{*}_{\mu,\nu} + {}_{[4]}\epsilon^{*}_{\nu,\mu} - \eta_{\mu\nu}\,{}_{[4]}\epsilon^{*,\rho}_{\rho}, $$

approximates solutions to Einstein's field equations at "distances" sufficiently far from the world lines of particles. A knowledge of ${}_{[4]}\gamma^{*I}_{(\mu\nu)}$ near the world line of each particle in an isolated region of the continuum is still useful, however, as it will allow us to determine ${}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu}$ throughout the entire region. A knowledge of ${}_{[4]}\gamma^{*I}_{(\mu\nu)}{}^{,\nu}$ throughout the entire region is all that is needed to determine the equations of motion of particles to the fourth order of approximation. See Sec. V and Sec. VI of paper I. See also paper III and Ref. 14 of this paper.

[20]These solutions are easily verified. See Appendix A of this paper.

[21]This is easy to verify. See Appendix A of this paper.

[22]It is easy to show that the field $_{[4]}\gamma_{(\mu\nu)}^{*I}$ in the neighborhood of the pth particle obtained here by means of the Lorentz-covariant procedure is equivalent, if we choose $_{[4]}\gamma_{(\mu\nu)}^{h}=0$, to the field $_{[4]}\gamma_{(\mu\nu)}^{I}$ of paper II Appendix B obtained by means of the noncovariant procedure. See Appendix B of this paper.

[23]To see how these equations follow from (56) and (57) see Sec. V and Sec. VI of paper I. Also see paper III.

[24]In the case where the spin is zero, Eqs. (58)–(62) are just the usual geodesic equations, to second order in κ, for the motion of a test particle in a gravitational field.

[25]In this Appendix we shall often suppress the superscript (p) for ease of writing.