PHYSICAL REVIEW D VOLUME 5, NUMBER 8 15 APRIL 1972

Solution of the Scalar Wave Equation in a Kerr Background by Separation of Variables*

Dieter R. Brill, Paul L. Chrzanowski, and C. Martin Pereira
University of Maryland, Department of Physics and Astronomy, College Park, Maryland 20742

and

Edward D. Fackerell
Monash University, Department of Mathematics, Clayton, Victoria, Australia

and

James R. Ipser
University of Chicago, Department of Astronomy and Astrophysics, Enrico Fermi Institute, Chicago, Illinois 60637
(Received 20 December 1971)

The effect of the Kerr gravitational field on wave phenomena is explored by examining the inhomogeneous wave equation for a scalar massive field in a Kerr background geometry. The equation is separated in Boyer-Lindquist coordinates. The angular functions are spheroidal harmonics, and the radial equation is reduced to a one-dimensional Schrödinger equation with an effective potential.

In recent years the physical importance of the Kerr geometry[1] for a rotating body in general relativity has been increasingly recognized. Rather than describing only a restricted class of rotating bodies, this geometry appears to be the universal limit reached by gravitational collapse of any body with nonvanishing angular momentum.[2] It is therefore important to understand how particles and waves behave in this geometry. Particle geodesics in the Kerr geometry have been studied[3,4] by exploiting the conservation laws which result from the Killing vectors and the Killing tensor which this geometry possesses. The analogous result for waves is the separability of the scalar wave equation in the Kerr family of geometries, as demonstrated by Carter.[3] Our purpose here is to give the details of this separation explicitly, specialized to the Kerr metric itself, which appears to be the physically most significant case.

We adopt the Boyer-Lindquist[5] coordinates which express the Kerr geometry as

$$ds^2=\frac{\rho^2}{\Delta}dr^2+\rho^2 d\theta^2+\left[(r^2+a^2)\sin^2\theta+\frac{2Mr}{\rho^2}a^2\sin^4\theta\right]d\phi^2-\frac{4Mr}{\rho^2}a\sin^2\theta\, d\phi\, dt-\left(1-\frac{2Mr}{\rho^2}\right)dt^2. \tag{1}$$

Here M and $J=+aM$ are the geometrized total Kerr mass and angular momentum,[6] respectively, and

$$\Delta=r^2-2Mr+a^2,\quad \rho^2=r^2+a^2\cos^2\theta. \tag{2}$$

The determinant of the metric tensor takes the simplified form

$$\det(g_{\mu\nu})\equiv g=-\rho^4\sin^2\theta. \tag{3}$$

The corresponding contravariant metric is

$$\left(\frac{\partial}{\partial s}\right)^2=\rho^{-2}\left\{\Delta\left(\frac{\partial}{\partial r}\right)^2+\left(\frac{\partial}{\partial\theta}\right)^2+(\sin^{-2}\theta-a^2\Delta^{-1})\left(\frac{\partial}{\partial\phi}\right)^2-\frac{4Mra}{\Delta}\frac{\partial}{\partial\phi}\frac{\partial}{\partial t}-[\Delta^{-1}(r^2+a^2)^2-a^2\sin^2\theta]\left(\frac{\partial}{\partial t}\right)^2\right\}. \tag{4}$$

The wave equation for a scalar field Φ of mass μ and source T,

$$\Box\Phi+\mu^2\Phi=4\pi T, \tag{5}$$

can now be written by using the familiar formula

$$\Phi=(-g)^{-1/2}\frac{\partial}{\partial x^\mu}\left(g^{\mu\nu}(-g)^{1/2}\frac{\partial\phi}{\partial x^\nu}\right). \tag{6}$$

The result is

$$\frac{\partial}{\partial r}\left(\Delta\frac{\partial\Phi}{\partial r}\right)-\frac{a^2}{\Delta}\frac{\partial^2\Phi}{\partial\phi^2}-\frac{4Mra}{\Delta}\frac{\partial^2\Phi}{\partial\phi\,\partial t}-\frac{(r^2+a^2)^2}{\Delta}\frac{\partial^2\Phi}{\partial t^2}+\mu^2r^2\Phi+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$
$$+a^2\sin^2\theta\frac{\partial^2\Phi}{\partial t^2}+\mu^2a^2\cos^2\theta\Phi=4\pi\rho^2T. \tag{7}$$

Here we have grouped terms so as to display explicitly the known[3] separability of this equation.

To solve the homogeneous equation ($T=0$) we make a separation ansatz in which the ϕ and t dependence is given by the usual eigenfunctions appropriate for an axially symmetric and stationary background geometry,

$$\Phi=R(r)\Theta(\theta)e^{im\phi}e^{-i\omega t}. \tag{8}$$

By the usual argument we then find the separated homogeneous equations, with separation constant Q, chosen so as to agree with Carter's[3] definition of the constant of the particle motion,

$$\Delta\frac{\partial}{\partial r}\left(\Delta\frac{\partial R}{\partial r}\right)+[a^2m^2-4Mram\omega+(r^2+a^2)^2\omega^2+\mu^2r^2\Delta]R=(Q+m^2+\omega^2a^2)\Delta R, \tag{9a}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left(a^2(\omega^2+\mu^2)\cos^2\theta-\frac{m^2}{\sin^2\theta}\right)\Theta=-(Q+m^2)\Theta. \tag{9b}$$

Consider first Eq. (9b). It depends only on the Kerr parameter a and not on the mass M of the background geometry. But in the case $M=0$, $a\neq0$, the Kerr solution represents flat Minkowski space, and the Boyer-Lindquist coordinates become the familiar oblate spheroidal coordinates. Equation (9b) is therefore the same as the flat-space angular spheroidal equation. Its eigenfunctions have been studied to some extent.[7-9] To connect with notation and "Meixner-Schärfke" normalization convention of Refs. 8 and 9 we define the eigenvalues

$$Q+m^2=\lambda_{ml} \tag{10}$$

and the eigenfunctions

$$\Theta=S_{ml}(-ic,\cos\theta) \tag{11}$$

with

$$c^2=a^2(\omega^2+\mu^2). \tag{12}$$

The eigenfunctions form a discrete set and go over into Legendre polynomials in the limit $c=0$. The integers l and m have their standard ranges, but the eigenvalues λ_{ml} cannot be analytically expressed in terms of l and m. Eigenvalues and eigenfunctions have been tabulated, and expansions for large and small c^2/l^2 are known.[8] As a consequence of the angular equation (9b) the eigenfunctions satisfy orthogonality relations. If we define the spheroidal harmonics

$$Z_l^m(\theta,\phi)=\left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2}S_{ml}(-ic,\cos\theta)e^{im\phi}, \tag{13}$$

these relations take the standard form

$$\int Z_l^{m*}Z_{l'}^{m'}d(\cos\theta)d\phi=\delta_{ll'}\delta_{mm'}. \tag{14}$$

The radial equation (9a) can be written as a one-dimensional equation with an effective potential by defining a new radial function u and a new measure of radial distance r^*,

$$u=(r^2+a^2)^{1/2}R,\quad dr^*=(r^2+a^2)\Delta^{-1}dr. \tag{15}$$

This r^* coordinate was chosen so that $t\mp r^*$ is the retarded (advanced) null Kerr coordinate. We then find

$$-d^2u/dr^{*2}+(V-E)u=0, \tag{16}$$

with

$$V-E=-\frac{\Delta\mu^2}{r^2+a^2}+\frac{4Mram\omega-a^2m^2+\Delta[\lambda_{ml}+(\omega^2+\mu^2)a^2]}{(r^2+a^2)^2}+\frac{\Delta(3r^2-4Mr+a^2)}{(r^2+a^2)^3}-\frac{3\Delta^2r^2}{(r^2+a^2)^4}-\omega^2. \tag{17}$$

This potential depends nontrivially on the energy ω^2 of the field. Appropriate boundary conditions for the solutions of (16) have been discussed by Matzner[10] for the Schwarzschild case ($a=0$). An analogous reasoning for the present case demands that only ingoing waves exist at the one-way membrane $r^* \to -\infty$.

To treat the case when a source is present we expand the solution in terms of spheroidal harmonics and harmonic time dependence,

$$\Phi(r,\theta,\phi,t)=(r^2+a^2)^{-1/2}\int\sum_{l'm'} u_{\omega' l' m'}(r) Z_{l'}^{m'}(\theta,\phi)e^{-i\omega' t}d\omega' . \tag{18}$$

The basis functions $Z_{l'}^{m'}e^{-i\omega' t}$ are orthogonal in the sense of the scalar product

$$(f,g)=\frac{1}{2\pi}\int f^* g\, d(\cos\theta)\,d\phi\, dt . \tag{19}$$

To obtain the inhomogeneous radial wave equation which describes the case where a source is present, we insert the expansion (18) into the wave equation (7) and take the scalar product with another basis function,

$$(Z_l^m e^{-i\omega t},\ \Delta\rho^2(\Box+\mu^2)\Phi)=4\pi(Z_l^m e^{-i\omega t},\ \Delta(r^2+a^2\cos^2\theta)T)$$

or

$$-\frac{d^2u_{\omega l m}}{dr^{*2}}+(V-E)u_{\omega l m}=-2\Delta(r^2+a^2)^{-3/2}\int Z_l^{m*}(\theta,\phi)e^{i\omega t}(r^2+a^2\cos^2\theta)T\,d(\cos\theta)\,d\phi\, dt . \tag{20}$$

Thus the problem of finding the scalar field due to an arbitrary given source in a Kerr background is reduced to one inhomogeneous ordinary differential equation.

This separation and reduction of the scalar equation is of interest not only in its own right, but also for the vector[11] and tensor (photon and graviton) wave equations. These equations are thought not to be separable in a Kerr background; however, in the limit of high orbital angular momentum (high l), the contribution of the spin would be expected to be negligible, as it is known to be in the Schwarzschild background case. Thus the high-l limit of Eq. (20) probably has more universal validity than the scalar case for which it was derived.

We want to thank C. W. Misner, R. A. Breuer, and H. G. Hughes III for helpful discussions and clarifications.

---

*Work supported in part by the National Science Foundation under Grant No. GP 25548 and by the Center for Theoretical Physics of the University of Maryland.

[1]R. P. Kerr, Phys. Rev. Letters 11, 237 (1963).
[2]B. Carter, Phys. Rev. Letters 26, 331 (1971).
[3]B. Carter, Phys. Rev. 174, 1559 (1968).
[4]J. M. Bardeen, Astrophys. J. 161, 103 (1970).
[5]R. H. Boyer and R. W. Lindquist, J. Math. Phys. 8, 265 (1967).
[6]Jeffrey M. Cohen, J. Math. Phys. 9, 905 (1968).
[7]C. Flammer, *Spheroidal Wave Functions* (Stanford Univ. Press, Stanford, Calif., 1957).
[8]See *Handbook of Mathematical Functions,* edited by Milton Abramowitz and Irene A. Stegun (Dover, New York, 1968).
[9]J. Meixner and F. W. Schärfke, *Mathieusche Funktionen und Sphäroidfunktionen* (Springer, Berlin, 1954).
[10]R. A. Matzner, J. Math. Phys. 9, 163 (1968).
[11]J. R. Ipser, Phys. Rev. Letters 27, 529 (1971).