tion, the scaling property can be shown to hold rigorously for a large class of relativistic field theories. (See comment E in Sec. V for further discussions.)

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[16]Cutkosky and Leon, Ref. 13.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 7 1 APRIL 1972

Radiative Corrections to the Decay $\pi^0\to\gamma e^+e^-$†

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(Received 15 December 1971)

We calculate the radiative corrections to the $\pi^0\to\gamma e^+e^-$ decay over the whole range of the Dalitz plot, with no restrictions on the radiative photon energy. The corrections are found to be negative and large in magnitude in the region of large invariant mass of the Dalitz pair, and small but positive for small values of the same quantity. The total correction to the decay rate, defined by $\Gamma^{\rm rad}(\pi^0\to\gamma e^+e^-)/\Gamma(\pi^0\to\gamma\gamma)$ is positive and agrees with the results of previous calculations.

I. INTRODUCTION

Radiative corrections to the general process

$$A\to Be^+e^-$$

were calculated recently by Lautrup and Smith,[1] using the soft-photon approximation. In this paper we concentrate on the specific decay

$$\pi^0\to\gamma e^+e^-$$

and include the hard-photon corrections so that our results are valid over the whole range of the Dalitz plot.

Interest in this process stems mainly from the possibility it affords of measuring the strong-interaction effects at the pion vertex. Since these effects are small, radiative corrections become important in such measurements.[2]

The branching ratio $\Gamma(\pi^0 \to \gamma e^+ e^-)/\Gamma(\pi^0 \to \gamma\gamma)$ was calculated by Dalitz[3] in 1951 with the result (neglecting strong-interaction effects):

$$\frac{\Gamma(\pi^0 \to \gamma e^+ e^-)}{\Gamma(\pi^0 \to \gamma\gamma)} = 0.01185. \tag{1.1}$$

Radiative corrections to the decay rate were calculated analytically in Ref. 1 and the result agrees with a previous numerical evaluation by Joseph[4]:

$$\frac{\Gamma^{\text{rad}}(\pi^0 \to \gamma e^+ e^-)}{\Gamma(\pi^0 \to \gamma\gamma)} = 1.05 \times 10^{-4}.$$

Experiments performed so far (or possible experiments at the Los Alamos Meson Facility) use, however, the differential decay rate, and hence require radiative corrections to the latter. Such experiments are designed to detect only the Dalitz pair, with no attempt at measuring the photon energy, which is necessary for the application of the soft-photon approximation in the radiative corrections.[1] The problem is further complicated by the ambiguity due to the identity of the radiative and decay photons. These difficulties are solved by removing the restriction that the radiative photon be soft. This we do in the present calculation.

In Sec. II we introduce our kinematical variables and present the lowest-order differential decay rate and the virtual-photon corrections to it. The discussion here follows almost identically the one in Ref. 1. Section III contains the matrix element for the bremsstrahlung corrections, and a discussion of phase-space integrals. We present our results in Sec. IV. In Appendix A we give the spin-and-polarization sum of the square of the bremsstrahlung diagrams, and list some phase-space integrals in Appendix B. Finally, in Appendix C we discuss the infrared-divergent integrals.

II. LOWEST-ORDER DIFFERENTIAL DECAY RATE AND VIRTUAL CORRECTIONS

In this section we quote the necessary definitions and main results of Secs. II–V A of Ref. 1 as applied to the pion decay.

The process $\pi^0 \to \gamma e^+ e^-$ is presented, in lowest order, in diagram (1) of Fig. 1. The momenta of the pion, photon, positron, and electron are denoted by p, k, q_1, and q_2, respectively. The invariant mass squared of the Dalitz pair (in units of m_π^2, m_π = pion mass) is

$$x = \frac{(q_1 + q_2)^2}{m_\pi^2}. \tag{2.1}$$

The variable y is defined as

$$y = \frac{2p \cdot (q_1 - q_2)}{m_\pi^2(1-x)}. \tag{2.2}$$

We also define the matrix element for the decay of π^0 into two photons (which may be virtual) by

$$\langle \gamma(k_1, \epsilon_1), \gamma(k_2, \epsilon_2) | T | \pi^0(p) \rangle = \frac{F}{m_\pi} f(p^2/m_\pi^2, k_1^2/m_\pi^2, k_2^2/m_\pi^2) \times \epsilon_{\mu\nu\rho\sigma} \epsilon_1^\mu \epsilon_2^\nu k_1^\rho k_2^\sigma,$$

where F is a dimensionless constant, and $f(p^2/m_\pi^2, k_1^2/m_\pi^2, k_2^2/m_\pi^2)$ is the "form factor" of the π^0 normalized to

$$f(1,0,0) = 1.$$

The decay rate into two real photons is given by

$$\Gamma_0 = \Gamma(\pi^0 \to \gamma\gamma) = \frac{m_\pi |F|^2}{64\pi}.$$

We can write for the lowest-order differential decay rate

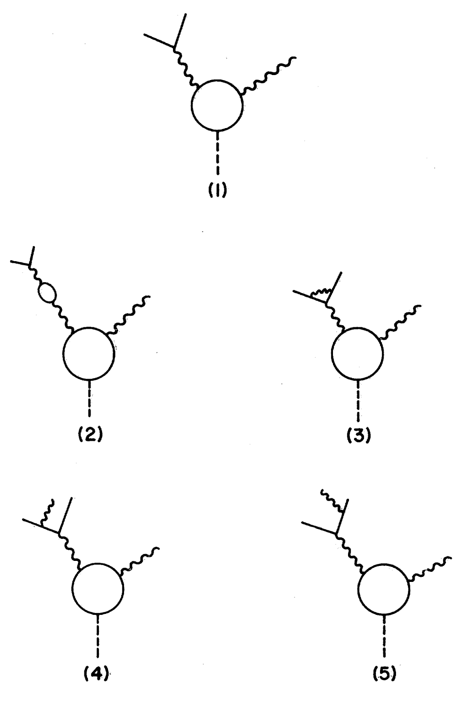

FIG. 1. (1) Lowest-order diagram for the decay $\pi^0 \to \gamma e^+ e^-$. (2), (3) Virtual corrections. (4), (5) Bremsstrahlung corrections.

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma(\pi^0\to\gamma e^+e^-)}{dxdy} = \frac{\alpha}{\pi}|f(1,0,x)|^2\frac{(1-x)^3}{4x}\left(1+y^2+\frac{r^2}{x}\right)$$

and (2.3)

$$\frac{1}{\Gamma_0}\frac{d\Gamma(\pi^0\to\gamma e^+e^-)}{dx} = \frac{\alpha}{\pi}|f(1,0,x)|^2\frac{2(1-x)^3}{3x}\beta\left(1+\frac{r^2}{2x}\right),$$

where (m = electron mass)

$$r^2=\frac{4m^2}{m_\pi{}^2},\quad \beta=\left(1-\frac{r^2}{x}\right)^{1/2}. \tag{2.4}$$

The limits on x and y are given by

$$r^2\le x\le 1,\quad -\beta\le y\le\beta.$$

The radiative corrections to $d^2\Gamma/dxdy$ are written as

$$\frac{d^2\Gamma^{\rm rad}}{dxdy}=\delta(x,y)\frac{d^2\Gamma}{dxdy}$$

and (2.5)

$$\frac{d\Gamma^{\rm rad}}{dx}=\delta(x)\frac{d\Gamma}{dx}.$$

δ is naturally divided into two parts,

$$\delta=\delta_{\rm virt}+\delta_{\rm brem}.$$

$\delta_{\rm virt}$ is obtained by considering the interference of diagram (1) with diagrams (2) and (3) of Fig. 1, which contribute through the second-order renormalized photon spectral function and the electromagnetic form factors of the electron.[5] The former is given by

$$\Pi^{(2)}(x)=\frac{\alpha}{\pi}\left[\tfrac{8}{9}-\tfrac{1}{3}\beta^2+\beta(\tfrac{1}{2}-\tfrac{1}{6}\beta^2)\ln\gamma\right], \tag{2.6}$$

with $\gamma=(1-\beta)/(1+\beta)$.

For the electromagnetic vertex in diagram (3) we write

$$-ie\bar{u}\left[\gamma_\mu(F_1^{(2)}+F_2^{(2)})-\frac{(q_2-q_1)_\mu}{2m}F_2^{(2)}\right]v,$$

with

$$F_1^{(2)}=\frac{\alpha}{\pi}\left[-1-\frac{1+2\beta^2}{4\beta}\ln\gamma - \frac{1+\beta^2}{2\beta}\left[\mathrm{Li}_2(1-\gamma)-\tfrac{1}{2}\pi^2+\tfrac{1}{4}\ln^2\gamma\right] + \left(1+\frac{1+\beta^2}{2\beta}\ln\gamma\right)\ln\frac{m}{\lambda}\right],$$

$$F_2^{(2)}=\frac{\alpha}{\pi}\frac{1-\beta^2}{4\beta}\ln\gamma,$$

where the dilogarithm $\mathrm{Li}_2(x)$ is defined as

$$\mathrm{Li}_2(x)=-\int_0^x\frac{\ln(1-t)}{t}dt,$$

and λ is the photon mass (infrared cutoff). [Note that in $F_1^{(2)}$ defined in Eq. (5.5) of Ref. 1 the factor $\frac{1}{4}\pi^2$ should read $\frac{1}{2}\pi^2$.]

In terms of these functions we have

$$\delta_{\rm virt}(x,y)=\delta_1(x)+\delta_2'(x)+\delta_2''(x,y), \tag{2.7}$$

where

$$\delta_1(x)=-2\Pi^{(2)}(x),$$

$$\delta_2'(x)=2(F_1^{(2)}+F_2^{(2)}),$$

$$\delta_2''(x,y)=2F_2^{(2)}(x)\frac{1-y^2-r^2/x}{1+y^2+r^2/x}.$$

[Equation (5.8) of Ref. 1 contains a misprint. It should have a factor of 2 on the right-hand side to agree with the equation for $\delta_2''(x,y)$.]

Similarly,

$$\delta_{\rm virt}(x)=\delta_1(x)+\delta_2'(x)+\delta_2''(x), \tag{2.8}$$

with

$$\delta_2''(x)=F_2^{(2)}(x)\frac{1-r^2/x}{1+r^2/2x}.$$

III. BREMSSTRAHLUNG CORRECTIONS

We compute $\delta_{\rm brem}$ through

$$\delta_{\rm brem}(x,y)=\frac{d^2\Gamma^{\rm brem}}{dxdy}\Big/\frac{d^2\Gamma}{dxdy}. \tag{3.1}$$

$d^2\Gamma^{\rm brem}/dxdy$ is obtained by considering diagrams (4) and (5) of Fig. 1, plus two others obtained by interchanging the radiative and decay photons. The amplitude is given by

$$M=\frac{e^2F}{m_\pi}\frac{f(1,0,l_1^2/m_\pi{}^2)}{l_1^2}\bar{u}(q_2)[\not{a}_2(-\not{q}_1-\not{k}_1-m)^{-1}\not{\epsilon}_1+\not{\epsilon}_1(\not{q}_2+\not{k}_1-m)^{-1}\not{a}_2]v(q_1)+(k_1\leftrightarrow k_2), \tag{3.2}$$

where $l_1=q_1+q_2+k_1$, $a_{2\mu}=\epsilon_{\mu\nu\rho\beta}\epsilon_2^\nu k_2^\rho l_1^\beta$, and $\epsilon_1^\alpha(k_1)$, $\epsilon_2^\alpha(k_2)$ are the polarization vectors of the photons with momenta k_1, k_2, respectively.

As discussed in the Introduction, the strong-interaction effects are expected to be small, hence we neglect the variation of $f(1,0,l_1^2/m_\pi^2)$ with respect to the photon momentum k_1, i.e.,

$$f(1,0,l_1^2/m_\pi^2)=f(1,0,x+2k_1\cdot(q_1+q_2)/m_\pi^2)\approx f(1,0,x).$$

In fact, one usually writes

$$f(1,0,x)=1+ax+\cdots,$$

where a is the "slope of the form factor"; hence we are neglecting terms of order $a\times\delta_{\rm brem}$, which is surely justifiable since a is a small number.[2]

The differential decay rate is given by

$$d\Gamma^{\rm brem}=\frac{1}{4m_\pi(2\pi)^8}\sum_{\rm spins,\,pol.}|M|^2d^4q_1\delta(q_1^2-m^2)d^4q_2\delta(q_2^2-m^2)d^4k_1\delta(k_1^2)d^4k_2\delta(k_2^2)\delta^4(p-q_1-q_2-k_1-k_2). \quad (3.3)$$

We have suppressed all the θ functions in Eq. (3.3).

We define Tr by

$$\sum_{\rm spins,\,pol.}|M|^2=\left(\frac{e^2Ff}{2m_\pi}\right)^2\mathrm{Tr}. \quad (3.4)$$

The structure of Tr, which is essentially the trace of the γ matrices occurring in $|M|^2$, is best seen by examining the various propagators involved. There are six of these:

$$A=k_1\cdot q_1,\quad B=k_1\cdot q_2,\quad C=k_2\cdot q_1,\quad D=k_2\cdot q_2,$$
$$E=(q_1+q_2+k_1)^2,\quad F=(q_1+q_2+k_2)^2,$$

not all of which are independent because

$$A+B-\tfrac{1}{2}E=C+D-\tfrac{1}{2}F.$$

As discussed below, the integration over phase space is considerably simplified by introducing the invariant mass squared of the two photons (in units of m_π^2),

$$x_\gamma=\frac{(k_1+k_2)^2}{m_\pi^2},$$

so that our five independent variables become x, y, x_γ, A, and B, since

$$C=L\cdot q_1-A,\qquad D=L\cdot q_2-B,$$
$$E=m_\pi^2x+2A+2B,\quad F=m_\pi^2(1-x_\gamma)-2A-2B, \quad (3.5)$$

where

$$L=p-q_1-q_2=k_1+k_2,$$
$$L\cdot q_1=\tfrac{1}{4}m_\pi^2[(1-x)(1+y)-x_\gamma],$$

and

$$L\cdot q_2=\tfrac{1}{4}m_\pi^2[(1-x)(1-y)-x_\gamma].$$

Because the two photons are identical, the absolute square of the sum of diagrams (4) and (5) in Fig. 1 is equal to the one obtained by interchanging the photons. Further simplification occurs when we use this symmetry on individual terms (e.g., $1/E=1/F$, $1/AE=1/CF$, etc.).

The trace was taken on the Brookhaven CDC 6600 using the SCHOONSCHIP program developed by M. Veltman. Only terms up to order m^2 were kept. The result, after applying the identities discussed above, is given in Appendix A.

We now turn to the phase-space integrals. Consider

$$\int d^4q_1\delta(q_1^2-m^2)d^4q_2\delta(q_2^2-m^2)d^4k_1\delta(k_1^2)d^4k_2\delta(k_2^2)\times\delta^4(p-q_1-q_2-k_1-k_2)\,\mathrm{Tr}(x,y,x_\gamma,A,B);$$

this can be written as

$$\frac{\pi^3m_\pi^4}{16}\int(1-x)dxdydx_\gamma J[\mathrm{Tr}(x,y,x_\gamma,A,B)],$$

where J is written as an operator

$$J=\frac{1}{2\pi}\int\frac{d^3k_1}{k_{10}}\frac{d^3k_2}{k_{20}}\delta^4(p-q_1-q_2-k_1-k_2),$$

and is essentially an integration over the direction of one photon momentum, affecting only A and B. The results, for various combinations of A and B, are given in Appendix B.

Equation (3.3) can thus be written as

$$\frac{1}{\Gamma_0}\frac{d^2\Gamma^{\rm brem}}{dxdy}=\frac{|f|^2}{64}\left(\frac{\alpha}{\pi}\right)^2(1-x)\times\int dx_\gamma J[\mathrm{Tr}(x,y,x_\gamma,A,B)]; \quad (3.6)$$

the limits on x_γ are

$$\frac{\lambda^2}{m_\pi^2}\le x_\gamma\le 1+x-\left(4x+\frac{y^2}{\beta^2}(1-x)^2\right)^{1/2},$$

where λ is again the photon mass which is set equal to zero except in the infrared-divergent integrals. There are three of these, since (see Appendix B) $J[1/A^2]$, $J[1/B^2]$, and $J[1/AB]$ behave like $1/x_\gamma$, so that there is a logarithmic divergence upon integration over x_γ (whenever no x_γ appears in the numerator). These integrals are evaluated in Appendix C with the results:

$$\int J\left[\frac{1}{A^2}\right]dx_\gamma = \frac{4}{m^2 m_\pi{}^2}\left\{\ln\frac{m}{\lambda} + \ln\frac{2x_\gamma^{\max}}{(1-x)(1+y)}\right\},$$

$$\int J\left[\frac{1}{B^2}\right]dx_\gamma = \frac{4}{m^2 m_\pi{}^2}\left\{\ln\frac{m}{\lambda} + \ln\frac{2x_\gamma^{\max}}{(1-x)(1-y)}\right\},$$

$$\int J\left[\frac{1}{AB}\right]dx_\gamma = -\frac{8}{m_\pi{}^4 x\beta}\ln\gamma\ln\frac{m}{\lambda} - \frac{4}{m_\pi{}^4 x\beta}\left\{2\ln\frac{2x_\gamma^{\max}}{1-x} + \tfrac{1}{2}\ln\frac{m_\pi{}^2 x}{m^2} - \ln\left(1-\frac{y^2}{\beta^2}\right)\right\}\ln\gamma$$
$$+\frac{4}{m_\pi{}^4 x\beta}\left\{\mathrm{Li}_2\left[\frac{1-\beta}{2}\right] - \mathrm{Li}_2\left[\frac{1+\beta}{2}\right] + \mathrm{Li}_2\left[\frac{y(1+\beta)}{y-\beta}\right] + \mathrm{Li}_2\left[\frac{y(1+\beta)}{y+\beta}\right] - \mathrm{Li}_2\left[\frac{y(1-\beta)}{y-\beta}\right] - \mathrm{Li}_2\left[\frac{y(1-\beta)}{y+\beta}\right]\right\}. \tag{3.7}$$

Substituting these expressions in Tr and extracting the divergent part of

$$\int J\left[\frac{1}{A^2E}\right]dx_\gamma \quad\text{and}\quad \int J\left[\frac{1}{B^2E}\right]dx_\gamma$$

through

$$\frac{1}{A^2E} = \frac{1}{m_\pi{}^2 x A^2} - \frac{2}{m_\pi{}^2 x}\left(\frac{1}{AE} + \frac{B}{A^2E}\right)$$

and a similar expression for $1/B^2E$, we see that the divergent part of $d^2\Gamma^{\text{brem}}/dxdy$ cancels against the divergent part of $d^2\Gamma^{\text{virt}}/dxdy = \delta_{\text{virt}} d^2\Gamma/dxdy$.

The integration over x_γ for the rest of the terms is performed numerically to obtain $d^2\Gamma^{\text{brem}}/dxdy$ and $\delta_{\text{brem}}(x,y)$ through Eq. (3.1), which, when combined with Eq. (2.7), yields $\delta(x,y) = \delta_{\text{virt}}(x,y) + \delta_{\text{brem}}(x,y)$, the radiative corrections to the Dalitz

TABLE I. $\delta(x,y)$ given in percent for a range of values of x and y (i.e., the Dalitz-plot corrections).

x \ y	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	0.99
0.01	2.77	2.72	2.61	2.46	2.28	2.07	1.79	1.41	0.81	−0.35	−5.64
0.02	2.77	2.74	2.64	2.50	2.32	2.09	1.79	1.37	0.72	−0.52	−5.82
0.03	2.70	2.67	2.58	2.45	2.27	2.04	1.73	1.30	0.62	−0.68	−6.07
0.04	2.60	2.57	2.49	2.36	2.19	1.96	1.65	1.20	0.51	−0.83	−6.30
0.05	2.48	2.46	2.39	2.27	2.10	1.87	1.56	1.10	0.39	−0.97	−6.51
0.06	2.37	2.35	2.29	2.17	2.00	1.78	1.46	1.00	0.28	−1.10	−6.71
0.07	2.25	2.23	2.17	2.06	1.90	1.67	1.36	0.90	0.17	−1.23	−6.89
0.08	2.14	2.12	2.05	1.95	1.79	1.57	1.25	0.79	0.05	−1.36	−7.07
0.09	2.02	2.00	1.94	1.84	1.69	1.46	1.15	0.68	−0.06	−1.48	−7.23
0.10	1.90	1.88	1.83	1.73	1.58	1.36	1.04	0.57	−0.17	−1.61	−7.39
0.15	1.33	1.31	1.26	1.17	1.03	0.82	0.51	0.03	−0.73	−2.20	−8.14
0.20	0.77	0.75	0.71	0.62	0.49	0.28	−0.04	−0.51	−1.28	−2.78	−8.83
0.25	0.21	0.20	0.15	0.07	−0.06	−0.27	−0.58	−1.06	−1.83	−3.36	−9.49
0.30	−0.35	−0.36	−0.41	−0.49	−0.62	−0.82	−1.13	−1.61	−2.39	−3.93	−10.14
0.35	−0.92	−0.93	−0.97	−1.05	−1.18	−1.39	−1.70	−2.18	−2.98	−4.52	−10.79
0.40	−1.50	−1.51	−1.56	−1.63	−1.76	−1.97	−2.28	−2.76	−3.56	−5.13	−11.45
0.45	−2.11	−2.12	−2.16	−2.24	−2.37	−2.58	−2.89	−3.38	−4.18	−5.75	−12.13
0.50	−2.74	−2.76	−2.80	−2.88	−3.01	−3.21	−3.53	−4.02	−4.82	−6.41	−12.84
0.55	−3.42	−3.43	−3.47	−3.55	−3.69	−3.89	−4.21	−4.70	−5.51	−7.11	−13.58
0.60	−4.15	−4.16	−4.20	−4.28	−4.42	−4.63	−4.95	−5.44	−6.26	−7.87	−14.37
0.65	−4.95	−4.96	−5.00	−5.09	−5.22	−5.43	−5.75	−6.25	−7.07	−8.69	−15.23
0.70	−5.84	−5.85	−5.90	−5.98	−6.11	−6.33	−6.65	−7.15	−7.98	−9.61	−16.18
0.75	−6.86	−6.88	−6.92	−7.00	−7.14	−7.35	−7.68	−8.18	−9.01	−10.65	−17.26
0.80	−8.08	−8.09	−8.14	−8.22	−8.36	−8.57	−8.90	−9.41	−10.24	−11.90	−18.53
0.85	−9.60	−9.61	−9.66	−9.74	−9.88	−10.10	−10.43	−10.94	−11.78	−13.43	−20.10
0.90	−11.67	−11.69	−11.73	−11.82	−11.96	−12.18	−12.51	−13.02	−13.87	−15.53	−22.23
0.95	−15.11	−15.13	−15.17	−15.26	−15.40	−15.62	−15.96	−16.47	−17.32	−19.00	−25.72
0.99	−22.86	−22.87	−22.92	−23.01	−23.15	−23.37	−23.71	−24.23	−25.08	−26.76	−33.51

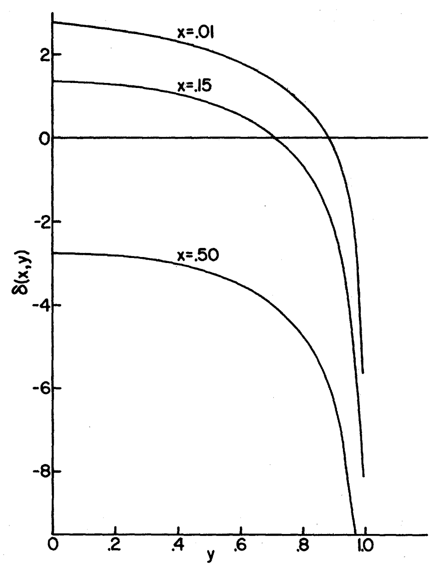

FIG. 2. $\delta(x,y)$ given in percent for $x=0.01$, 0.15, and 0.50.

plot in x and y.

A further integration of Eq. (3.6) over y gives $d\Gamma^{\text{brem}}/dx$ and thus $\delta_{\text{brem}}(x)$ defined by

$$\delta_{\text{brem}}(x)=\frac{d\Gamma^{\text{brem}}}{dx}\bigg/\frac{d\Gamma}{dx}\ ;$$

adding to this the right-hand side of Eq. (2.8), we obtain $\delta(x)$.

Finally, we integrate Eq. (2.5) to obtain Γ^{rad}, the correction to the decay rate of $\pi^0\to\gamma e^+e^-$.[6] We discuss our results in the following section.

IV. RESULTS AND DISCUSSION

Table I gives the radiative corrections to the Dalitz plot and we draw $\delta(x,y)$ for various values of x in Fig. 2 (only positive values of y are used

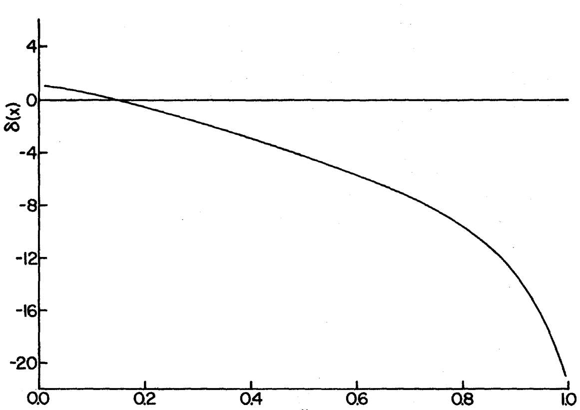

FIG. 3. $\delta(x)$ given in percent as a function of x.

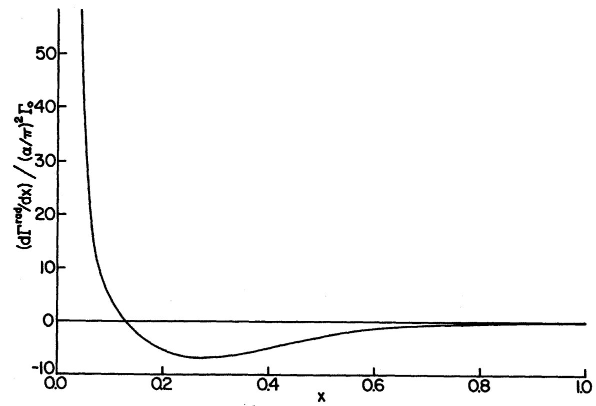

FIG. 4. Plot of $(d\Gamma^{\text{rad}}/dx)/(\alpha/\pi)^2\Gamma_0$ as a function of x.

since the corrections are symmetric under $y\to -y$). We see that $\delta(x,y)$ changes sign (for small values of x) as y increases and becomes more negative for larger values of x.

In Fig. 3 we plot $\delta(x)$. For $x\gtrsim 0.15$, $\delta(x)$ is negative and rather large in absolute value. However, as suggested in Ref. 1, the corrections become positive for small values of x (where one could not use the soft-photon approximation) and this region is important in calculating the radiative corrections to the decay rate, since most of the contributions to $\int(d\Gamma^{\text{rad}}/dx)dx$ come from this region. The reason for this is, of course, the well-known $1/x$ behavior of the nonradiative differential decay rate [Eq. (2.3)], which more than compensates for the small values of $\delta(x)$ in the small-x region. This is seen clearly in Fig. 4, where we plot $(d\Gamma^{\text{rad}}/dx)/(\alpha/\pi)^2\Gamma_0$ versus x [the extra factor of $(\alpha/\pi)^2$ is included for convenience]. The $1/x$ enhancement and $(1-x)^3$ suppression (for large x) is obvious.

As a check on our calculation, a final integration over $(1/\Gamma_0)(d\Gamma^{\text{rad}}/dx)$ [removing the extra factor of $(\alpha/\pi)^2$] yielded the value of 0.95×10^{-4} as the correction to the branching ratio, which compares favorably with 1.05×10^{-4} quoted in the Introduction. The 5% difference is due to numerical inaccuracy in computing the three-dimensional integral which is very badly behaved for small values of x. However, the values given in Table I involve only the integration over x_γ and are accurate to 1%. The value 1.05×10^{-4} was obtained using the fourth-order corrections to the photon propagator. This calculation neglected the interference between dia-

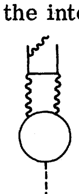

FIG. 5. Example of a two-photon-exchange diagram which is proportional to the electron mass and can be neglected in our calculation.

grams (4) plus (5) in Fig. 1 and the corresponding photon-exchanged diagrams. In Ref. 1 it was argued that the contribution of this interference term is negligible since it reduces to a two-photon-exchange process which, as has been shown by Brown,[7] has no divergence in the lepton mass. We have checked this explicitly by omitting the interference terms and found that Γ^{rad} changes only by about 0.3%.

It is for the same reason that we neglect two-photon-exchange diagrams like the one in Fig. 5, whose interference with the basic diagram (1) of Fig. 1 is of the same order in α as the corrections we have considered. The contribution of Fig. 5, however, is well known to be proportional to the lepton mass, so we are justified in neglecting it completely.

ACKNOWLEDGMENTS

We wish to thank W. P. Trower for interesting discussions on the experimental aspects of this and related processes and Brookhaven National Laboratory for their hospitality. One author (J. S.) wishes to thank B. E. Lautrup for his contributions in a preliminary investigation of the hard-photon effects in this decay.

APPENDIX A

Here we list the expression for the trace of the bremsstrahlung diagrams as defined in Eq. (3.4).

$$\mathrm{Tr}=32+16\frac{B}{A}+16\frac{A}{B}+8m^2m_\pi{}^2(1+xy-y-x_\gamma)\frac{1}{A^2}+8m^2m_\pi{}^2(y\to-y)\frac{1}{B^2}+8\frac{m^2m_\pi{}^2}{x}[(1-x)^2(1-y^2)+x_\gamma{}^2]\frac{1}{AB}$$

$$+4m_\pi{}^4[(1-x)^2(1+y^2)+x_\gamma{}^2]\frac{1}{AB}-4m^2m_\pi{}^4\{1+[y(1-x)-x+x_\gamma]^2\}\frac{1}{A^2E}-4m^2m_\pi{}^4\{y\to-y\}\frac{1}{B^2E}$$

$$+4m^2m_\pi{}^2\left(18-xy+4xy^2+4\frac{y^2}{x}-4\frac{x_\gamma{}^2}{x}-x-\frac{4}{x}+y-8y^2+3x_\gamma\right)\frac{1}{AE}$$

$$+4m^2m_\pi{}^2(y\to-y)\frac{1}{BE}-22\frac{m^2m_\pi{}^2}{AE}-m_\pi{}^4(5+2xyx_\gamma+6xy-7xx_\gamma-12x+2x^2y+3x^2-2yx_\gamma-8y+9x_\gamma+6x_\gamma{}^2)\frac{1}{AE}$$

$$-m_\pi{}^4(11.25-2.25xyx_\gamma+0.5xy-3.5xy^2-7.75xx_\gamma-10x-2x^2y+1.75x^2y^2$$

$$+2.75x^2+2.25yx_\gamma+1.5y+1.75y^2+8.25x_\gamma+6.5x_\gamma{}^2)\frac{1}{BE}$$

$$-16\frac{m^2B}{A^2}-16m^2\frac{A}{B^2}-32\frac{m^2}{B}-32\frac{m^2}{A}-4m_\pi{}^2(3+xy-3x-y-3x_\gamma)\frac{1}{A}-4m_\pi{}^2(y\to-y)\frac{1}{B}$$

$$+4m^2m_\pi{}^2(2+xy-x-y+7x_\gamma)\frac{1}{ED}+4m^2m_\pi{}^2(y\to-y)\frac{1}{EC}-22\frac{m^2m_\pi{}^2}{EC}$$

$$-m^2m_\pi{}^4(13+5xy-19x-5y+11x_\gamma)\frac{1}{EFC}-m^2m_\pi{}^4(1-2xy-2xy^2-2xx_\gamma+x^2y^2-x^2+2y+y^2-x_\gamma{}^2)\frac{1}{AEC}$$

$$-m^2m_\pi{}^4(4.5-6xyx_\gamma-6.5xy-8xx_\gamma-5.5x+2x^2y+2x^2+6yx_\gamma+4.5y+1.5x_\gamma+6x_\gamma{}^2)\frac{1}{AED}$$

$$-m^2m_\pi{}^4(y\to-y)\frac{1}{BEC}+m^2m_\pi{}^6[1+x+x_\gamma+y(1-x)]^2\frac{1}{AEFC}+m^2m_\pi{}^6[y\to-y]^2\frac{1}{BEFD}$$

$$+m^2m_\pi{}^6(4.5-9xy^2-xx_\gamma-3x+4.5x^2y^2+0.5x^2+4.5y^2+3x_\gamma+0.5x_\gamma{}^2)\frac{1}{AEFD}$$

$$-m_\pi{}^4(5.25-2.25xyx_\gamma+0.5xy+0.5xy^2+4.25xx_\gamma+2x-2x^2y-0.25x^2y^2$$

$$-3.25x^2+2.25yx_\gamma+1.5y-0.25y^2+4.25x_\gamma+0.5x_\gamma{}^2)\frac{1}{ED}$$

$$+m_\pi{}^4(1-2xyx_\gamma-6xy-4xy^2-5xx_\gamma-2x^2y+2x^2y^2+3x^2+2yx_\gamma+8y+2y^2-5x_\gamma)\frac{1}{EC}$$

$$-m_\pi{}^6(2xyx_\gamma-2xyx_\gamma{}^2+4xy+2xy^2x_\gamma+2xy^2+10xx_\gamma{}^2+2x-4x^2y-x^2y^2x_\gamma-4x^2y^2$$

$$-7x^2x_\gamma+2x^3y^2+2x^3-2yx_\gamma+2yx_\gamma{}^2-y^2x_\gamma-x_\gamma+2x_\gamma{}^2-5x_\gamma{}^3)\frac{1}{EFC}$$

$$+\tfrac{1}{4}m_\pi^6(1+xyx_\gamma^2+3xy-2xy^2x_\gamma+xy^2-3xy^3-2xx_\gamma+3xx_\gamma^2-x-2x^2yx_\gamma-3x^2y+x^2y^2x_\gamma+x^2y^2+3x^2y^3$$
$$-3x^2x_\gamma-x^2+x^3y-x^3y^2-x^3y^3+x^3+2yx_\gamma-yx_\gamma^2-y+y^2x_\gamma-y^2+y^3-3x_\gamma+3x_\gamma^2-x_\gamma^3)\frac{1}{AED}$$
$$+\tfrac{1}{4}m_\pi^6(y\to -y)\frac{1}{BEC}+m_\pi^4(13+9xyx_\gamma+43xx_\gamma+32x-9x^2y-13x^2-9yx_\gamma+9y-15x_\gamma-30x_\gamma^2)\frac{1}{EF}$$
$$-\tfrac{1}{4}m_\pi^8(3xyx_\gamma-7xyx_\gamma^3-3xy^2x_\gamma+6xy^2x_\gamma^2-3xy^3x_\gamma+3xx_\gamma-6xx_\gamma^2+7xx_\gamma^3-3x^2yx_\gamma+6x^2yx_\gamma^2-3x^2y^2x_\gamma$$
$$-3x^2y^2x_\gamma^2+3x^2y^3x_\gamma+3x^2x_\gamma-3x^2x_\gamma^2-3x^3yx_\gamma+3x^3y^2x_\gamma-x^3y^3x_\gamma+x^3x_\gamma+3yx_\gamma-6yx_\gamma^2+7yx_\gamma^3$$
$$+3y^2x_\gamma-3y^2x_\gamma^2+y^3x_\gamma+x_\gamma-5x_\gamma^4-3x_\gamma^2+7x_\gamma^3)\frac{1}{AEFC}-\tfrac{1}{4}m_\pi^8(y\to -y)\frac{1}{BEFD}$$
$$-m_\pi^6(-10.25+4.5xyx_\gamma+1.75xyx_\gamma^2-10xy-1.5xy^2x_\gamma+6.75xy^2+3.5xx_\gamma+8.75xx_\gamma^2-6.25x+0.25x^2yx_\gamma$$
$$+1.5x^2y+0.75x^2y^2x_\gamma-5.25x^2y^2-6.5x^2x_\gamma-1.75x^2+2x^3y+1.25x^3y^2+2.25x^3$$
$$-4.75yx_\gamma-1.75yx_\gamma^2+6.5y+0.75y^2x_\gamma-2.75y^2+6x_\gamma+0.75x_\gamma^2-4.5x_\gamma^3)\frac{1}{BEF}$$
$$-2m^2m_\pi^4(6-5xy-2x+5y-6x_\gamma)\frac{1}{BEF}+m_\pi^6(4-6xy-3xy^2+4x^2y+3x^2y^2+2x^3y-x^3y^2+y^2)\frac{1}{AEF}$$
$$+m^2m_\pi^4(23-5xy+7x+5y+x_\gamma)\frac{1}{AEF}-8\frac{m_\pi^6x}{AE^2}-32\frac{m^2m_\pi^4}{AE^2}-8\frac{m_\pi^6x}{BE^2}-32\frac{m^2m_\pi^4}{BE^2}$$
$$-32\frac{m_\pi^4}{E^2}-16m_\pi^2(1+x-x_\gamma)\frac{A}{EF}+2m_\pi^2(7xy+11x-7y-32x_\gamma-43)\frac{1}{E}$$
$$-m^2m_\pi^4(1+2xy-2xy^2-2xx_\gamma+x^2y^2-x_\gamma^2-2y+y^2-x^2)\frac{1}{BED}$$
$$-2m^2m_\pi^2(1-xy+x+y+x_\gamma)\frac{1}{AC}-2m^2m_\pi^2(y\to -y)\frac{1}{BD}.$$

APPENDIX B

Here we give the basic integrals

$$J[f(A,B)]=\frac{1}{2\pi}\int\frac{d^3k_1}{k_{10}}\frac{d^3k_2}{k_{20}}\times\delta^4(p-q_1-q_2-k_1-k_2)f(A,B)$$

with fixed x, y, and x_γ. The photon mass is set equal to zero because the divergent integrals are treated in the next appendix.

The general procedure is to evaluate the integral in the rest frame of the particles whose momenta appear in the denominator of $f(A,B)$. We shall need the following definitions:

$$L_{01}=\frac{L\cdot q_1}{m};\quad L_1=(L_{01}^2-m_\pi^2x_\gamma)^{1/2};$$

$$L_{02}=\frac{L\cdot q_2}{m};\quad L_2=(L_{02}^2-m_\pi^2x_\gamma)^{1/2};$$

$$L\cdot p=\tfrac{1}{2}m_\pi^2(1-x+x_\gamma);\quad L_0=\frac{m_\pi}{2\sqrt{x}}(1-x-x_\gamma);$$

$$L=(L_0^2-m_\pi^2x_\gamma)^{1/2};$$

$$p_0=\frac{m_\pi^2}{4m}[1+x-x_\gamma+y(1-x)];$$

$$p_z=\frac{L_{01}p_0-L\cdot p}{L_1};\quad q_0=\tfrac{1}{2}m_\pi\sqrt{x};\quad q=(q_0^2-m^2)^{1/2};$$

$$q_z=\frac{-m_\pi^2y(1-x)}{4L};\quad q_{10}=\frac{m_\pi^2x}{2m}-m;$$

$$q_1=(q_{10}^2-m^2)^{1/2};\quad q_{1z}=\frac{L_{02}q_{10}-L\cdot q_1}{L_2};$$

$$q_{20}=q_{10};\quad q_{2z}=\frac{L_{01}q_{20}-L\cdot q_2}{L_1};$$

$$c_1=\frac{m_\pi^8x^2}{16m^2}[(1-x)(1+y)+x_\gamma]^2-m_\pi^6xx_\gamma;$$

$$b_1=m_\pi^4x\left[2L_{01}(1-x+x_\gamma)-\frac{m_\pi^2}{m}x_\gamma(1+x-x_\gamma)\right];$$

$$c_3=m_\pi^4[(L_{01}-x_\gamma p_0)^2-m_\pi^2xx_\gamma];$$

$$b_3=\frac{m_\pi^6}{2m}\{-(1-x_\gamma)^3-y(1-x)[(1-x_\gamma)^2-x(1+x_\gamma)]+x(1-x_\gamma)(2+2x_\gamma-x)\};$$

$$c_5=(L_2L\cdot q_1)^2-m_\pi^2x_\gamma L_2L\cdot q_1q_{1z}+\tfrac{1}{4}m_\pi^4x_\gamma^2q_1^2;$$

$$b_5 = -L_2(2L_2 L\cdot q_1 q_{10} - 2L_{02}L\cdot q_1 q_{1z} - m_\pi^2 x_\gamma q_{10}q_{1z}) - m_\pi^2 x_\gamma L_{02} q_1^2;$$

$$n_1 = m_\pi^2 x_\gamma\left[\tfrac{1}{2}b_1 + 2\sqrt{c_1}\,L_1\left(\frac{m_\pi^2 x}{2m} + q_{2z}\right)\right] + 2(L_{01}+L_1)(c_1 + m_\pi^2 x L_1\sqrt{c_1});$$

$$d_1 = m_\pi^2 x_\gamma\left[\tfrac{1}{2}b_1 + 2\sqrt{c_1}\,L_1\left(\frac{m_\pi^2 x}{2m} - q_{2z}\right)\right] + 2(L_{01}-L_1)(c_1 + m_\pi^2 x L_1\sqrt{c_1});$$

$$n_3 = m_\pi^2 x_\gamma[\tfrac{1}{2}b_3 - 2\sqrt{c_3}\,L_1(p_0 + p_z)] + 2(L_{01}+L_1)(c_3 + m_\pi^2 L_1\sqrt{c_3});$$

$$d_3 = m_\pi^2 x_\gamma[\tfrac{1}{2}b_3 - 2\sqrt{c_3}\,L_1(p_0 - p_z)] + 2(L_{01}-L_1)(c_3 + m_\pi^2 L_1\sqrt{c_3});$$

$$n_5 = m_\pi^2 x_\gamma[\tfrac{1}{2}b_5 - \sqrt{c_5}\,L_2(q_{10}+q_{1z})] + 2(L_{02}+L_2)(c_5 + \sqrt{c_5}\,L_2 L\cdot q_1);$$

$$d_5 = m_\pi^2 x_\gamma[\tfrac{1}{2}b_5 - \sqrt{c_5}\,L_2(q_{10}-q_{1z})] + 2(L_{02}-L_2)(c_5 + \sqrt{c_5}\,L_2 L\cdot q_1)\,.$$

With the above definitions the basic integrals are given by

$$J[1] = 1\,,$$

$$J\left[\frac{1}{A}\right] = \frac{1}{mL_1}\ln\frac{L_{01}+L_1}{L_{01}-L_1}\,,$$

$$J\left[\frac{1}{E}\right] = \frac{1}{2m_\pi\sqrt{x}\,L}\ln\frac{m_\pi\sqrt{x}+L_0+L}{m_\pi\sqrt{x}+L_0-L}\,,$$

$$J\left[\frac{1}{E^2}\right] = \frac{1}{m_\pi^4 x}\,,$$

$$J\left[\frac{1}{AB}\right] = \frac{4}{m_\pi^3\sqrt{x}\,x_\gamma q}\ln\frac{q_0+q}{q_0-q}\,,$$

$$J\left[\frac{1}{A^2}\right] = \frac{4}{m^2 m_\pi^2 x_\gamma}\,,$$

$$J\left[\frac{B}{A}\right] = \frac{1}{m}\left(q_{20} - q_{2z}\frac{L_{01}}{L_1}\right) + \frac{m_\pi^2 x_\gamma q_{2z}}{2mL_1^2}\ln\frac{L_{01}+L_1}{L_{01}-L_1}\,,$$

$$J\left[\frac{A}{E}\right] = \frac{1}{4} - \frac{q_z L_0}{2m_\pi\sqrt{x}\,L} + \frac{m_\pi}{8xL}\left(\frac{q_z}{L}(1-x+x_\gamma) - x\right)\ln\frac{m_\pi\sqrt{x}+L_0+L}{m_\pi\sqrt{x}+L_0-L}\,,$$

$$J\left[\frac{1}{AE}\right] = \frac{1}{m\sqrt{c_1}}\ln\frac{n_1}{d_1}\,,$$

$$J\left[\frac{1}{EC}\right] = \frac{1}{m\sqrt{c_3}}\ln\frac{n_3}{d_3}\,,$$

$$J\left[\frac{1}{BC}\right] = \frac{1}{m\sqrt{c_5}}\ln\frac{n_5}{d_5}\,,$$

$$J\left[\frac{B}{A^2}\right] = \frac{1}{m^2 L_1}\left[2q_{2z} + \left(q_{20} - q_{2z}\frac{L_{01}}{L_1}\right)\ln\frac{L_{01}+L_1}{L_{01}-L_1}\right],$$

$$J\left[\frac{1}{A^2E}\right] = \frac{4L_1}{m^2 c_1 x_\gamma}(xL_1 + x_\gamma q_{2z}) - \frac{b_1}{2mc_1}J\left[\frac{1}{AE}\right],$$

$$J\left[\frac{1}{AE^2}\right] = \frac{-b_1}{2mm_\pi^4 x c_1} + m_\pi^2(xL_1 + x_\gamma q_{2z})\frac{L_1}{c_1}J\left[\frac{1}{AE}\right].$$

Letting $y \to -y$ one obtains corresponding expressions for $q_1 \leftrightarrow q_2$, e.g., $J[A/B] = J[B/A]_{y\to -y}$.

Relations (3.5) can be used to obtain the integrals not listed above, e.g.,

$$J\left[\frac{1}{AEFC}\right] = \frac{2}{L\cdot q_1}J\left[\frac{1}{EFC}\right] = \frac{2}{m_\pi^2(1+x-x_\gamma)L\cdot q_1}\left(J\left[\frac{1}{AE}\right] + J\left[\frac{1}{EC}\right]\right).$$

APPENDIX C

We now evaluate the infrared-divergent integrals. First let us consider

$$\int J\left[\frac{1}{AB}\right]dx_\gamma = \frac{1}{\pi}\int dx_\gamma \int\frac{d^3k}{k_0}\,\frac{\delta(m_\pi^2 x_\gamma - 2L\cdot k + \lambda^2)}{k\cdot q_1\, k\cdot q_2} = \int_0^1 d\alpha\, K(q_1, q_2, \alpha)\,,$$

where

$$K(q_1,q_2,\alpha) = \frac{1}{\pi}\int dx_\gamma\int\frac{d^3k}{k_0}\,\frac{\delta(m_\pi^2 x_\gamma - 2m_\pi\sqrt{x_\gamma}\,k_0 + \lambda^2)}{\{[\alpha q_1 + (1-\alpha)q_2]\cdot k\}^2}$$

(we are working in the $\vec{L} = \vec{p} - \vec{q}_1 - \vec{q}_2 = 0$ frame).

Writing

$$[\alpha q_1 + (1-\alpha)q_2]\cdot k = k_0 A_1 - A_2 kz\,,$$

where

$$A_1 = \alpha q_{10} + (1-\alpha)q_{20}, \quad A_2 = |\alpha\vec{q}_1 + (1-\alpha)\vec{q}_2|\,,$$

$$q_{10} = \frac{L\cdot q_1}{m_\pi\sqrt{x_\gamma}}, \quad q_{20} = \frac{L\cdot q_2}{m_\pi\sqrt{x_\gamma}},$$

z = cosine of the angle between $\vec{k}$ and $[\alpha\vec{q}_1 + (1-\alpha)\vec{q}_2]$,

$k = |\vec{k}|$,

we get

$$K(q_1,q_2,\alpha) = 2\int dx_\gamma\int\frac{k^2 dk}{k_0}\,\delta(m_\pi^2 x_\gamma - 2m_\pi\sqrt{x_\gamma}\,k_0 + \lambda^2)\int_{-1}^{1}\frac{dz}{(k_0 A_1 - A_2 kz)^2}$$

$$=4\int\frac{dx_\gamma(m_\pi{}^2x_\gamma-\lambda^2)}{(A_1{}^2-A_2{}^2)(m_\pi{}^2x_\gamma-\lambda^2)^2+4m_\pi{}^2x_\gamma A_1{}^2\lambda^2}\,.$$

Now,

$$A_1{}^2-A_2{}^2=m^2+m_\pi{}^2\alpha(1-\alpha)\left(x-\frac{4m^2}{m_\pi{}^2}\right) \tag{C1}$$

and is independent of x_γ, while

$$4m_\pi{}^2x_\gamma A_1{}^2\lambda^2=\tfrac{1}{4}m_\pi{}^4\lambda^2[(1-x)(2\alpha y+1-y)-x_\gamma]^2\,;$$

however, the λ^2 terms are important only for $x_\gamma\approx\lambda^2/m_\pi{}^2$. Hence we can write

$$K(q_1,q_2,\alpha)=\frac{4}{m_\pi{}^2}\int_{\lambda^2/m_\pi{}^2}^{x_\gamma^{\max}}\frac{x_\gamma dx_\gamma}{(A_1{}^2-A_2{}^2)x_\gamma{}^2+\frac{1}{4}\lambda^2(1-x)^2(2\alpha y+1-y)^2}$$

$$=\frac{2}{m_\pi{}^2(A_1{}^2-A_2{}^2)}\ln\frac{4(x_\gamma^{\max})^2(A_1{}^2-A_2{}^2)}{\lambda^2(1-x)^2(2\alpha y+1-y)^2}\,. \tag{C2}$$

At this stage we calculate $\int J[1/B^2]\,dx_\gamma$ by putting $\alpha=0$ in the above equation, $A_1{}^2-A_2{}^2=m^2$, to obtain

$$\int J\left[\frac{1}{B^2}\right]dx_\gamma=\frac{4}{m^2m_\pi{}^2}\left[\ln\frac{m}{\lambda}+\ln\frac{2x_\gamma^{\max}}{(1-x)(1-y)}\right].$$

Similarly,

$$\int J\left[\frac{1}{A^2}\right]dx_\gamma=\frac{4}{m^2m_\pi{}^2}\left[\ln\frac{m}{\lambda}+\ln\frac{2x_\gamma^{\max}}{(1-x)(1+y)}\right].$$

Returning to the evaluation of $\int J[1/AB]\,dx_\gamma$, we write for Eq. (C2)

$$K(q_1,q_2,\alpha)=K_{\rm DIV}+K_{\rm CONV}\,,$$

where

$$K_{\rm DIV}=\frac{4}{m_\pi{}^2(A_1{}^2-A_2{}^2)}\left(\ln\frac{m}{\lambda}+\ln\frac{2x_\gamma^{\max}}{1-x}\right),$$

$$K_{\rm CONV}=\frac{2}{m_\pi{}^2(A_1{}^2-A_2{}^2)}\ln\frac{A_1{}^2-A_2{}^2}{m^2(2\alpha y+1-y)^2}\,.$$

Using Eq. (C1) we obtain

$$\int_0^1 d\alpha\;K_{\rm DIV}=\frac{-8}{m_\pi{}^4x\beta}\left(\ln\frac{m}{\lambda}+\ln\frac{2x_\gamma^{\max}}{1-x}\right)\ln\gamma\,, \tag{C3}$$

where β and γ are defined in Eqs. (2.4) and (2.6).

Observing that $\int_0^1 d\alpha K_{\rm CONV}$ is symmetric under $y\to -y$, we can write [letting $\alpha\to\frac{1}{2}(1+\alpha)$]

$$\int_0^1 d\alpha K_{\rm CONV}=\frac{4}{m_\pi{}^4x}\int_{-1}^1\frac{d\alpha}{1-\alpha^2\beta^2}\ln\frac{m_\pi{}^2x}{4m^2}\,\frac{1-\alpha^2\beta^2}{1-\alpha^2y^2}$$

$$=\frac{4}{m_\pi{}^4x\beta}\left\{\left[\ln\left(1-\frac{y^2}{\beta^2}\right)-\frac{1}{2}\ln\frac{m_\pi{}^2x}{m^2}\right]\ln\gamma+{\rm Li}_2\left[\frac{1-\beta}{2}\right]-{\rm Li}_2\left[\frac{1+\beta}{2}\right]\right.$$

$$\left.+{\rm Li}_2\left[\frac{y(1+\beta)}{y-\beta}\right]+{\rm Li}_2\left[\frac{y(1+\beta)}{y+\beta}\right]-{\rm Li}_2\left[\frac{y(1-\beta)}{y-\beta}\right]-{\rm Li}_2\left[\frac{y(1-\beta)}{y+\beta}\right]\right\}. \tag{C4}$$

Adding Eqs. (C 3) and (C 4) we obtain Eq. (3.7) for $\int J[1/AB]\,dx_\gamma$.

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†Work supported in part under Contract No. AT(30-1)-3668B of the U. S. Atomic Energy Commission.

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PHYSICAL REVIEW D VOLUME 5, NUMBER 7 1 APRIL 1972

Time-Reversal Invariance and the Phase of the ρ-ω Mixing*

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(Received 28 December 1970; revised manuscript received 29 September 1971)

We use the unitarity sum to rederive in a simple way an explicit formula for the ρ-ω mixing phase in terms of $|\epsilon|$, Γ_ω, Γ_ρ, and $m_\omega^2 - m_\rho^2$. We then present a systematic study of the condition imposed by T (or CPT) invariance and unitarity on the relative phase and strength of the mixing. It is shown that the departure from T invariance in the ρ-ω system is, in principle, directly demonstrable through the measurement of the ρ-ω mixing phase in $e^+e^- \to \pi^+\pi^-$. The connection between the phase discrepancy and the breakdown of microscopic reversibility is also discussed. Finally, two possible graphical representations for the ρ-ω mixing parameters which exhibit directly the condition imposed by unitarity and time-reversal invariance on the relative phases and strengths of the ρ-ω mixing are given.

I. INTRODUCTION

The discovery of the G-parity-violating effect[1,2] $\omega \to 2\pi$ seen as an interference dip in the ρ-ω spectrum has reactivated the problem of ρ-ω mixing in recent years.[3-5] In this paper, we wish to report a systematic study of the condition imposed by time-reversal invariance and unitarity on the phase and strength of the ρ-ω mixing and present a simple method of calculating the mixing phase from the so-called unitarity sum.

Although the problem of ρ-ω mixing has often been claimed[4] to be equivalent to the neutral-kaon-decay problem and the phase of the ρ-ω mixing has been estimated previously,[3,5] nevertheless, we feel that the connection between the two problems has not yet been thoroughly explored and the role of the unitarity sum in the determination of the ρ-ω mixing phase has not yet been fully exploited. Here, we wish to reveal gradually through our discussion this close connection between the two mixing problems and then use the Bell-Steinberger unitarity sum to obtain an explicit formula for the ρ-ω mixing phase in terms of the ρ and ω masses and widths, assuming CPT invariance. The result obtained by us as well as those reported recently[3] seem to show that the theoretical estimated phase of the ρ-ω mixing is not in excellent agreement with its earlier experimental value.[6] There has been an attempt[5] to correct this discrepancy by considering (i) not only the mass mixing of ρ and ω vector mesons but also their vector mixing[7] and (ii) the energy dependence of the ρ width, but the situation is not improved much. On the contrary, the new estimate might even widen the discrepancy, taking into account the fact that in the latter estimate, the ρ-ω mixing phase may be smaller than 112° which is the value estimated in Ref. 3. We are, therefore, content with the mass mixing alone. In view of this difficulty, we believe that it would be interesting to consider a detailed investigation of the condition imposed by unitarity and time-reversal invariance on the mixing phase. This condition might serve as a guideline for future experiments measuring the phase concerned. The results of this investigation will be reported in a subsequent section of this paper. Besides, we show that the departure from T invariance in the electromagnetic interaction would, in principle, be directly demonstrable through the measured phase of the ρ-ω mixing in $e^+e^- \to \pi^+\pi^-$ alone, should the accuracy of the determination of that