PHYSICAL REVIEW D
PARTICLES AND FIELDS

THIRD SERIES, VOL. 5, NO. 8 — 15 April 1972

Ginzburg-Landau Theory of Anisotropic Superfluid Neutron-Star Matter

R. W. Richardson*
Physics Department, New York University, Washington Square, New York, New York 10003
(Received 18 October 1971)

The Gor'kov procedure is used to obtain a generalized Ginzburg-Landau theory for anisotropic superfluid neutron-star matter. The resulting equations are solved for the case of superfluid flow past a plane boundary and the case of an isolated vortex line. The structure of these solutions and their connection with rotating neutron stars is discussed.

I. INTRODUCTION

The theory of ultradense matter has received renewed impetus from the identification of pulsars as rotating neutron stars.[1] The density of matter in the case of such a star may be as high as 10^{15} g/cm^3. Thus a theory of neutron stars must include a theory of matter for all densities up to 10^{15} g/cm^3. For densities less than about 4×10^{11} g/cm^3, the ground state of matter consists of a lattice of nuclei immersed in an electron sea.[2] At a density slightly greater than 4×10^{11} g/cm^3, the neutrons start to leave the nuclei and form a free neutron gas in the space between the nuclei.[2] This situation persists as the density is increased until the nuclei dissolve at about nuclear matter density 3×10^{14} g/cm^3. For densities greater than this the dense matter consists of a uniform fluid made up of neutrons plus a few percent protons and electrons. At even higher densities, 10^{15} g/cm^3 or more, various hyperons begin to appear and the matter becomes a mixture of various species.

In the density range from 4×10^{11} g/cm^3 to 10^{15} g/cm^3, a free neutron gas or fluid is an important constituent of dense matter and the study of its properties is an important element in the theory of neutron stars. The properties of normal, i.e., nonsuperfluid, neutron matter have been studied using the techniques of nuclear-matter theory[3] by several authors.[4] Superfluid neutron matter has been studied by Yang and Clark.[5] They considered the possibility of superfluidity of neutron matter due to pairing correlations induced by the attractive interaction between two neutrons that is effective in the $^1S_0$ partial wave of relative motion. Such pairing is of the conventional type studied by Bardeen, Cooper, and Schrieffer[6] in their theory of superconductivity. They found that this pairing was a significant contributor to the energy of the neutron fluid in the low-density regime. However, due to the increasing importance of the short-range repulsion in the $^1S_0$ partial wave, this effect decreased rapidly with increasing density. At about the density of neutrons in normal nuclear matter, the effects of pairing in the $^3P_2$ state of the two-neutron system become comparable to those of the $^1S_0$ pairing, and a transition from an isotropic $^1S_0$ superfluid to an anisotropic $^3P_2$ superfluid has been predicted to occur at this density.[7] This transition can be qualitatively understood on the basis of the neutron-neutron phase shifts. The $^1S_0$ and $^3P_2$ phase shifts cross at a wave vector that is characteristic of the Fermi momentum of neutron matter with a density of about 1.5×10^{14} g/cm^3. We therefore have the result that the ground state of neutron matter will always be that of a superfluid and that for densities less than that of nuclear matter, this superfluid will be a conventional isotropic one, while for higher densities it will be an anisotropic superfluid.

The ground-state and transition temperature of anisotropic superfluid neutron matter have been studied in detail by Hoffberg[8] and by Tamagaki.[9] These works are a direct outgrowth of the work of Balian and Werthamer[10] on isotropic superfluidity due to $^3P_0$ pairing. The complexity of these two studies attests to the many complications that arise in an anisotropic superfluid in which the order parameter or gap is a second-rank tensor rather than a scalar as is the case for isotropic superfluids.

In this paper we extend the study of anisotropic superfluid neutron matter to nonuniform states at temperatures just below the transition temperature. This is done by developing an appropriately generalized Ginzburg-Landau theory[11] for the gap tensor. This theory is developed by generalizing Gor'kov's[12] derivation of the original Ginzburg-Landau theory. The generalization involves two major points. First the $^1S_0$ contact interaction used by Gor'kov to obtain local equations for the thermal Green functions is generalized to a $^3P_2$ contact interaction. This introduces a derivative coupling in the interaction. The second point involves the use of a Cartesian tensor representation of the gap tensor rather than a spherical tensor representation which might at first appear to be the more natural representation. These two points folded into the Gor'kov procedure for deriving the Ginzburg-Landau theory yield a nonlinear tensor field equation for the gap tensor. This equation is the appropriate generalization of the Ginzburg-Landau equation to anisotropic superfluid neutron matter. This equation is then applied to situations in which the system has a nonuniform state such as flow past a plane boundary or a single vortex line[13] such as may be found in a rotating neutron star.

An outline of the paper is as follows: In Sec. II we discuss the model Hamiltonian of the system, derive the Gor'kov equations[14] for the thermal Green functions, and solve them for a uniform system. Then, in Sec. III, we derive a Ginzburg-Landau equation for the gap tensor from the Gor'kov equations and show that it reproduces the results of Sec. II for the uniform system. In Sec. IV we consider two solutions of the Ginzburg-Landau equations. We first consider superfluid flow past a plane boundary and show that the properties of the superfluid are the result of a very delicate balance between effects produced by the flow and effects produced by the boundary condition. The next solution we study is the one representing an isolated vortex line. We reduce such a solution to radial equations and derive the major qualitative features of the vortex from them. In the conclusion, Sec. V, we discuss our results and indicate what numerical and analytical work is yet to be done on the problem of nonuniform states of anisotropic superfluids.

II. MICROSCOPIC THEORY

Our microscopic theory of $^3P_2$ superfluid neutron matter is based upon a simple model Hamiltonian with a zero-range, attractive $^3P_2$ force between the neutrons. We begin this section with a discussion of this Hamiltonian. We then develop a theory for the thermal Green's functions of this system using the usual Gor'kov factorization[14] to truncate the hierarchy of equations. This theory is then applied to a translationally invariant system for which the physical significance of the theory is explicitly displayed.

We consider a system of neutrons interacting through a zero-range $^3P_2$ force which can be described by the Hamiltonian

$$H=\int\left[\psi^\dagger(\vec{r}\sigma)\left(-\frac{\nabla^2}{2m}-\mu+U(\vec{r})\right)\psi(\vec{r}\sigma)\right.$$
$$\left.-\tfrac{1}{2}g\,T^\dagger_{\alpha\beta}(\vec{r})T_{\alpha\beta}(\vec{r})\right]d^3r\,, \tag{2.1}$$

where $\psi^\dagger(\vec{r}\sigma)$ and $\psi(\vec{r}\sigma)$ are the neutron field operators at position $\vec{r}$ and spin projection $\sigma=\pm\frac{1}{2}$, and we have included an external potential $U(\vec{r})$ along with the kinetic energy and chemical potential in the first term. In the second term of (2.1), g is the positive interaction strength and the $T^\dagger_{\alpha\beta}$ and $T_{\alpha\beta}$ are the *Cartesian* components of $^3P_2$ pair creation and annihilation operators given by

$$T^\dagger_{\alpha\beta}(\vec{r})=\psi^\dagger(\vec{r}\sigma)t^*_{\alpha\beta}(\overleftarrow{\nabla}\sigma\sigma')\psi(\vec{r}\sigma')\,, \tag{2.2a}$$

$$T_{\alpha\beta}(\vec{r})=[t_{\alpha\beta}(\overleftrightarrow{\nabla}\sigma\sigma')\psi(\vec{r}\sigma')]\psi(\vec{r}\sigma)\,, \tag{2.2b}$$

where

$$t_{\alpha\beta}(\vec{\nabla}\sigma\sigma')=\tfrac{1}{2}[S_\alpha(\sigma\sigma')\nabla_\beta+S_\beta(\sigma\sigma')\nabla_\alpha]$$
$$-\tfrac{1}{3}\delta_{\alpha\beta}\vec{S}(\sigma\sigma')\cdot\vec{\nabla} \tag{2.3}$$

and

$$S_x=\frac{1}{\sqrt{2}}\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad S_y=\frac{i}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad S_z=\frac{1}{\sqrt{2}}\begin{pmatrix}0&1\\1&0\end{pmatrix}. \tag{2.4}$$

In Eqs. (2.1)–(2.3), we use the summation convention for repeated spinor indices, which are written on line as arguments of functions, and repeated tensor indices, which are written as subscripts. We will follow these conventions throughout this paper. The vector $\vec{S}$ in (2.4) is the vector operator that couples the two spins to the triplet state. Its components may be easily obtained from vector coupling coefficients. The tensor t in (2.3) is the symmetric traceless tensor formed from the two vectors $\vec{S}$ and $\vec{\nabla}$ and therefore has the desired $^3P_2$

transformation properties. The interaction in (2.1) is therefore an attractive, zero-range $^3P_2$ force which is written in terms of the Cartesian rather than spherical components of the tensor T.

In order to derive equations for the thermal Green functions generated by the Hamiltonian H, we introduce the Heisenberg field operators for imaginary time defined by

$$\Psi(\vec{r}\sigma\tau)=e^{\tau H}\psi(\vec{r}\sigma)e^{-\tau H} \tag{2.5a}$$

and

$$\overline{\Psi}(\vec{r}\sigma\tau)=e^{\tau H}\psi^\dagger(\vec{r}\sigma)e^{-\tau H}\,. \tag{2.5b}$$

These operators satisfy the equations of motion

$$\frac{\partial\Psi}{\partial\tau}(\vec{r}\sigma\tau)=\left(\frac{\nabla^2}{2m}+\mu-U(\vec{r})\right)\Psi(\vec{r}\sigma\tau)+g\{[t^*_{\alpha\beta}(\vec{\nabla}\sigma\sigma')\overline{\Psi}(\vec{r}\sigma'\tau)]T_{\alpha\beta}(\vec{r}\tau)+\tfrac{1}{2}\overline{\Psi}(\vec{r}\sigma'\tau)[t^*_{\alpha\beta}(\vec{\nabla}\sigma\sigma')T_{\alpha\beta}(\vec{r}\tau)]\} \tag{2.6a}$$

and

$$\frac{\partial\overline{\Psi}}{\partial\tau}(\vec{r}\sigma\tau)=\left(-\frac{\nabla^2}{2m}-\mu+U(\vec{r})\right)\overline{\Psi}(\vec{r}\sigma\tau)-g\{\overline{T}_{\alpha\beta}(\vec{r}\tau)[t_{\alpha\beta}(\vec{\nabla}\sigma\sigma')\Psi(\vec{r}\sigma'\tau)]+\tfrac{1}{2}[t_{\alpha\beta}(\vec{\nabla}\sigma\sigma')\overline{T}_{\alpha\beta}(\vec{r}\tau)]\Psi(\vec{r}\sigma'\tau)\}, \tag{2.6b}$$

where $\overline{T}_{\alpha\beta}(\vec{r}\tau)$ and $\overline{T}_{\alpha\beta}(\vec{r}\tau)$ are given by (2.2) with Ψ replacing ψ and $\overline{\Psi}$ replacing $\psi^\dagger$. These equations of motion may now be used to derive equations for the thermal Green's functions.

The one-particle thermal Green's function is defined by

$$\mathcal{G}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')\equiv-\langle T_\tau[\Psi(\vec{r}\sigma\tau)\overline{\Psi}(\vec{r}'\sigma'\tau')]\rangle\,, \tag{2.7}$$

where T_τ is the usual time-ordering operator on the imaginary time axis, and the angular brackets indicate an ensemble average in the grand canonical ensemble. Using (2.7), we obtain the equation of motion for $\mathcal{G}$,

$$\begin{aligned}\frac{\partial\mathcal{G}}{\partial\tau}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')=&-\delta(\sigma,\sigma')\delta(\vec{r}-\vec{r}')\delta(\tau-\tau')+\left(\frac{\nabla^2}{2m}+\mu-U(\vec{r})\right)\mathcal{G}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')\\&-g\langle T_\tau[\{[t^*_{\alpha\beta}(\vec{\nabla}\sigma\sigma'')\overline{\Psi}(\vec{r}\sigma''\tau)]T_{\alpha\beta}(\vec{r}\tau)+\tfrac{1}{2}\overline{\Psi}(\vec{r}\sigma''\tau)[t^*_{\alpha\beta}(\vec{\nabla}\sigma\sigma'')T_{\alpha\beta}(\vec{r}\tau)]\}\overline{\Psi}(\vec{r}'\sigma'\tau')]\rangle\,.\end{aligned} \tag{2.8}$$

We truncate the hierarchy of Green's-function equations by factoring the last term in (2.8). In so doing, we ignore the Hartree-Fock contributions to the single-particle energies. Those contributions may be taken into account by using an effective-mass approximation for the kinetic energy. These approximations then lead to the equation for $\mathcal{G}$,

$$\left(-\frac{\partial}{\partial\tau}+\frac{\nabla^2}{2m}+\mu-U(\vec{r})\right)\mathcal{G}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')-\hat{\Delta}(\vec{r}\sigma\sigma'')\overline{\mathcal{F}}(\vec{r}\sigma''\tau;\vec{r}'\sigma'\tau')=\delta(\sigma,\sigma')\delta(\vec{r}-\vec{r}')\delta(\tau-\tau')\,, \tag{2.9}$$

where

$$\hat{\Delta}(\vec{r}\sigma\sigma'')=\Delta_{\alpha\beta}(\vec{r})t^*_{\alpha\beta}(\vec{\nabla}\sigma\sigma'')+\tfrac{1}{2}[t^*_{\alpha\beta}(\vec{\nabla}\sigma\sigma'')\Delta_{\alpha\beta}(\vec{r})] \tag{2.10}$$

and

$$\Delta_{\alpha\beta}(\vec{r})=g\langle T_{\alpha\beta}(\vec{r})\rangle\,. \tag{2.11}$$

In (2.10), the function $\overline{\mathcal{F}}$ is defined by

$$\overline{\mathcal{F}}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')\equiv\langle T_\tau[\overline{\Psi}(\vec{r}\sigma\tau)\overline{\Psi}(\vec{r}'\sigma'\tau')]\rangle\,. \tag{2.12}$$

The equation of motion for $\overline{\mathcal{F}}$ derived from (2.7) is

$$\begin{aligned}\frac{\partial\overline{\mathcal{F}}}{\partial\tau}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')=&\left(-\frac{\nabla^2}{2m}-\mu+U(\vec{r})\right)\overline{\mathcal{F}}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')\\&-g\langle T_\tau[\{\overline{T}_{\alpha\beta}(\vec{r}\tau)[t_{\alpha\beta}(\vec{\nabla}\sigma\sigma'')\Psi(\vec{r}\sigma''\tau)]+\tfrac{1}{2}[t_{\alpha\beta}(\vec{\nabla}\sigma\sigma'')\overline{T}_{\alpha\beta}(\vec{r}\tau)]\Psi(\vec{r}\sigma''\tau)\}\overline{\Psi}(\vec{r}'\sigma'\tau')]\rangle\,.\end{aligned} \tag{2.13}$$

Using the same approximations to evaluate the last term of (2.13) that were used in obtaining (2.9), we obtain the second equation for $\overline{\mathcal{F}}$ and $\mathcal{G}$,

$$\left[\frac{\partial}{\partial\tau}+\frac{\nabla^2}{2m}+\mu-U(\vec{r})\right]\overline{\mathcal{F}}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau')-\hat{\Delta}^*(\vec{r}\sigma\sigma'')\mathcal{G}(\vec{r}\sigma''\tau;\vec{r}'\sigma'\tau')=0\,. \tag{2.14}$$

Note from (2.10) that $\hat{\Delta}$ is a differential operator and not just a number as in the original Gor'kov theory.

Equations (2.9) and (2.14) are completed by writing (2.11) in terms of $\overline{\mathcal{F}}$. This yields the relation

$$\Delta_{\alpha\beta}(\vec{r}) = -g\left[t_{\alpha\beta}(\vec{\nabla}\sigma'\sigma)\mathfrak{F}^*(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau)\right]_{\vec{r}'=\vec{r}}\,. \tag{2.15}$$

Equations (2.9), (2.14), and (2.15) are the Gor'kov equations for the $^3P_2$ superfluid neutron system. As usual, we expand the functions $\mathcal{G}$ and $\mathfrak{F}$ in a Fourier series in the variable $\tau-\tau'$,

$$\begin{aligned}\mathcal{G}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau') &= T\sum_n \mathcal{G}(\vec{r}\sigma,\vec{r}'\sigma';\omega_n)e^{-i\omega_n(\tau-\tau')}\,,\\ \mathfrak{F}(\vec{r}\sigma\tau;\vec{r}'\sigma'\tau') &= T\sum_n \mathfrak{F}(\vec{r}\sigma,\vec{r}'\sigma';\omega_n)e^{-i\omega_n(\tau-\tau')}\,,\end{aligned} \tag{2.16}$$

where we are using energy units for the temperature T and $\omega_n=(2n+1)\pi T$. We then have the equations

$$\left[i\omega+\nabla^2/2m+\mu-U(\vec{r})\right]\mathcal{G}(\vec{r}\sigma,\vec{r}'\sigma';\omega)-\hat{\Delta}(\vec{r}\sigma\sigma'')\mathfrak{F}(\vec{r}\sigma'',\vec{r}'\sigma';\omega)=\delta(\sigma,\sigma')\delta(\vec{r}-\vec{r}')\,, \tag{2.17}$$

$$\left[-i\omega+\nabla^2/2m+\mu-U(\vec{r})\right]\mathfrak{F}(\vec{r}\sigma,\vec{r}'\sigma';\omega)-\hat{\Delta}^*(\vec{r}\sigma\sigma'')\mathcal{G}(\vec{r}\sigma'',\vec{r}'\sigma';\omega)=0\,, \tag{2.18}$$

and

$$\Delta_{\alpha\beta}(\vec{r}) = -gT\sum_n\left[t_{\alpha\beta}(\vec{\nabla}\sigma'\sigma)\mathfrak{F}^*(\vec{r}\sigma,\vec{r}'\sigma';\omega)\right]_{\vec{r}=\vec{r}'}\,. \tag{2.19}$$

These equations are our starting point for obtaining Ginzburg-Landau equations for $\Delta_{\alpha\beta}(\vec{r})$ in Sec. III. However, before we derive the Ginzburg-Landau equations, we will discuss the solutions of (2.17), (2.18), and (2.19) for a uniform system.

For a uniform system, we have $U(\vec{r})=0$ and $\Delta_{\alpha\beta}(\vec{r})$ independent of $\vec{r}$. The functions $\mathcal{G}$ and $\mathfrak{F}$ then depend upon the variable $\vec{r}-\vec{r}'$ and we can Fourier transform with respect to this variable,

$$\begin{aligned}\mathcal{G}(\vec{r}\sigma,\vec{r}'\sigma';\omega) &= \frac{1}{(2\pi)^3}\int d^3k\;\mathcal{G}(\sigma\sigma';\vec{k}\omega)e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\,,\\ \mathfrak{F}(\vec{r}\sigma,\vec{r}'\sigma';\omega) &= \frac{1}{(2\pi)^3}\int d^3k\;\mathfrak{F}(\sigma\sigma';\vec{k}\omega)e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\,.\end{aligned} \tag{2.20}$$

We then have the equations

$$\begin{aligned}&(i\omega-\epsilon_k)\mathcal{G}(\sigma\sigma';\vec{k}\omega)-i\Delta(\vec{k}\sigma\sigma'')\mathfrak{F}(\sigma''\sigma';\vec{k}\omega)=\delta(\sigma,\sigma')\,,\\ &(-i\omega-\epsilon_k)\mathfrak{F}(\sigma\sigma';\vec{k}\omega)-i\Delta^*(\vec{k}\sigma\sigma'')\mathcal{G}(\sigma''\sigma';\vec{k}\omega)=0\,,\end{aligned} \tag{2.21}$$

where

$$\epsilon_k=\frac{k^2}{2m}-\mu\,,\qquad \Delta(\vec{k}\sigma\sigma'')=\Delta_{\alpha\beta}t^*_{\alpha\beta}(\vec{k}\sigma\sigma'')\,, \tag{2.22}$$

and

$$t^*_{\alpha\beta}(\vec{k}\sigma\sigma'')=\tfrac{1}{2}\left[S_\alpha(\sigma\sigma'')k_\beta+S_\beta(\sigma\sigma'')k_\alpha\right]-\tfrac{1}{3}\delta_{\alpha\beta}\vec{S}(\sigma\sigma'')\cdot\vec{k}\,. \tag{2.23}$$

We can rewrite Eqs. (2.21) as

$$\begin{aligned}&\mathfrak{F}(\sigma\sigma';\vec{k}\omega)=i\Delta^*(\vec{k}\sigma\sigma_1)\mathcal{G}(\sigma_1\sigma';\vec{k}\omega)/(-i\omega-\epsilon_k)\,,\\ &(\omega^2+\epsilon_k^{\,2})\mathcal{G}(\sigma\sigma';\vec{k}\omega)+\Delta(\vec{k}\sigma\sigma_1)\Delta^*(\vec{k}\sigma_1\sigma_2)\mathcal{G}(\sigma_2\sigma';\vec{k}\omega)=-\delta(\sigma,\sigma')(i\omega+\epsilon_k)\,.\end{aligned} \tag{2.24}$$

In solving (2.24), we follow Balian and Werthamer[10] and require that the gap $\Delta(\vec{k}\sigma\sigma')$ be a unitary matrix in spin space. This requirement has been shown to be equivalent to the requirement of time-reversal invariance on the trial density matrix.[8] We therefore write

$$\Delta(\vec{k}\sigma\sigma_1)\Delta^*(\vec{k}\sigma_1\sigma')=D^2(\vec{k})\delta(\sigma,\sigma')\,. \tag{2.25}$$

If we perform the indicated matrix multiplications in (2.25) using (2.22) and (2.23), we find that (2.25) can be written as $\vec{A}\times\vec{A}^*=0$, where the components of the vector $\vec{A}$ are defined by $A_\alpha=\Delta_{\alpha\beta}k_\beta$. This means that the real and imaginary parts of the vector $\vec{A}$ must be collinear and, furthermore, this must be true for all vectors $\vec{k}$. Thus, the tensor $\Delta_{\alpha\beta}$ must be real up to an arbitrary over-all phase factor. We can therefore satisfy (2.25) by requiring $\Delta_{\alpha\beta}$ to be real and then

$$D^2(\vec{k})=\tfrac{1}{2}k_\alpha\Delta_{\alpha\beta}\Delta_{\beta\gamma}k_\gamma\,. \tag{2.26}$$

Returning to Eqs. (2.24), we find that those solutions that satisfy (2.25) can be written as

$$\mathcal{G}(\sigma\sigma';\vec{k}\omega) = -\frac{i\omega+\epsilon_k}{\omega^2+\epsilon_k^2+D^2(\vec{k})}\,\delta(\sigma,\sigma')\,,$$
$$\overline{\mathcal{F}}(\sigma\sigma';\vec{k}\omega) = \frac{i\Delta^*(\vec{k}\sigma\sigma')}{\omega^2+\epsilon_k^2+D^2(\vec{k})}\,, \tag{2.27}$$

and the gap equation (2.19) becomes

$$\Delta_{\alpha\beta} = \frac{gT}{(2\pi)^3}\sum_n \int d^3k\, \frac{t_{\alpha\beta}(\vec{k}\sigma'\sigma)t^*_{\gamma\delta}(\vec{k}\sigma\sigma')\Delta_{\gamma\delta}}{\omega_n^2+\epsilon_k^2+D^2(\vec{k})}\,, \tag{2.28}$$

where we have used (2.20) and (2.27). The spin sums can be done using (2.23) and

$$S_\alpha(\sigma'\sigma)S^*_\beta(\sigma\sigma') = \delta_{\alpha\beta}\,, \tag{2.29}$$

yielding the result

$$\Delta_{\alpha\beta} = \frac{gT}{(2\pi)^3}\sum_n \int d^3k \left\{\tfrac{1}{4}[k_\alpha(\Delta_{\beta\gamma}+\Delta_{\gamma\beta})k_\gamma + k_\beta(\Delta_{\alpha\gamma}+\Delta_{\gamma\alpha})k_\gamma] - \tfrac{1}{3}\delta_{\alpha\beta}k_\gamma\Delta_{\gamma\delta}k_\delta\right\}/[\omega_n^2+E^2(\vec{k})]\,, \tag{2.30}$$

where we have discarded terms on the right-hand side which vanish due to the vanishing of the trace of $\Delta_{\alpha\beta}$. We have also defined $E(\vec{k})$ by

$$E(\vec{k}) = [\epsilon_k^2 + D^2(\vec{k})]^{1/2}\,. \tag{2.31}$$

We can now perform the sum on n in (2.30) to get a standard Bardeen, Cooper, and Schrieffer (BCS) type of integral equation for the gap tensor,

$$\Delta_{\alpha\beta} = \frac{g}{2(2\pi)^3}\int d^3k\, \frac{\tanh[E(\vec{k})/2T]}{E(\vec{k})}\left\{\tfrac{1}{4}[k_\alpha(\Delta_{\beta\gamma}+\Delta_{\gamma\beta})k_\gamma + k_\beta(\Delta_{\alpha\gamma}+\Delta_{\gamma\alpha})k_\gamma] - \tfrac{1}{3}\delta_{\alpha\beta}k_\gamma\Delta_{\gamma\delta}k_\delta\right\}. \tag{2.32}$$

In this equation $\Delta_{\alpha\beta}$ is real and manifestly symmetric and traceless.

The gap tensor is a real symmetric traceless tensor. It may be specified by giving the orientation of its principal axes and its two independent diagonal elements in its principal-axis coordinate system. For a uniform system which is rotationally invariant there is no external field that would specify the orientation of the principal-axis coordinate system. Therefore, this orientation is arbitrary. However, within this arbitrary coordinate system, we can find two simple solutions of (2.32). For the first solution, we have the nonvanishing components of Δ given by

$$\Delta_{xx} = \Delta_{yy} = -\tfrac{1}{2}\Delta_{zz} = d_1\,, \tag{2.33}$$

where d_1 is a real and positive constant which is determined by the equation

$$1 = \frac{g}{12(2\pi)^3}\int d^3k \tanh\left(\frac{E(\vec{k})}{2T}\right)\frac{\vec{k}^2+3k_z^2}{E(\vec{k})}\,, \tag{2.34}$$

with

$$E(\vec{k}) = [\epsilon_k^2 + \tfrac{1}{2}d_1^2(\vec{k}^2+3k_z^2)]^{1/2}\,. \tag{2.35}$$

This solution with its characteristic $1+3\cos^2\theta$ angular dependence of the energy gap has been reported earlier.[7] It has been studied in detail by Hoffberg[8] and has been shown by him to be the lowest-energy solution at zero temperature. In the second solution, the nonvanishing components of the gap tensor are given by

$$\Delta_{xx} = -\Delta_{yy} = d_2\,, \tag{2.36}$$

where d_2 is a real and positive constant which is determined by the equation

$$1 = \frac{g}{4(2\pi)^3}\int d^3k \tanh\left(\frac{E(\vec{k})}{2T}\right)\frac{k_x^2+k_y^2}{E(\vec{k})}\,, \tag{2.37}$$

with

$$E(\vec{k}) = [\epsilon_k^2 + \tfrac{1}{2}d_2^2(k_x^2+k_y^2)]^{1/2}\,. \tag{2.38}$$

This solution has a characteristic $\sin^2\theta$ angular dependence of the energy gap. Note that these two solutions both have the same transition temperature which is given by

$$1 = \frac{g}{6(2\pi)^3}\int d^3k \tanh\left(\frac{\epsilon_k}{2T_c}\right)\frac{\vec{k}^2}{\epsilon_k}\,. \tag{2.39}$$

This degeneracy in the neighborhood of the transition temperature will be shown to be a source of problems when we consider the solutions of the Ginzburg-Landau equations in the following sections.

The physical significance of the gap tensor may be easily obtained by rewriting (2.11) in terms of momentum states for the translationally invariant system that we have just considered. This leads to the expression

$$\Delta_{\alpha\beta} = \frac{-ig}{(2\pi)^3}\int d^3k\, t_{\alpha\beta}(\vec{k}\sigma\sigma')\langle a_{-\vec{k}\sigma'}a_{\vec{k}\sigma}\rangle\,. \tag{2.40}$$

Thus, $\Delta_{\alpha\beta}$ is just the amplitude for finding the pairs condensed into a $^3P_2$ bound state whose internal, magnetic quantum numbers are given in our Cartesian representation by the indices $\alpha\beta$. We may use this interpretation to give a physical picture of the two solutions of the preceding paragraph. The first solution (2.33) represents a condensation of the pairs into the state with $M=0$ and the second solution represents a condensation of the pairs into the two states with $|M|=2$. These are the two natural possibilities that are consistent with the restrictions that we have placed upon the gap. For, we have required that the state be invariant under time reversal. This requires that states with magnetic quantum numbers $\pm|M|$ must be populated with equal likelihood. We have also required that the gap tensor be diagonal. This second requirement excludes the possibility of populating states with $|M|=1$. Therefore, the two remaining possibilities $M=0$ and $|M|=2$ are indeed the two solutions to (2.32). The solution which leads to a condensation into the state with $M=0$ is energetically favorable at zero temperature because the condensation is into one magnetic substate rather than two. This implies a higher level of coherence in the first solution than there is in the second.

III. $^3P_2$ GINZBURG-LANDAU EQUATIONS

The basic equations for a $^3P_2$ superfluid neutron system are given by (2.17)–(2.19) and (2.10). For spatially nonhomogeneous systems, these equations are intractable. We therefore follow Gor'kov and confine our attention to temperatures near the transition temperature where $\Delta_{\alpha\beta}$ may be treated as a perturbation and a Ginzburg-Landau type of differential equation for the gap tensor may be derived. This gives a much more transparent and physical picture of the condensation in systems where the gap depends upon position. Furthermore, the qualitative features of the gap obtained from such a theory should remain valid at lower temperatures where $\Delta_{\alpha\beta}$ cannot be treated as a perturbation. We complete this section by writing down a variational principle for the Ginzburg-Landau equations which may be used to develop approximations for the gap tensor.

Following Gor'kov, we convert Eqs. (2.17) and (2.18) into integral equations using the thermal Green's function for the normal system $\mathcal{G}_0$ defined by

$$[i\omega+\nabla^2/2m+\mu-U(\vec{r})]\,\mathcal{G}_0(\vec{r}\sigma,\vec{r}\,'\sigma';\omega)=\delta(\sigma,\sigma')\delta(\vec{r}-\vec{r}\,')\,. \tag{3.1}$$

In order to simplify some of the expressions that follow, we will write the solution of (3.1) as

$$\mathcal{G}_0(\vec{r}\sigma,\vec{r}\,'\sigma';\omega)=\delta(\sigma,\sigma')\,\mathcal{G}_0(\vec{r},\vec{r}\,';\omega)\,. \tag{3.2}$$

Equations (2.17) and (2.18) can then be written as

$$\mathcal{G}(\vec{r}\sigma,\vec{r}\,'\sigma';\omega)=\delta(\sigma,\sigma')\,\mathcal{G}_0(\vec{r},\vec{r}\,';\omega)+\int d^3r_1\,\mathcal{G}_0(\vec{r},\vec{r}_1;\omega)\hat{\Delta}(\vec{r}_1\sigma\sigma_1)\overline{\mathcal{F}}(\vec{r}_1\sigma_1,\vec{r}\,'\sigma';\omega) \tag{3.3}$$

and

$$\overline{\mathcal{F}}(\vec{r}\sigma,\vec{r}\,'\sigma';\omega)=\int d^3r_1\,\mathcal{G}_0(\vec{r},\vec{r}_1;\omega)\hat{\Delta}^*(\vec{r}_1\sigma\sigma_1)\,\mathcal{G}(\vec{r}_1\sigma_1;\vec{r}\,'\sigma';\omega)\,. \tag{3.4}$$

We now substitute (3.3) into (3.4) and treat $\hat{\Delta}$ as a small perturbation. Iterating the resulting integral equation for $\overline{\mathcal{F}}$ once yields the expression

$$\begin{aligned}\overline{\mathcal{F}}(\vec{r}\sigma,\vec{r}\,'\sigma';\omega)=&\int d^3r_1\,\mathcal{G}_0(\vec{r},\vec{r}_1;-\omega)\hat{\Delta}^*(\vec{r}_1\sigma\sigma')\,\mathcal{G}_0(\vec{r}_1,\vec{r}\,';\omega)\\&+\int d^3r_1d^3r_2d^3r_3\,\mathcal{G}_0(\vec{r},\vec{r}_1;-\omega)\hat{\Delta}^*(\vec{r}_1\sigma\sigma_1)\,\mathcal{G}_0(\vec{r}_1,\vec{r}_2;\omega)\hat{\Delta}(\vec{r}_2\sigma_1\sigma_2)\,\mathcal{G}_0(\vec{r}_2,\vec{r}_3;-\omega)\hat{\Delta}^*(\vec{r}_3\sigma_2\sigma')\,\mathcal{G}_0(\vec{r}_3,\vec{r}\,';\omega)\,.\end{aligned} \tag{3.5}$$

This relates $\overline{\mathcal{F}}$ to the gap operator $\hat{\Delta}$. We may now substitute (3.5) into (2.19) to obtain the equation for the gap tensor:

$$\begin{aligned}\Delta_{\alpha\beta}(\vec{r})=-gT\sum_n\Big\{&\int d^3r_1\,[t_{\alpha\beta}(\vec{\nabla}\sigma'\sigma)\,\mathcal{G}_0(\vec{r},\vec{r}_1;\omega)]\hat{\Delta}(\vec{r}_1\sigma\sigma')\,\mathcal{G}_0(\vec{r}_1,\vec{r};-\omega)\\&+\int d^3r_1d^3r_2d^3r_3\,[t_{\alpha\beta}(\vec{\nabla}\sigma'\sigma)\,\mathcal{G}_0(\vec{r},\vec{r}_1;\omega)]\hat{\Delta}(\vec{r}_1\sigma\sigma_1)\,\mathcal{G}_0(\vec{r}_1,\vec{r}_2;-\omega)\\&\times\hat{\Delta}^*(\vec{r}_2\sigma_1\sigma_2)\,\mathcal{G}_0(\vec{r}_2,\vec{r}_3;\omega)\hat{\Delta}(\vec{r}_3\sigma_2\sigma')\,\mathcal{G}_0(\vec{r}_3,\vec{r};-\omega)\Big\}\,,\end{aligned} \tag{3.6}$$

where the gap operator $\hat{\Delta}$ is related to the gap tensor Δ by (2.10).

In evaluating the terms in (3.6), we use two approximations. We first use the $\mathcal{G}_0$ that is appropriate for the uniform system, i.e.,

$$\mathcal{G}_0(\vec{r}, \vec{r}\,'; \omega) = \frac{1}{(2\pi)^3} \int d^3k \;\; \mathcal{G}_0(\vec{k}, \omega) e^{i\vec{k}\cdot(\vec{r}-\vec{r}\,')}, \tag{3.7}$$

where

$$\mathcal{G}_0(\vec{k}, \omega) = (i\omega - \epsilon_k)^{-1}.$$

For temperatures near T_c, the spatial range of $\mathcal{G}_0$ is $0.2\times\hbar v_F/k_B T_c \sim 10^{-12}$ cm for typical neutron-star material. This leads us to our second approximation which is the assumption that $\Delta_{\alpha\beta}$ changes slowly compared to $\mathcal{G}_0$. This should be a good approximation for neutron-star matter for which the average interparticle spacing is about 2×10^{-13} cm at a density of 10^{14} g/cm^3.

We now apply these approximations to the linear term on the right-hand side of (3.6). The spin sums can be simplified by using

$$\mathcal{T}_{\alpha\beta\alpha'\beta'} \equiv \tfrac{1}{2}(\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \delta_{\alpha\beta'}\delta_{\beta\alpha'}) - \tfrac{1}{3}\delta_{\alpha\beta}\delta_{\alpha'\beta'} \tag{3.8}$$

and writing

$$t_{\alpha\beta}(\vec{\nabla}\sigma\sigma') = \mathcal{T}_{\alpha\beta\alpha'\beta'} S_{\alpha'}(\sigma\sigma') \frac{\partial}{\partial r_{\beta'}}, \quad \text{etc.}$$

The linear term in (3.6) then becomes

$$\begin{aligned} L_{\alpha\beta}(\vec{r};\omega) &\equiv \int d^3r_1 \, [t_{\alpha\beta}(\vec{\nabla}\sigma'\sigma)\mathcal{G}_0(\vec{r},\vec{r}_1;\omega)]\hat{\Delta}(\vec{r}_1\sigma\sigma')\mathcal{G}_0(\vec{r}_1,\vec{r};-\omega) \\ &= \mathcal{T}_{\alpha\beta\alpha'\beta'}\,\mathcal{T}_{\gamma\delta\gamma'\delta'} S_{\alpha'}(\sigma'\sigma) S^*_{\gamma'}(\sigma\sigma') \int d^3r_1 \frac{\partial \mathcal{G}_0}{\partial r_{\beta'}}(\vec{r},\vec{r}_1;\omega)\left[\Delta_{\gamma\delta}(\vec{r}_1)\frac{\partial \mathcal{G}_0}{\partial r_{1\delta'}}(\vec{r}_1,\vec{r};\omega) + \tfrac{1}{2}\Delta_{\gamma\delta,\delta'}(\vec{r}_1)\mathcal{G}_0(\vec{r}_1,\vec{r};\omega)\right], \end{aligned} \tag{3.9}$$

where we have used (2.10) for $\hat{\Delta}$, (3.8) for $t^*_{\alpha\beta}$, and are using the notation

$$\Delta_{\gamma\delta,\delta'}(\vec{r}) = \frac{\partial \Delta_{\gamma\delta}(\vec{r})}{\partial r_{\delta'}}. \tag{3.10}$$

The spin sum in (3.9) can be done using (2.29) and the second term in the integral can be simplified by an integration by parts yielding

$$L_{\alpha\beta}(\vec{r};\omega) = \mathcal{T}_{\alpha\beta\alpha'\beta'}\,\mathcal{T}_{\gamma\delta\gamma'\delta'}\frac{1}{2}\int d^3r_1\left[\frac{\partial \mathcal{G}_0}{\partial r_{\beta'}}(\vec{r},\vec{r}_1;\omega)\frac{\partial \mathcal{G}_0}{\partial r_{1\delta'}}(\vec{r}_1,\vec{r};-\omega) - \frac{\partial^2 \mathcal{G}_0}{\partial r_{\beta'}\partial r_{1\delta'}}(\vec{r},\vec{r}_1;\omega)\mathcal{G}_0(\vec{r}_1,\vec{r};-\omega)\right]\Delta_{\gamma\delta}(\vec{r}_1). \tag{3.11}$$

Now, assuming that $\Delta_{\gamma\delta}$ changes slowly compared to $\mathcal{G}_0$, we make the expansion

$$\Delta_{\gamma\delta}(\vec{r}_1) = \Delta_{\gamma\delta} + \Delta_{\gamma\delta,\mu}(r_1 - r)_\mu + \tfrac{1}{2}\Delta_{\gamma\delta,\mu\nu}(r_1-r)_\mu(r_1-r)_\nu, \tag{3.12}$$

where we have suppressed the argument $\vec{r}$ of $\Delta_{\gamma\delta}$ on the right-hand side and substitute it into (3.11). The second term on the right-hand side of (3.12) gives no contribution to (3.11) due to the spherical symmetry of $\mathcal{G}_0$. The remaining two terms in (3.12) lead to the result

$$\begin{aligned} L_{\alpha\beta}(\vec{r};\omega) = &-\left(\frac{1}{3(2\pi)^3}\int d^3k\, k^2 \mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)\right)\Delta_{\alpha\beta} - \left(\frac{1}{30(2\pi)^3}\int d^3k\,[\vec{k}\cdot\vec{\nabla}_k\,\mathcal{G}_0(\vec{k},\omega)][\vec{k}\cdot\vec{\nabla}_k\,\mathcal{G}_0(\vec{k},-\omega)]\right)\Delta_{\alpha\beta,\mu\mu} \\ &-\left(\frac{1}{30(2\pi)^3}\int d^3k\,[\vec{k}\cdot\vec{\nabla}_k\,\mathcal{G}_0(\vec{k},\omega)][\vec{k}\cdot\vec{\nabla}_k\,\mathcal{G}_0(\vec{k},-\omega)] - \frac{1}{2(2\pi)^3}\int d^3k\,\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)\right) \\ &\times(\Delta_{\alpha\mu,\beta\mu} + \Delta_{\beta\mu,\alpha\mu} - \tfrac{2}{3}\delta_{\alpha\beta}\Delta_{\mu\nu,\mu\nu}). \end{aligned} \tag{3.13}$$

We now turn to the evaluation of the cubic term in (3.6). In this term, we will neglect the spatial variation of $\Delta_{\alpha\beta}$ and therefore we can neglect the second term in (2.10). We then have

$$C_{\alpha\beta}(\vec{r};\omega)\equiv\int d^3r_1d^3r_2d^3r_3\,[t_{\alpha\beta}(\vec{\nabla}\sigma'\sigma)\mathcal{G}_0(\vec{r},\vec{r}_1;\omega)]\hat{\Delta}(\vec{r}_1\sigma\sigma_1)$$
$$\times\,\mathcal{G}_0(\vec{r}_1,\vec{r}_2;-\omega)\hat{\Delta}^*(\vec{r}_2\sigma_1\sigma_2)\mathcal{G}_0(\vec{r}_2,\vec{r}_3;\omega)\hat{\Delta}(\vec{r}_3\sigma_2\sigma')\mathcal{G}_0(\vec{r}_3,\vec{r};-\omega)$$
$$=\mathcal{T}_{\alpha\beta\alpha'\beta'}\,\mathcal{T}_{\gamma\delta\gamma'\delta'}\,\mathcal{T}_{\lambda\mu\lambda'\mu'}\,\mathcal{T}_{\nu\rho\nu'\rho'}\,S_{\alpha'}(\sigma'\sigma)S^*_{\gamma'}(\sigma\sigma_1)S_{\lambda'}(\sigma_1\sigma_2)S^*_{\nu'}(\sigma_2\sigma')$$
$$\times\,\Delta_{\gamma\delta}\Delta^*_{\lambda\mu}\Delta_{\nu\rho}\int d^3r_1d^3r_2d^3r_3\,\frac{\partial\mathcal{G}_0}{\partial r_{\beta'}}(\vec{r},\vec{r}_1;\omega)\frac{\partial\mathcal{G}_0}{\partial r_{1\delta'}}(\vec{r}_1,\vec{r}_2;-\omega)\frac{\partial\mathcal{G}_0}{\partial r_{2\mu'}}(\vec{r}_2,\vec{r}_3;\omega)\frac{\partial\mathcal{G}_0}{\partial r_{3\rho'}}(\vec{r}_3,\vec{r};-\omega)\,. \tag{3.14}$$

The tensor contractions in this expression can be simplified using $\mathcal{T}_{\gamma\delta\gamma'\delta'}\Delta_{\gamma\delta}=\Delta_{\gamma'\delta'}$. The trace over the spin variables can be done using the relation

$$S_\alpha S^*_\beta=\tfrac{1}{2}(\delta_{\alpha\beta}I+i\epsilon_{\alpha\beta\gamma}\sigma_\gamma)\,, \tag{3.15}$$

where I is the unit matrix and the σ_γ are the Pauli matrices. This yields the result for the spin trace

$$\mathrm{tr}\{S_{\alpha'}S^*_{\gamma'}S_{\lambda'}S^*_{\nu'}\}=\tfrac{1}{2}(\delta_{\alpha'\gamma'}\delta_{\lambda'\nu'}-\delta_{\alpha'\lambda'}\delta_{\gamma'\nu'}+\delta_{\alpha'\nu'}\delta_{\gamma'\lambda'})\,. \tag{3.16}$$

The spatial integral can be done in momentum space with the results

$$\int d^3r_1d^3r_2d^3r_3\,\frac{\partial\mathcal{G}_0}{\partial r_{\beta'}}(\vec{r},\vec{r}_1;\omega)\frac{\partial\mathcal{G}_0}{\partial r_{1\delta'}}(\vec{r}_1,\vec{r}_2;-\omega)\frac{\partial\mathcal{G}_0}{\partial r_{2\mu'}}(\vec{r}_2,\vec{r}_3;\omega)\frac{\partial\mathcal{G}_0}{\partial r_{3\rho'}}(\vec{r}_3,\vec{r};-\omega)$$
$$=\frac{1}{(2\pi)^3}\int d^3k\,k_{\beta'}k_{\delta'}k_{\mu'}k_{\rho'}[\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)]^2$$
$$=\frac{1}{15(2\pi)^3}\int d^3k\,k^4[\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)]^2(\delta_{\beta'\delta'}\delta_{\mu'\rho'}+\delta_{\beta'\mu'}\delta_{\delta'\rho'}+\delta_{\beta'\rho'}\delta_{\delta'\mu'})\,.$$

Combining these results, we have

$$C_{\alpha\beta}(\vec{r};\omega)=\mathcal{T}_{\alpha\beta\alpha'\beta'}\Delta_{\gamma'\delta'}\Delta^*_{\lambda'\mu'}\Delta_{\nu'\rho'}(\delta_{\alpha'\gamma'}\delta_{\lambda'\nu'}-\delta_{\alpha'\lambda'}\delta_{\gamma'\nu'}+\delta_{\alpha'\nu'}\delta_{\gamma'\lambda'})$$
$$\times\,(\delta_{\beta'\delta'}\delta_{\mu'\rho'}+\delta_{\beta'\mu'}\delta_{\delta'\rho'}+\delta_{\beta'\rho'}\delta_{\delta'\mu'})\frac{1}{30(2\pi)^3}\int d^3k\,[\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)]^2k^4\,.$$

Performing the remaining tensor contractions in this expression yields

$$C_{\alpha\beta}(\vec{r};\omega)=\frac{1}{30(2\pi)^3}\int d^3k\,k^4[\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)]^2\;(2\Delta_{\alpha\beta}\Delta^*_{\gamma\delta}\Delta_{\gamma\delta}-\Delta^*_{\alpha\beta}\Delta_{\gamma\delta}\Delta_{\gamma\delta}+2\Delta_{\alpha\gamma}\Delta^*_{\gamma\delta}\Delta_{\delta\beta}-\tfrac{2}{3}\delta_{\alpha\beta}\Delta_{\gamma\delta}\Delta^*_{\delta\lambda}\Delta_{\lambda\gamma})\,. \tag{3.17}$$

This completes the evaluation of $C_{\alpha\beta}$.

We now combine (3.6), (3.13), and (3.17) to obtain the equation

$$A\Delta_{\alpha\beta}-B_1\Delta_{\alpha\beta,\mu\mu}-B_2(\Delta_{\alpha\mu,\beta\mu}+\Delta_{\beta\mu,\alpha\mu}-\tfrac{2}{3}\delta_{\alpha\beta}\Delta_{\mu\nu,\mu\nu})$$
$$+C(2\Delta_{\alpha\beta}\Delta^*_{\gamma\delta}\Delta_{\gamma\delta}-\Delta^*_{\alpha\beta}\Delta_{\gamma\delta}\Delta_{\gamma\delta}+2\Delta_{\alpha\gamma}\Delta^*_{\gamma\delta}\Delta_{\delta\beta}-\tfrac{2}{3}\delta_{\alpha\beta}\Delta_{\gamma\delta}\Delta^*_{\delta\lambda}\Delta_{\lambda\gamma})=0\,, \tag{3.18}$$

where the constants are given by

$$A=1-\frac{gT}{3(2\pi)^3}\sum_n\int d^3k\,k^2\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)\,, \tag{3.19}$$

$$B_1=\frac{gT}{30(2\pi)^3}\sum_n\int d^3k\,[\vec{k}\cdot\vec{\nabla}_k\,\mathcal{G}_0(\vec{k},\omega)][\vec{k}\cdot\vec{\nabla}_k\,\mathcal{G}_0(\vec{k},-\omega)]\,, \tag{3.20}$$

$$B_2=B_1-\frac{gT}{2(2\pi)^3}\sum_n\int d^3k\,\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)\,, \tag{3.21}$$

and

$$C=\frac{gT}{30(2\pi)^3}\sum_n\int d^3k\,k^4[\mathcal{G}_0(\vec{k},\omega)\mathcal{G}_0(\vec{k},-\omega)]^2\,. \tag{3.22}$$

This equation is the desired Ginzburg-Landau equation for the gap tensor.

The constants may be evaluated using (3.19)–(3.22) and standard techniques with the results

$$
\begin{aligned}
A &= (T - T_c)/2T_c\,,\\
B_1 &= \frac{7\zeta(3)}{40\pi^2}\left(\frac{\epsilon_F}{T_c}\right)^2 \frac{1}{k_F^{\,2}} \cong 150\ \mathrm{fm}^2\,,\\
B_2 &= B_1 - \frac{3}{2k_F^{\,2}} \cong B_1\,,\\
C &= m^2 B_1 \cong \tfrac{1}{10}\ (\mathrm{MeV\,fm})^{-2}\,,
\end{aligned}
\tag{3.23}
$$

where $\zeta(3)$ is the ζ function and we have used (3.19) to estimate g with the result

$$
g = \frac{3\pi}{mk_F^{\,3}} \cong 40\ \mathrm{MeV\,fm^5}\,. \tag{3.24}
$$

The numerical estimates in (3.23) and (3.24) are for a density of 6×10^{14} g/cm^3. In what follows, we will ignore the small difference between B_1 and B_2 and we will use B to denote their common value. With this approximation, g is a factor common to A, B, and C and the estimate (3.24) does not affect the equations.

In order to develop approximate solutions of Eqs. (3.18) we will now write down a variational principle for the gap tensor. Following Ginzburg and Landau, we will identify this variational principle with the one for the free-energy difference between the normal and superfluid state. Taking advantage of the symmetry of $\Delta_{\alpha\beta}$ and its vanishing trace, we obtain the following expression for the free-energy difference:

$$
\Delta\mathfrak{F} \equiv \mathfrak{F}_s - \mathfrak{F}_n = \frac{1}{g}\int d^3r\,\{A\Delta^*_{\alpha\beta}\Delta_{\alpha\beta} + B(\Delta^*_{\alpha\beta,\mu}\Delta_{\alpha\beta,\mu} + 2\Delta^*_{\alpha\mu,\mu}\Delta_{\alpha\nu,\nu}) + \tfrac{1}{2}C[2(\Delta^*_{\alpha\beta}\Delta_{\alpha\beta})^2 - |\Delta_{\alpha\beta}\Delta_{\alpha\beta}|^2 + 2\Delta^*_{\alpha\beta}\Delta_{\beta\gamma}\Delta^*_{\gamma\delta}\Delta_{\delta\alpha}]\}\,, \tag{3.25}
$$

where we have ignored some surface terms. These surface terms imply the boundary conditions

$$
\Delta^*_{\alpha\beta}\Delta_{\alpha\beta,\mu}n_\mu = n_\alpha\Delta^*_{\alpha\beta}\Delta_{\beta\mu,\mu} = 0 \tag{3.26}
$$

on the surface, where $\vec{n}$ is the normal to the surface. These conditions are satisfied if $\Delta_{\alpha\beta}=0$ or $\Delta_{\alpha\beta,\mu}=0$ on the surface. However, they must be checked for each particular geometry. The free-energy difference $\Delta\mathfrak{F}$ is to be minimized subject to the boundary conditions (3.26). When $\Delta_{\alpha\beta}$ satisfies Eqs. (3.18), the value of $\Delta\mathfrak{F}$ is given by

$$
\Delta\mathfrak{F} = -\frac{C}{2g}\int d^3r\,[2(\Delta^*_{\alpha\beta}\Delta_{\alpha\beta})^2 - |\Delta_{\alpha\beta}\Delta_{\alpha\beta}|^2 + 2\Delta^*_{\alpha\beta}\Delta_{\beta\gamma}\Delta^*_{\gamma\delta}\Delta_{\delta\alpha}]\,. \tag{3.27}
$$

Otherwise, the full expression (3.25) must be used.

For a uniform system, we know the form of $\Delta_{\alpha\beta}$ which is given by (2.33) or (2.36). We may use (3.18) to obtain the values of d_1 and d_2 for these two solutions with the results

$$
d_1 = \left(\frac{|A|}{12C}\right)^{1/2}, \quad d_2 = \left(\frac{|A|}{4C}\right)^{1/2}. \tag{3.28}
$$

The free-energy densities for these two solutions are the same and are given by

$$
\Delta f = -\frac{A^2}{4gC} \cong -\left(1-\frac{T}{T_c}\right)^2 1.5\times 10^{-2}\ \mathrm{MeV\,fm^{-3}}\,. \tag{3.29}
$$

Note that not only do these two solutions have the same transition temperature, but they are also degenerate to this order in the Ginzburg-Landau expansion. In Sec. IV we turn to solutions of the equations for nonuniform systems.

IV. SOLUTIONS OF THE GINZBURG-LANDAU EQUATIONS

We now turn to the study of the explicit structure of two solutions of Eqs. (3.18). The first solution describes flow of the superfluid past a plane boundary and is included mainly to exhibit the structure of the equations. The second solution describes an isolated vortex line in the superfluid and is applicable to the theory of rotating neutron stars.

A. Flow Past a Plane Boundary

We consider the situation of superfluid flow in the y direction past a boundary formed by the yz plane. The gap tensor then depends upon y through a phase describing the flow and on x due to the boundary condition that the gap vanish at $x=0$. Therefore, the gap has the form

$$\Delta_{\alpha\beta}(\vec{r})=\left(\frac{|A|}{12C}\right)^{1/2} D_{\alpha\beta}(\xi)e^{i\kappa\eta}, \tag{4.1}$$

where the first factor is taken out for dimensional reasons [see (3.28)], $D_{\alpha\beta}$ is a dimensionless tensor to be determined, ξ and η are dimensionless distances in the x and y directions given by

$$\xi=\left(\frac{|A|}{B}\right)^{1/2} x, \quad \eta=\left(\frac{|A|}{B}\right)^{1/2} y, \tag{4.2}$$

and κ is the dimensionless wave vector describing the flow,

$$\kappa=\left(\frac{B}{|A|}\right)^{1/2} k. \tag{4.3}$$

Substituting (4.1) into (3.18) yields the equation

$$D_{\alpha\beta}''+\delta_{\alpha 1}(D_{\beta 1}''+i\kappa D_{\beta 2}')+\delta_{\beta 1}(D_{\alpha 1}''+i\kappa D_{\alpha 2}')+i\kappa(D_{\beta 1}'+i\kappa D_{\beta 2})\delta_{\alpha 2}+i\kappa\delta_{\beta 2}(D_{\alpha 1}'+i\kappa D_{\alpha 2})$$
$$-\tfrac{2}{3}\delta_{\alpha\beta}(D_{11}''+2i\kappa D_{12}'-\kappa^2 D_{22})+(1-\kappa^2)D_{\alpha\beta}-\tfrac{1}{12}(2D_{\alpha\beta}D^*_{\gamma\delta}D_{\gamma\delta}-D^*_{\alpha\beta}D_{\gamma\delta}D_{\gamma\delta}+2D_{\alpha\gamma}D^*_{\gamma\delta}D_{\delta\beta}-\tfrac{2}{3}\delta_{\alpha\beta}D_{\gamma\delta}D^*_{\delta\lambda}D_{\lambda\gamma})=0 \tag{4.4}$$

for the tensor D. In (4.4), the primes refer to differentiation with respect to ξ. Study of this equation shows that the tensor D can be written as

$$D(\xi)=\begin{pmatrix} f_1 & i\kappa g & 0 \\ i\kappa g & f_2 & 0 \\ 0 & 0 & -f_1-f_2 \end{pmatrix}, \tag{4.5}$$

where f_1, f_2, and g are real functions of ξ. Substituting (4.5) into (4.4), we get the equations

$$7f_1''-2\kappa^2 g'+3(1-\kappa^2)f_1+2\kappa^2 f_2-(f_1^2+f_1f_2+f_2^2)f_1-\tfrac{1}{3}(7f_1-2f_2)\kappa^2 g^2=0, \tag{4.6a}$$

$$3f_2''-2f_1''-2\kappa^2 g'+(3-7\kappa^2)f_2-(f_1^2+f_1f_2+f_2^2)f_2-\tfrac{1}{3}(7f_2-2f_1)\kappa^2 g^2=0, \tag{4.6b}$$

$$2g''+f_1'+f_2'+(1-2\kappa^2)g-\tfrac{1}{3}(2f_1^2+f_1f_2+2f_2^2+\kappa^2 g^2)g=0, \tag{4.6c}$$

where in the last equation, we have factored out an over-all factor of κ.

We first consider the solutions of (4.6) when there is no flow and $\kappa=0$. Equations (4.6a) and (4.6b) then become

$$\begin{aligned} &7f_1''+3f_1-(f_1^2+f_1f_2+f_2^2)f_1=0,\\ &3f_2''-2f_1''+3f_2-(f_1^2+f_1f_2+f_2^2)f_2=0. \end{aligned} \tag{4.7}$$

There are two solutions to these equations corresponding to the two solutions for the uniform system given by (3.28). They are $f_1=-2f_2$ and

$$D^{(1)}(\xi)=\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tanh(\tfrac{3}{14})^{1/2}\xi, \tag{4.8}$$

and $f_1=0$ and

$$D^{(2)}(\xi)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \sqrt{3}\tanh\frac{\xi}{\sqrt{2}}. \tag{4.9}$$

For large ξ, these two solutions approach those given by (3.28) for the uniform system with the one difference that the role of the x axis has been interchanged with that of the z axis. These two solutions have the following properties: (1) They are degenerate for the uniform system to this order in the Ginzburg-Landau expansion. (2) Solution one has lower bulk free energy when higher-order terms in the expansion are kept, i.e., at lower temperatures. (3) Solution two has the shorter coherence length and therefore lower surface energy. As a consequence of these properties, we see that surface effects will make the superfluid transition a two-stage transition as the temperature is lowered through T_c. The first stage will be to solution two which has a lower surface energy, but the same bulk energy as solution one at the transition temperature. The second stage will be from solution two to one as the contribution to the bulk energy of the higher-order terms in the Ginzburg-Landau expansion dominates the surface energy. This should give an interesting structure to the fluctuations in the neighborhood of $T=T_c$.

The other limiting case of Eqs. (4.6) is for $\xi\gg 1$ and $\kappa\neq 0$. In this limit, we assume that f_1 and f_2 approach constants a and b and $g\to 0$. We then have the equations for a and b

$$[3-3\kappa^2-(a^2+ab+b^2)]a+2\kappa^2 b=0\,,$$
$$[3-7\kappa^2-(a^2+ab+b^2)]b=0\,. \qquad (4.10)$$

These equations have two solutions which yield the results

$$D^{(1)}\to(1-\tfrac{7}{3}\kappa^2)^{1/2}\begin{pmatrix}1&0&0\\0&-2&0\\0&0&1\end{pmatrix},\quad \xi\to\infty \qquad (4.11)$$

and

$$D^{(2)}\to[3(1-\kappa^2)]^{1/2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix},\quad \xi\to\infty\,. \qquad (4.12)$$

Comparing (4.8) and (4.9) with (4.11) and (4.12), we see that the boundary condition at $\xi=\infty$ changes discontinuously at $\kappa=0$. The tensors (4.8) and (4.9) have their preferred direction along the x axis which is normal to the boundary, while the tensors (4.11) and (4.12) have their preferred direction along the y axis which is the direction of flow. This competition between flow effects and boundary effects is an important feature of the structure of a vortex line which we now turn to.

B. Isolated Vortex Line

From the preceding section, we see that in general the gap tensor is not real or diagonal in any coordinate system for a situation in which there is flow. We can, however, describe the gap tensor in a local coordinate system in which the real part is diagonal and the off-diagonal part is purely imaginary. The orientation of this local coordinate system may depend upon position, and this is the case for an isolated vortex whose properties we will now discuss.

For an isolated vortex, we seek a solution of the equations with cylindrical symmetry that has a phase of $m\varphi$, where m measures the circulation about the vortex line. The considerations of the previous subsection suggest the form

$$\Delta_{\alpha\beta}(\vec{r})=\left(\frac{|A|}{12C}\right)^{1/2}R_{\alpha\alpha'}(\varphi)R_{\beta\beta'}(\varphi)D_{\alpha'\beta'}(\rho)e^{im\varphi}\,, \qquad (4.13)$$

where the first factor is taken out for dimensional reasons, the R's are rotation matrices which rotate the tensor D from axes oriented along the ρ, φ, and z directions to the fixed xyz directions,

$$R(\varphi)=\begin{pmatrix}\cos\varphi&-\sin\varphi&0\\\sin\varphi&\cos\varphi&0\\0&0&1\end{pmatrix}, \qquad (4.14)$$

D is the gap tensor described in its natural frame of reference with axes along the ρ, φ, and z directions, ρ is the dimensionless distance from the vortex line,

$$\rho=[(|A|/B)(x^2+y^2)]^{1/2}\,, \qquad (4.15)$$

and the final factor is the phase factor specifying the circulation about the vortex. Substitution of (4.13) into (3.18) leads to the form

$$D(\rho)=\begin{pmatrix}f_1&img&0\\img&f_2&0\\0&0&-f_1-f_2\end{pmatrix}, \qquad (4.16)$$

where f_1, f_2, and g are real functions satisfying the equations

$$7f_1''+\frac{1}{\rho}(5f_1'-4f_2'-2m^2g')+3\left(1-\frac{m^2+4}{\rho^2}\right)f_1+\frac{2(m^2+6)}{\rho^2}f_2$$
$$+\frac{22m^2}{\rho^2}g-(f_1^2+f_1f_2+f_2^2)f_1-\tfrac{1}{3}(7f_1-2f_2)m^2g^2=0\,, \qquad (4.17a)$$

$$3f_2''-2f_1''+\frac{1}{\rho}(2f_1'+5f_2'-2m^2g')+\frac{12}{\rho^2}f_1+\left(3-\frac{7m^2+12}{\rho^2}\right)f_2$$
$$-\frac{26m^2}{\rho^2}g-(f_1^2+f_1f_2+f_2^2)f_2-\tfrac{1}{3}(7f_2-2f_1)m^2g^2=0\,, \qquad (4.17b)$$

and

$$2g''+\frac{1}{\rho}(f_1'+f_2'+2g')+\frac{3}{\rho^2}f_1-\frac{5}{\rho^2}f_2+\left(1-\frac{2m^2+8}{\rho^2}\right)g-\tfrac{1}{3}(2f_1^2+f_1f_2+2f_2^2+m^2g^2)g=0\,, \qquad (4.17c)$$

where we have factored out an over-all factor of m in (4.17c). These equations determine the behavior of the "radial" tensor D. The free energy per unit length of vortex relative to that of the uniform system is given by (3.27) which can be written as

$$\frac{\pi|A|B}{18gC}\int_0^{\rho_{\max}}\rho\,d\rho[9-(f_1^2+f_1f_2+f_2^2)^2-2m^2(2f_1^2+f_1f_2+2f_2^2)g^2-m^4g^4]\,. \tag{4.18}$$

In order to gain some familiarity with Eqs. (4.17), we first consider the special case with $m=0$. The equations then become

$$7f_1''+\frac{1}{\rho}(5f_1'-4f_2')+3\left(1-\frac{4}{\rho^2}\right)f_1+\frac{12}{\rho^2}f_2-(f_1^2+f_1f_2+f_2^2)f_1=0\,, \tag{4.19a}$$

$$3f_2''-2f_1''+\frac{1}{\rho}(2f_1'+5f_2')+\frac{12}{\rho^2}f_1+3\left(1-\frac{4}{\rho^2}\right)f_2-(f_1^2+f_1f_2+f_2^2)f_2=0\,, \tag{4.19b}$$

and $g=0$. Clearly, one solution of these equations is $f_1=f_2=1$. This corresponds to the first solution for the uniform system given by (2.33). The second solution for the uniform system given by (2.36) is not allowed as a solution of (4.19) because the resulting gap tensor is not independent of φ.

For nonuniform solutions of (4.19), we investigate the behavior of the solutions for small and large ρ. For small ρ, we have, in addition to the uniform solution, a solution that behaves like

$$f_1\to a\rho^2\,,\quad f_2\to -3a\rho^2\,,\quad \rho\to 0\,, \tag{4.20}$$

where a is a constant that must be determined by numerical integration of Eqs. (4.19). For large ρ, there are two solutions in addition to the uniform solution. They are given by

$$f_1^{(1)}\to 1-4\sqrt{3}/\rho^2\,,\quad f_2^{(1)}\to 1+4\sqrt{3}/\rho^2\,,\quad \rho\to\infty\,, \tag{4.21}$$

and

$$f_1^{(2)}\to\sqrt{3}(1-4/\rho^2)\,,\quad f_2^{(2)}\to -\sqrt{3}(1-4/\rho^2)\,,\quad \rho\to\infty\,. \tag{4.22}$$

The first of these two solutions is asymptotically the solution (2.33) of the uniform system. The second becomes asymptotically the solution (2.36) of the uniform system in the local coordinate system formed by the ρ, φ, and z directions. However, it does not become a uniform solution itself.

The free energy of a vortex in a neutral superfluid characteristically diverges like $\ln\rho_{\max}$ in (4.18). However, due to a cancellation of terms, the first solution (4.21) does not have a logarithmically divergent free energy and, even though it approaches the uniform limit slowly like $1/\rho^2$, it will have a finite free energy in an infinite system. The second solution does have a logarithmically divergent free energy which is given by

$$\frac{2\pi|A|B}{gC}\ln\rho_{\max}\cong 110\left(1-\frac{T}{T_c}\right)\ln\rho_{\max}\ \mathrm{MeV/fm}\,. \tag{4.23}$$

The vortices with $|m|=1$ are the physically interesting ones, and we shall now study this case in some detail. For $m=1$, Eqs. (4.17) become

$$7f_1''+\frac{1}{\rho}(5f_1'-4f_2'-2g')+\frac{1}{\rho^2}(-15f_1+14f_2+22g)+[3-(f_1^2+f_1f_2+f_2^2)]f_1-\tfrac{1}{3}(7f_1-2f_2)g^2=0\,, \tag{4.24a}$$

$$3f_2''-2f_1''+\frac{1}{\rho}(2f_1'+5f_2'-2g')+\frac{1}{\rho^2}(12f_1-19f_2-26g)+[3-(f_1^2+f_1f_2+f_2^2)]f_2-\tfrac{1}{3}(7f_2-2f_1)g^2=0\,, \tag{4.24b}$$

and

$$2g''+\frac{1}{\rho}(f_1'+f_2'+2g')+\frac{1}{\rho^2}(3f_1-5f_2-10g)+[1-\tfrac{1}{3}(2f_1^2+f_1f_2+2f_2^2+g^2)]g=0\,. \tag{4.24c}$$

We can get a fairly clear picture of the solutions of these equations by studying their behavior for $\rho\to 0$ and $\rho\to\infty$.

For $\rho\to 0$, Eqs. (4.24) have solutions that vanish like ρ and ρ^3. This is a special case of the more general situation for Eqs. (4.17) which has solutions which vanish like $\rho^{|m|}$ and $\rho^{|m|+2}$. For $m=1$, the solution that vanishes like ρ is doubly degenerate. Two linearly independent solutions are

$$f_1=f_2=a\rho\,,\quad g=0\,, \tag{4.25a}$$

and

$$f_1 = -f_2 = g = a\rho . \tag{4.25b}$$

The precise form of the solution and the numerical value of the constant a can only be determined by numerical integration of Eqs. (4.24). We have chosen the solutions (4.25) for their resemblance to the solutions of the uniform system. Thus (4.25a) corresponds to (2.33) and (4.25b) to (2.36).

For $\rho \to \infty$, we have a much more complicated situation and we must carefully treat the nonlinear terms in (4.25) in order to get the asymptotic behavior of the solutions. The method of expansion is as follows: The expansions

$$f_1 = a + a_1/\rho^2 + a_2/\rho^4, \quad f_2 = b + b_1/\rho^2 + b_2/\rho^4, \quad g = c_1/\rho^2 \tag{4.26}$$

are first substituted into Eqs. (4.24) and the coefficients of the various powers of $1/\rho^2$ are equated to zero. This yields the system of equations for the coefficients:

$$a^2 + ab + b^2 = 3, \tag{4.27a}$$

$$a(2a+b)a_1 + a(a+2b)b_1 = -15a + 14b, \tag{4.27b}$$

$$b(2a+b)a_1 + b(a+2b)b_1 = 12a - 19b, \tag{4.27c}$$

$$a(2a+b)a_2 + a(a+2b)b_2 = 17a_1 + 22b_1 + 26c_1 - (3a+b)a_1^2 - 2(a+b)a_1b_1 - ab_1^2 - \tfrac{1}{3}(7a-2b)c_1^2, \tag{4.27d}$$

$$b(2a+b)a_2 + b(a+2b)b_2 = -4a_1 - 11b_1 - 22c_1 - ba_1^2 - 2(a+b)a_1b_1 - (a+3b)b_1^2 - \tfrac{1}{3}(7b-2a)c_1^2, \tag{4.27e}$$

$$c_1 = (9a - 15b)/(a^2+b^2). \tag{4.27f}$$

The lowest-order equation (4.27a) does not determine the leading terms a and b in f_1 and f_2. However, the first-order equations for a_1 and b_1, (4.27b) and (4.27c), are singular and the requirement that a_1 and b_1 be nonzero imposes an additional requirement on a and b which then allows their determination. The two equations for a and b are

$$\begin{aligned} &a^2 + ab + b^2 = 3, \\ &-6a^2 + 2ab + 7b^2 = 0. \end{aligned} \tag{4.28}$$

The solutions to these equations are

$$\begin{aligned} &a = [2 \mp 5(43)^{-1/2}]^{1/2}, \\ &b = \pm[2 \mp 8(43)^{-1/2}]^{1/2}. \end{aligned} \tag{4.29}$$

In order to determine the first-order corrections, we need to impose the requirement that the singular equations for a_2 and b_2, (4.27d) and (4.27e), have a solution. The resulting algebraic equations uniquely determine a_1 and b_1 once the values of a and b are chosen. Thus, there are two uniquely determined asymptotic solutions to Eqs. (4.24). For the first solution, we have

$$\begin{aligned} &f_1 \to 1.112 - 85.65/\rho^2, \\ &f_2 \to 0.8832 + 91.12/\rho^2, \\ &g \to -1.604/\rho^2, \end{aligned} \tag{4.30}$$

and for the second solution

$$\begin{aligned} &f_1 \to 1.662 - 8.051/\rho^2, \\ &f_2 \to -1.794 + 9.238/\rho^2, \\ &g \to 7.000/\rho^2. \end{aligned} \tag{4.31}$$

The free-energy density to be used in (4.19) is

$$\frac{\pi|A|B}{18gC}\rho[9 - (f_1^2 + f_1f_2 + f_2^2)^2 - 2(2f_1^2 + f_1f_2 + 2f_2^2)g^2 - g^4]. \tag{4.32}$$

For large ρ this becomes

$$\frac{\pi|A|B}{3gC}[17 \mp (43)^{1/2}]/\rho, \tag{4.33}$$

where the signs go with those in (4.29). Thus, both solutions have logarithmically divergent free energies of the form

$$200\left(1 - \frac{T}{T_c}\right)\ln\rho_{\max}\ \text{MeV/fm}$$

and

$$440\left(1 - \frac{T}{T_c}\right)\ln\rho_{\max}\ \text{MeV/fm}.$$

However, the coefficient of the divergent term is smaller for the first solution (4.30) than it is for the second one (4.31). Therefore, we conclude that the solution with the upper signs in (4.30) describes the vortex with lower free energy.

The two asymptotic solutions to the radial equations discussed in the preceding paragraph exhibit a number of interesting properties. First, neither of the solutions approach a solution of the uniform system. This is a manifestation of the degeneracy in the solutions for the uniform system. For the vortex line, we find that the centrifugal barrier splits this degeneracy, and the effects of this splitting persist out to $\rho = \infty$. On the other hand, the solutions almost make it to those of the uniform system. That is, the first solution (4.30) almost makes it to the first solution of the uniform system for which $f_1 = f_2 = 1$, and the second solution (4.31) almost makes it to the second solution of the uniform system in the local ρ, φ, z frame, for which $f_1 = -f_2 = \sqrt{3}$. This effect is a manifestation of the competition between flow properties and boundary-condition properties that was exhibited in the preceding section. From the second terms in the asymptotic expansion, we see that the first solution has a longer correlation length than the second which is in accord with our results on the plane boundary given in the preceding section. From these qualitative considerations one might expect that the first solution with the longer correlation length would have the higher free energy. However, the logarithmically divergent part of the free energy comes from the cross term resulting from the first and second terms in the asymptotic expansions of f_1 and f_2, and in the first solution these terms interfere destructively and in the second solution they interfere constructively. Thus, the solution with the longer correlation length and destructive interference has lower free energy than the solution with the shorter correlation length and constructive interference. The detailed numerical integration of Eqs. (4.24) and a variational treatment of $m=1$ vortices will be presented in a subsequent publication.

V. CONCLUSION

In this paper, we have derived Ginzburg-Landau equations for anisotropic superfluid neutron-star matter with $^3P_2$ pairing, and have studied their solutions for superfluid flow past a boundary and an isolated vortex. Since the validity of the Ginzburg-Landau expansion of the free energy is restricted to the temperature region near the transition temperature, these results are not immediately applicable to neutron stars which are essentially at zero temperature. However, the general features of vortices as given by the Ginzburg-Landau theory can be expected to hold for lower temperatures. Therefore, the study of such vortices within the framework of the Ginzburg-Landau theory is a necessary first step toward a complete theory of rotating neutron stars. Within this framework, there remains a good deal of numerical work to be done. Thus, the radial equations for f_1, f_2, and g can be integrated numerically to obtain some insight into the radial structure of the vortex and a better estimate of the free energy required to create a vortex. This numerical work will be reported elsewhere.

The basic defect in the present theory is the degeneracy that is a concomitant of the Ginzburg-Landau expansion of the free energy. This degeneracy does not exist at low temperatures.[8] Therefore, a proper theory of vortices in neutron stars must be a low-temperature theory that is not degenerate in lowest order. Such a theory might start with an adaptation of the variational principle of Eilenberger[15] or an extension of the work of Bardeen *et al.*[16] Such investigations are now in progress.

---

*Work supported in part by the National Science Foundation.

[1]T. Gold, Nature 218, 731 (1968); 221, 25 (1969).

[2]G. Baym, H. A. Bethe, and C. J. Pethick, Nucl. Phys. A175, 225 (1971).

[3]See, for example, B. D. Day, Rev. Mod. Phys. 39, 719 (1967).

[4]J. Nemeth and D. W. L. Sprung, Phys. Rev. 176, 1496 (1968); C. G. Wang, W. K. Rose, and S. L. Schlenker, Astrophys. J. 160, L117 (1970); E. Østgaard, Nucl. Phys. A154, 202 (1970); P. J. Siemens (unpublished).

[5]C.-H. Yang and J. W. Clark, Nucl. Phys. A174, 49 (1971).

[6]J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. 108, 1175 (1957).

[7]M. Hoffberg, A. E. Glassgold, R. W. Richardson, and M. Ruderman, Phys. Rev. Letters 24, 775 (1970).

[8]M. Hoffberg, thesis, New York University, 1970 (unpublished).

[9]R. Tamagaki, Progr. Theoret. Phys. (Kyoto) 44, 905 (1970).

[10]R. Balian and N. R. Werthamer, Phys. Rev. 131, 1553 (1963).

[11]V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. 20, 1064 (1950).

[12]L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959); 37, 1407 (1959) [Soviet Phys. JETP 9, 1364 (1959); 10, 998 (1960)].

[13]R. W. Richardson, Phys. Letters 35A, 342 (1971).

[14]L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 34, 735 (1958) [Soviet Phys. JETP 7, 505 (1958)].

[15]G. Eilenberger, Z. Physik 184, 427 (1965); 190, 142 (1966).

[16]J. Bardeen, R. Kümmel, A. E. Jacobs, and L. Tewordt, Phys. Rev. 187, 556 (1969).