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# Pion-Nucleon *P*-Wave Scattering Amplitudes and Their *S*-Matrix Calculations

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The effective-range formulas for the $P$-wave scattering amplitudes, $I=\frac{1}{2}$ and $I=\frac{3}{2}$, of pions on nucleons are derived for the low-energy region using the requirements of analyticity, unitarity, and crossing symmetry and a sum rule. The $S$-matrix calculations are carried out by the generalized one-channel Chew-Mandelstam method in which the denominator function $D$ may, generally, contain a Castillejo-Dalitz-Dyson (CDD) pole, while the function $N$ is considered in the three-pole effective-range approximation. The effective-range parameters are then determined by the usual procedure in which the input values are the $\rho$-meson and (3,3)-isobar $N^*$ discontinuities and the high-energy contributions in the crossed channels, and the nucleon pole in the direct channel. The amplitudes $P_{13}$ and $P_{33}$, derived in this way as functions of the CDD-pole term, are then compared with experiment by means of the calculated and experimental values of the scattering lengths $a_{13}$ and $a_{33}$. It is shown, as one of the main results, that the low-energy $P_{13}$ amplitude does not contain any CDD pole and produces a scattering length which is in excellent agreement with the experimental value, while the $P_{33}$ amplitude necessarily contains a nonzero pole term. These results are in agreement with the conclusions made on the basis of a quite different approach – the "crankshaft analysis" – in the papers by Atkinson *et al*.

## I. INTRODUCTION

$S$-matrix calculations enable us generally to derive analytically the partial-wave scattering amplitudes in strong interactions, which are consistent with the requirements of analyticity, unitarity, and crossing symmetry.

An approximative approach which starts from these requirements and determines the partial-wave amplitudes in a certain low-energy region has been worked out as the $N/D$ effective-range method by Balázs.[1]

The determining left-hand cut in the $s$-plane is replaced in this approximation by several (in our case two) poles $s_i$ so that the numerator function $N$ is given as a sum of $n$ fractions $b_i/(s-s_i)$ with unknown residues $b_i$.

To solve the problem of finding the $N/D$ amplitudes, we determine these residues in our self-consistent calculations by comparison of the

$N(s)/D(s)$ integral representation and its derivatives at the point $s_R$ in the unphysical region with the expressions for the partial-wave amplitudes and their derivatives derived theoretically on the basis of pole theory.

In the case of pion-nucleon scattering the amplitudes in the unphysical region are well represented by several discontinuities, discussed in detail below, which are further used as the input values.

However, having in mind the general theorems of the Herglotz type[2] for analytic functions, according to which Castillejo-Dalitz-Dyson (CDD) poles may occur in the course of our one-channel calculations in the $D$ function[3] as the expressions $\gamma_{il}/(s-s_{il})$, we must devote special attention to these poles.

The role of CDD poles is particularly important wherever there is a binding activity to other channels; it will manifest itself by the term belonging to a dynamical resonance or bound states ($s_{il}=s_{\rm res}$). This has been widely studied with the aid of Levinson's theorem in the nonelastic scattering problem in relation to one-channel (the $N/D$ methods of Frye and Warnock, and of Chew and Mandelstam) and many-channel calculations in the papers by Atkinson *et al.*[4-6] As a general result it has been found that a multichannel dynamical resonance (bound state) will appear as a CDD pole in the single-channel inelastic amplitude for at least one channel.

In the applications[6] of this so-called "crankshaft analysis" to pion-nucleon scattering in the SU(6)-symmetric model, Capps, Atkinson, and Halpern came to the conclusion that the partial waves $P_{33}$ and $P_{11}$ of this process should have a CDD pole in the $D$ function.

Although our one-channel calculations are carried out by the method of Chew and Mandelstam,[7] for which the "crankshaft analysis" is more complicated than for the Frye-Warnock equations[8] (the character of which in respect to the CDD poles can even be opposite[9]), we retain in the course of calculations a general pole term $C(s)$ for CDD poles in the $N/D$ integral representation, for reasons of possible binding among the channels. This will enable us to study the possible influence of CDD poles in the $P_{13}$ and $P_{33}$ amplitudes in the numerical equations for pion-nucleon scattering.

In the case of elastic pion-nucleon scattering, we thus obtain the possibility of studying the phenomena beyond the scope of the principle of maximal analyticity of the second kind.[10] According to this principle, it is assumed that the lower partial waves in the elastic region should be attainable from the higher ones by analytic continuation. More concretely, this continuation should be the $N/D$ partial-wave amplitudes without CDD poles in the elastic region.

The principle of maximal analyticity of the second kind is, however, valid according to Levinson's theorem only under the assumption that the difference between the phase shifts at infinity and threshold $s_{\rm thr}$ is equal to the number $M_l$ of the zero points of the $D$ function:

$$\delta_l(\infty)-\delta_l(s_{\rm thr})=-\pi M_l,$$

which should be verified by experiment and would correspond to the fact that all particles are composite. In fact, the situation is more complicated; in case this condition is not fulfilled, then even with the maximal analyticity of the second kind, the CDD poles may occur in some channels even for the elastic scattering.

The purpose of this paper is to derive, on the basis of $S$-matrix calculations by the Chew-Mandelstam method, the analytical expressions for the $N/D$ $P$-wave amplitudes $P_{13}$ and $P_{33}$ which would depend on the possible CDD poles. In this way it is also possible to get some of the answers to the above-mentioned complicated problems in the pion-nucleon scattering. This is achieved in the course of this work by passing in the elastic approximation to self-consistent calculations in which we put the CDD pole term $C(s)$ identically equal to zero and compare the obtained results for some of the physical quantities (the scattering length or a resonance) with the experimental value.

One of the results of our self-consistent calculations is the observation that the low-energy pion-nucleon $P_{13}$ amplitude does not contain any CDD pole in the $D$ function and produces the correct experimental value of the scattering length $a_{13}$, while the $P_{33}$ amplitude does need such a pole in order to reproduce the experimental length $a_{33}$.

This fully confirms the previous conclusion of Atkinson *et al.*[5,6] that in contrast to the $P_{13}$ state, a certain dynamical resonance (bound state) should be present in one-channel calculations in the $P_{33}$ wave. On the basis of our one-channel calculations for the $P_{33}$ pion-nucleon wave, we argue at the same time that the special condition for the difference of phase shifts in the above-given form is not valid. Let us complete the description of our method for calculating the residues by mentioning that we use as the input values the discontinuities of the $\rho$ meson and pion-nucleon (3,3)-isobar $N^*$ in the crossed channels and the nucleon pole in the direct channel. The expressions for these contributions, including the residue for the nucleon pole,[11,12] have been self-consistently calculated in the pole theory by Frautschi and Walecka.[11] The high-energy contributions in the cross channels $u$ and $t$ are newly taken into account. These can be expressed with the help of the relation between the high-energy states of the high angular momenta in the crossed channels and the low-energy resonance in the direct channel.

Finding the residues in the function $N$ represents the solution of the whole problem. In this paper we derive the determining numerical equations for these residues, in a form which retains the additional dependence on the CDD pole term $C(s)$.

These equations for the $N/D$ amplitude of the $P$ state with isospin $I=\frac{1}{2}$ are found from comparison with the experimental scattering length $a_{13}$ to be valid without the CDD poles $[C(s)\equiv 0]$ and, therefore, we find an explicit expression for this amplitude in the low-energy region.

The partial-wave amplitude with isospin $I=\frac{3}{2}$ is given by its dependence on the pole term $C(s)$. The role of the CDD poles, at present unclear, is the subject of further studies.

As the input values in the calculations, we use the noncorrelated physical constants: the coupling constant $g$, the nucleon mass $m$, the mass $W_R$ and the width $\gamma_{33}$ of the (3,3) resonance in the natural system of units with the pion mass equal to unity $(\hbar=c=\mu=1)$.

## II. EFFECTIVE-RANGE THEORY AND THE $P_{13}$ AND $P_{33}$ AMPLITUDES

Starting from the fundamental axioms of the analytic theory of the $S$ matrix, i.e., analyticity, unitarity, and crossing symmetry, we want to derive analytical expressions of the $N/D$ integral representation for the partial $P$-wave amplitude $I=\frac{1}{2}$ and $\frac{3}{2}$, respectively, of the pion-nucleon scattering. At present, however, it is only possible to find the scattering amplitudes with these properties in the limited low-energy regions where they are expressible in the form of the effective-range formula.[1]

With regard to the applicability of the general theorems of analytic functions, we shall consider in our self-consistent calculations a certain generalization of the effective-range theory especially in relation to possible CDD poles in the $N/D$ amplitudes. We shall decide about their existence as an analytical property of functions (as is always the case in the technique of dispersion relations) on the basis of comparison with the available experimental data, in the conclusion of this section. We normalize the $P$-wave amplitudes in the form

$$^{I}f_{1+}(s)=\frac{W^2}{q^3}\exp(i\delta_{1+})\sin^I\delta_{1+},\quad I=\tfrac{1}{2},\tfrac{3}{2} \tag{1}$$

in which $W$ indicates the energy of the incident particle in the c.m. system, $s=W^2$, $q$ is the three-momentum, $\delta_{1+}$ is the corresponding phase shift, and $I$ is the isotopic index.

The integral representation for the $N/D$ amplitudes $P_{13}$ and $P_{33}$ in one-channel calculations is generally dependent on the CDD poles and may be written as

$$^{I}f_l(s)=\frac{^{I}N_l(s)}{^{I}D_l(s)}, \tag{2}$$

where the angular momentum $l$ equals $l_\pm=J\mp\frac{1}{2}=1+$ in case of a $P$ wave (spin $J=\frac{3}{2}$) and $^{I}f_l\,(s)$ denotes the partial-wave amplitude, free of kinematic singularities, in the point $s$. This amplitude can be found theoretically from pole theory, as in our case, or experimentally. In the $S$-matrix theory, where the Mandelstam representation is used for expressing analyticity in two independent variables $s$ and $t$ of the partial-wave amplitude $^{I}f_l(s,t)$, the latter is a function of the invariant Mandelstam amplitudes $^{I}A_l^{\pm}$, $^{I}B_l^{\pm}$, $^{I}f_l(^{I}A_l^{\pm}(s),{}^{I}B_l^{\pm}(s))$. For such a determination of the amplitude $^{I}f_l$ see, e.g., Eq. (2.26) and Sec. IV of Ref. 11. For pion-nucleon scattering, the main contributions in the complex $W$ plane were calculated in the paper by Frautschi and Walecka[11] of which, in a certain sense (as far as the input values are concerned), the present paper is a continuation. We use, therefore, similar notation in the pion-mass units. Let us consider the numerator function $N$ in Eq. (2) in the three-pole approximation

$$^{I}N_l(s)=\sum_{i=1}^{n}\frac{b_i}{s-s_i},\quad n=3,\quad i=1,2,3. \tag{3}$$

Here the $b_i$ denotes residues which are to be determined and the $s_i$ denote the corresponding poles. For the unknown residues–effective-range parameters of the $N/D$ amplitudes–we shall find numerical equations.

The denominator function $D$ in the integral form with one subtraction is generally given by

$$^{I}D_l(s)=1-\frac{s-s_0}{\pi}\int_{s_t}^{\infty}ds'\,\frac{\rho_l(s')R_l(s')}{(s'-s_0)(s'-s)}\,{}^{I}N_l(s')+C_l(s-s_{il}), \tag{4}$$

where $s_t$ is the threshold energy squared and $s_0$ is the subtraction point. The term $C_l(s)$ expresses the possible dependence on the $l$ CDD poles by

$$C_l(s)=C_l(s-s_{il})=\frac{\gamma_{il}}{s-s_{il}},\quad i=1,\ldots,l. \tag{5}$$

In our case $l=1$, which corresponds to one CDD pole at the most, with the parameter of position $s_{i1}$ and the residue $\gamma_{i1}$.

In the elastic approximation with which we shall deal further, the parameter of inelasticity $R$ is

$$R_l(s)=\frac{\sigma^{\text{tot}}(s)}{\sigma^{\text{el}}(s)}=\frac{\text{Im}f_l(s)}{|f_l(s)|^2}\frac{1}{\rho_l(s)}=1, \tag{6}$$

so that the unitarity relation is

$$\mathrm{Im}[f_l(s)]^{-1} = -\rho_l(s)R_l(s) = -\rho_l(s) = \frac{q^{2l+1}(s)}{s}. \quad (7)$$

Here the square of the three-momentum $q^2(s)$ may be derived in an explicit form:

$$q^2(s) = (4s)^{-1}(s-\alpha_+)(s-\alpha_-), \quad \alpha_\pm = (m\pm 1)^2 \quad (8)$$

which is identical to relation (2.9) of Ref. 11 and in which $m$ denotes the nucleon mass in pion units.

Further, we limit ourselves to the $P$-wave scattering; therefore, the angular momentum $l$ in all previous expressions is equal to one (1+).

We put one of the poles ($s_3$) in the numerator function $N$ [Eq. (3)] in the nucleon pole, in which the residue has been calculated self-consistently,[11,12] and found to be

$$b_3 = \tfrac{8}{3} f^2 m^3\, {}^I D_{1+}(m^2), \quad f^2 = (g/2m)^2 = 0.081. \quad (9)$$

In this way there remain two effective-range parameters $b_1$ and $b_2$ to be determined. We choose their respective poles in the points $s_1 = -m^2$ and $s_2 = -16m^2$ so that the $N$ function takes the form

$$\begin{aligned} {}^I N_{1+}(s) &= \frac{b_1}{s+m^2} + \frac{b_2}{s+16m^2} + \frac{b_3}{s-m^2}, \\ b_3 &= \tfrac{8}{3} f^2 m^3\, {}^I D_{1+}(m^2). \end{aligned} \quad (10)$$

In the $D_{1+}$ function [Eq. (4)], in which also the $N_{1+}$ function occurs, we choose the subtraction point $s_0 = \alpha_- = (m-1)^2$, while the threshold energy squared $s_t$ is equal to $s_t = \alpha_+ = (m+1)^2$ and the discontinuity $\rho_l(s)R_l(s)$ is given by relation (7).

The process of determination of the effective-range parameters $b_1$ and $b_2$, the so-called matching procedure, runs as follows: The system of, generally, the $n$ determining equations for $n$ unknown parameters, $b_i$, is represented in the matching point $s = s_R$ by the equation

$$\left(\frac{{}^I N_l(s_R)}{{}^I D_l(s_R)}\right)^{(j)} = {}^I f_l^{(j)}(s_R), \quad j = 0, 1, \ldots n-1; \quad l = 1+ \quad (11)$$

where $j$ denotes the degree of the derivative. In our case we have to determine only two unknown parameters $b_i$, $i = 1, 2$, so that the first derivative of both sides of Eq. (2) will do for obtaining the determining equations.

The comparison must be made in a point in the unphysical region in which the following hold: (1) The partial-wave amplitude converges. (2) The analytically continuable contributions from the physical region which are not taken into account are vanishingly small.

In this paper we choose the matching point $s_R$ in $\alpha_-$.

Further, we shall deal with the self-consistent calculation of the amplitude on the left-hand side in Eq. (2), which will enable us to derive the determining equations for both parameters $b_i$ ($i = 1, 2$).

For this purpose it is necessary to know from pole theory the amplitudes ${}^I f_{1+}(s)$ and their first derivatives at the point $s = \alpha_-$. Generally, we shall get these by projecting [with the help of Eq. (26) of Ref. 11] the invariant amplitudes $A^i(s,u,t)$ (${}^I A^i \equiv {}^I A^\pm, {}^I B^\pm$, $i = 1, 2, 3, 4,$) in which $s, u, t$ are the Mandelstam squared-energy variables in the corresponding channels.

The invariant amplitudes $A^i$ satisfy the fixed-$s$ dispersion relations

$$\begin{aligned} A^i(s,u,t) = \frac{R_s^i}{m^2-s} + \frac{R_u^i}{m^2-u} &+ \frac{1}{\pi}\int_{\alpha_+}^{\infty} \frac{A_u^i(u',s)}{u'-u}\,du' \\ &+ \frac{1}{\pi}\int_4^{\infty} \frac{A_t^i(t',s)}{t'-t}\,dt'. \end{aligned} \quad (12)$$

Of all states in channels $u$ and $t$ in the low-energy region, we retain only the $N^*$ resonance in the first integral and the $\rho$ meson in the second. Both these states are well known from previous study,[11] as is the nucleon pole in the direct channel.

The high-energy contributions (index $h$) from the remaining parts of the integrals may be considered in the sense of the strip approximation and expressed in terms of the low-energy resonances in the direct channel as (see Ref. 13)

$$\begin{aligned} \frac{1}{\pi}\int_{\mathrm{high}\,u} \frac{A_u(u',s)}{u'-u}\,du' &+ \frac{1}{\pi}\int_{\mathrm{high}\,t} \frac{A_t(t',s)}{t'-t}\,dt' \\ &\simeq \frac{1}{\pi}\int_{(3,3)} \frac{A_s(s',t)}{s'-s}\,ds'. \end{aligned} \quad (13)$$

This expression has been achieved on the basis of the relation between the high-energy states of high angular momenta in the crossed channels and the low-energy resonance in the direct channel; it completes our initial assumption.

Thus, we get the amplitude (12) distributed into all three channels $s$, $u$, and $t$ as follows:

$$\begin{aligned} {}^I A^i(s,u,t) \simeq \frac{R_s^i}{m^2-s} + \frac{R_u^i}{m^2-u} &+ \frac{1}{\pi}\int_{(3,3)} \frac{A_u^i(u',s)}{u'-u}\,du' \\ &+ \frac{1}{\pi}\int_{\rho} \frac{A_t^i(t',s)}{t'-t}\,dt' + \frac{1}{\pi}\int_{(3,3)} \frac{A_s^i(s',t)}{s'-s}\,ds'. \end{aligned} \quad (14)$$

Projections of the invariant amplitudes ${}^I A^i$ and analytical continuation into the point $s = \alpha_-$ in the unphysical region yield then a sufficiently high approximation for the amplitudes ${}^I f_{1+}(s)$:

$$\begin{aligned} &{}^I f_{1+}(s) = {}^I f_{1+}^{(N)}(s) + {}^I f_{1+}^{(N^*)}(s) + {}^I f_{1+}^{(\rho)}(s) + {}^I f_{1+}^{(h)}(s), \\ &I = \tfrac{1}{2} \text{ or } \tfrac{3}{2}; \quad s = s_R = \alpha_- = (m-1)^2. \end{aligned} \quad (15)$$

The first three terms have been determined by

Frautschi and Walecka (see errata in Ref. 13) and the fourth can be calculated (from the high-energy contributions in the crossed channels $u$ and $t$); we then obtain for $I=\frac{3}{2}$ (upper numbers in the first brackets) and $I=\frac{1}{2}$ (lower numbers) the relations ($s=x^2$, $x\equiv W$)

$$f_{1+}^{(N)}(x)=\begin{pmatrix}1\\-\frac{1}{2}\end{pmatrix}\frac{g^2}{4}\left((x-m)\beta_+\frac{Q_1(\alpha_1)}{q^4}+(x+m)\beta_-\frac{Q_2(\alpha_1)}{q^4}\right), \tag{16}$$

$$f_{1+}^{(N^*)}(x)=-\begin{pmatrix}1\\4\end{pmatrix}\frac{2W_R^2}{9}\mathfrak{f}^2 q_T^2\Big([3x^*(W_R+2m-x)\beta_{R+}^{-1}+(W_R-2m+x)\beta_{R-}^{-1}]\beta_+\frac{Q_1(\alpha_2)}{q^4}$$
$$+[3x^*(W_R+2m+x)\beta_{R+}^{-1}+(W_R-2m-x)\beta_{R-}^{-1}]\beta_-\frac{Q_2(\alpha_2)}{q^4}\Big), \tag{17}$$

$$f_{1+}^{(\rho)}(x)=\begin{pmatrix}-1\\2\end{pmatrix}\frac{3}{8\pi}\left[\left(\frac{\gamma_2}{m}(x^2+\tfrac{1}{2}t_R-m^2-1)-(x-m)(\gamma_1+2\gamma_2)\right)\beta_+\frac{Q_1(\alpha_3)}{q^4}\right.$$
$$\left.+\left(\frac{\gamma_2}{m}(x^2+\tfrac{1}{2}t_R-m^2-1)+(x+m)(\gamma_1+2\gamma_2)\right)\beta_-\frac{Q_2(\alpha_3)}{q^4}\right], \tag{18}$$

$$f_{1+}^{(h)}(x)=\begin{pmatrix}1\\4\end{pmatrix}\frac{4}{3}\mathfrak{f}^2\frac{W_R^2}{W_R-x}\frac{\beta_+}{\beta_{R+}}. \tag{19}$$

Here, the three-momentum $q$ as well as the quantity $q_T$ are given by the expression (8); in the latter case the variable $s$ is replaced by the square $W_R^2$, i.e., the position of the (3,3) resonance in the complex $s$ plane. The quantities $Q_i$ ($i=1,2$) are Legendre polynomials of the second kind of order $i$. The symbols $\beta_{R\pm}$, $\beta_\pm$, $\alpha_i$, $\xi_i$ ($i=1,2,3$), and $x^*$ denote

$$\beta_{R\pm}=(W_R\pm m)^2-1,\quad \beta_\pm=(x\pm m)^2-1, \tag{20}$$

$$\alpha_k=\frac{\xi_k}{q^2}-1\quad(k=1,2),\quad 2\xi_1=x^2-m^2-2,$$
$$2\xi_2=x^2-2m^2-2+W_R^2,$$
$$\alpha_3=\frac{\xi_3}{q^2}+1,\qquad 2\xi_3=t_R, \tag{21}$$
$$-x^*=\frac{\xi_2}{q_t^2}-1,$$

where $W_R$ indicates the position of the (3,3) resonance with the width $\gamma_{33}=\frac{4}{3}\mathfrak{f}^2$ and $t_R$ is the position of the $\pi$-$\pi$ resonance. For the factors $\gamma_1$ and $\gamma_2$, we take theoretical values from the study[14] of the electromagnetic structure of the nucleon, $\gamma_1=-4.91$, $\gamma_2=-11.7$.

Substituting the relations (16)–(19) into Eq. (15) and then Eq. (15) into Eq. (2), we obtain one of the determining equations for the unknown parameters $b_1$ and $b_2$. Afterwards we get the second one by differentiation of both sides of Eq. (1). To determine the effective-range parameters $b_i$, we use the above expressions at the point $s=\alpha_-$.

Through tedious calculations which include nine numerical integrals of the type (4) in the coefficients, we get for the residues $b_i$ ($i=1,2$) the determining equations

$$^I\alpha_1\,^Ib_1+{}^I\alpha_2\,^Ib_2={}^I\gamma_1(1+C)+{}^I\delta_1(1+C(m^2)),$$
$$^I\beta_1\,^Ib_1+{}^I\beta_2\,^Ib_2={}^I\gamma_2(1+C)+{}^I\delta_2(1+C(m^2))+{}^I\epsilon_2(1+C'), \tag{22}$$

where $C\equiv C(\alpha_-)$, $C'=(d/ds)C(s)|_{s=\alpha_-}$ and the index $I$ is the isospin quantum number. The numerical values of the separate coefficients are given by

$$^I\alpha_1=\begin{pmatrix}136.176\\136.176\end{pmatrix}\times10^{-4},\qquad {}^I\alpha_2=\begin{pmatrix}16.021\\16.021\end{pmatrix}\times10^{-4},$$
$$^I\gamma_1=\begin{pmatrix}1.265\,45\\4.811\,79\end{pmatrix},\qquad {}^I\delta_1=\begin{pmatrix}5.235\,01\\5.235\,01\end{pmatrix};$$
$$^I\beta_1=\begin{pmatrix}-0.5258\\-0.021\,95\end{pmatrix}\times10^{-4},\qquad {}^I\beta_2=\begin{pmatrix}0.2806\\0.4285\end{pmatrix}\times10^{-4},$$
$$^I\gamma_2=\begin{pmatrix}-0.302\,07\\0.187\,28\end{pmatrix},\qquad {}^I\delta_2=\begin{pmatrix}-0.413\,21\\-0.406\,54\end{pmatrix},$$
$$^I\epsilon_2=\begin{pmatrix}1.265\,45\\4.811\,79\end{pmatrix}. \tag{23}$$

The upper row is here again valid for isospin $I=\frac{3}{2}$ and the lower one for $I=\frac{1}{2}$. These calculations have been carried out using the noncorrelated physical quantities,

$$g^2=14.97,\quad m=6.7974,\quad W_R=8.9674 \tag{24}$$

(in pion units), given by experiment. The problem of self-consistency in the $P_{33}$ wave is mentioned below.

The effective-range parameters $b_i$ ($i=1,2$) are determined in this way for both cases of isospin, with the help of the determinant of system $\mathfrak{D}$ and the determinant $\mathfrak{D}_i$ in which the $i$th column is replaced by the column on the right-hand side of Eq. (22), in the form

$$ {}^I b_i = {}^I\mathfrak{D}_i / {}^I\mathfrak{D}, \quad I = \tfrac{1}{2}, \tfrac{3}{2}, \quad i = 1, 2. \tag{25} $$

The calculated parameters $b_i$ then provide the $N/D$ $P$-wave amplitudes for the scattering of $\pi$ mesons on nucleons. The influence of CDD poles is included in this calculation in the pole term $C(s)$ which may generally contain adjustable parameters, discussed below.

We now apply the effective-range theory, as specified in Sec. II, in more detail to the calculation of the scattering length $a_{13}$ of pion-nucleon scattering. This will give us a possible test of the partial $P$-wave $N/D$ amplitude (2) with isospin $I=\frac{1}{2}$.

We shall show that this amplitude may be considered as belonging to the class of functions which do not contain any CDD pole so that it is possible to put the term $C$ in the $D$ function (2.4) equal to zero.

The scattering length $a_{2I,2J}$ is generally defined by Eq. (2) as

$$ a_{2I,2J} = \frac{{}^I N_l(\alpha_+)}{\alpha_+ {}^I D_l(\alpha_+)}, \quad \alpha_+ = (m+1)^2. \tag{26} $$

With the help of the parameters $b_i$ $(i=1,2)$, given by Eq. (25) for $I=\frac{1}{2}$, which equal

$$ b_1 = -12\,996.0, \quad b_2 = 116\,733.0 \quad (b_3 = -208.7357), \tag{27} $$

the $P_{13}$ amplitude is determined by the $N$ function (10) and the denominator function $D$ of the form (4) for $C_l(s) \equiv 0$, where

$$ l = 1_+, \quad \rho(s)R(s) = \frac{q^3(s)}{s} = (4s^2)^{-1}(s-\alpha_+)^{3/2}(s-\alpha_-)^{3/2}, $$
$$ s_0 = \alpha_-, \quad s_t = \alpha_+, \quad \alpha_\pm = (m \pm 1)^2. \tag{28} $$

On the basis of the $N/D$ amplitude with $I=\frac{1}{2}$ defined in this way, scattering length $a_{13}$ is calculated through Eq. (26) and is found to be[15]

$$ a_{13} = -0.019. \tag{29} $$

The comparison with the experimental value of Barnes *et al.*,[16]

$$ a_{13} = -0.016 \pm 0.008, \tag{30} $$

shows excellent agreement.

This agreement of both scattering lengths seems to confirm the deductions, achieved by an entirely different method with the help of the "crankshaft analysis" of Atkinson *et al.*,[5,6] that the $P_{13}$ amplitude does not contain any CDD pole in the function $D$ in the one-channel calculation based on the Chew-Mandelstam method.

An analogous approach for the $P_{33}$ wave is more complicated. First of all, by using our Eqs. (22) and (25) in which we put the term $C$ equal to zero, $C \equiv 0$, it is possible to make sure through determining the scattering length $a_{33}$ that the CDD term here differs from zero.[17] This is also in agreement with the previous deductions of Atkinson *et al.*[5,6] On the other hand, the CDD pole which is responsible for the binding activity between channels involves two parameters or only one in the case that the position of the dynamical resonance [e.g., (3, 3) resonance] is known.

In order that our calculations become self-consistently closed this parameter is not arbitrary, but must take a special value which together with the noncorrelated input quantities (24) would give the same values for the positions and the widths of the (3,3) resonance in the $P_{33}$ amplitude.

In this sense, the role of the CDD pole in the $P_{33}$ amplitude and the physical consequences of the derivation of the $P_{13}$ amplitude, especially as far as comparison with experiment is concerned, are further studied with the aid of the Eqs. (22) and (25).

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[16]S. W. Barnes *et al.*, Phys. Rev. 117, 116 (1960); 117, 238 (1960).

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# Eikonal Regge Model for the Processes $\pi^\pm p \to A_1^\pm p$ at Small Momentum Transfers

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An eikonal Regge representation is set up for the helicity amplitudes in the process $\pi p \to A_1 p$. These amplitudes are then used to calculate the density-matrix elements of $A_1(1^+)$, which are in agreement with the data.

## I. INTRODUCTION

The power of the eikonal or absorption model of generating the Regge-cut effects has been used extensively with considerable success in the elastic[1] and also inelastic[2] scatterings. These cuts are found to satisfy properties very similar to those one expects from the study of Feynman graphs.[3] Phenomenologically, a definite evidence for the existence of the Regge cuts, in our opinion, lies not so much in their fitting the data on cross sections as in their giving the correct predictions on measurements like polarizations, density-matrix elements, etc. This is because the Regge cuts have the characteristic slow logarithmic factors appearing in the amplitudes to which the cross sections may in general not be so sensitive.[4] The cuts with a suitable mechanism (like the eikonal or absorption model, which determines their weight functions in terms of the basic input parameters of the poles, would give a better test for their showing up in the polarization and density-matrix-element data[2] which are normalized with respect to the data on the differential cross section.

In this paper we have tried to analyze the data on the density-matrix elements of $A_1(1^+, 1070\text{ MeV})$ in the process $\pi p \to A_1 p$, using the 8-GeV high-energy data[5] on the differential cross section of $\pi^+ p \to A_1^+ p$. For high-energy scattering we use the Regge eikonal model in which, to the Regge-pole contributions coming from the Pomeranchukon ($P$) and the $\rho$ Regge trajectories, are added the $P-P$, $P-\rho$, and $\rho-\rho$ Regge cuts. We assume here that at high energies the $P$ couples only to the helicity-nonflip amplitude in the problem.[6]

The plan of the paper is as follows. Section II deals with some notation, kinematics, the Regge representations of the independent helicity amplitudes in the problem, and the eikonal representation of these amplitudes. In Sec. III we calculate the integrals occurring in Sec. II, and in Sec. IV we give the results of our computations and discuss and compare these with the available data.

## II. FORMALISM

### A. Kinematics, Notation, and Isospin Decomposition for $\pi N \to A_1 N$

We define the following kinematical variables in