its divergence.

But the strange thing that we observe is that values of α (and consequently of β) as obtained from various combinations of Eqs. (6) are not consistent.[3] The results[4] are

(6a) and (6b) give $\alpha = 4.5$, $\beta = 4.8$;
(6a) and (6c) give $\alpha = -2$, $\beta = -2.1$;
(6b) and (6c) give $\alpha = -1.5$, $\beta = -1.58$.

We find this result surprising.

Recently, it has been shown[5] that in the limit of conserved $SU(2)\times SU(2)$ symmetry, the contribution of the $(1, 8+8, 1)$ representation is not as small as originally thought. In fact the corresponding parameter is as large as unity. But this result will not make any change in our conclusion because the only specific information on the symmetry-breaking Hamiltonian that we are using is SU(3) octet breaking.

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[1]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[2]We take the matrix elements at zero momentum transfer squared.

[3]One trivial solution which is common to all is of course $\alpha = \beta = -3$. The inconsistency in the values of α is evident if we consider the determinant formed by the coefficients of the homogeneous equations (3) and (4) which does not vanish.

[4]On the other hand, if we take different combinations of Eqs. (4) and (5) to solve for α we obtain better results. For example, if we divide (5a) by (4) and (5b) by (5c) and solve these two resulting equations simultaneously, we obtain $\alpha = \pm 1.82$ and $\beta = {}^{+1.66}_{-2}$. Similarly, if we divide (5b) by (4) and (5c) by (5a) and again solve, we get $\alpha = \pm 1.73$ and $\beta = \pm 1.76$. Not only are these values consistent with each other, but the magnitude of α is in agreement with generally accepted values of A_d/A_f.

[5]K. Schilcher, Phys. Rev. D 4, 237 (1971).

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Gauge Invariance, Chiral Symmetry, and $\gamma+\gamma\to$ Odd Numbers of Soft Pions*

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Using electromagnetic gauge invariance, chiral symmetry, and the requirement that an amplitude involving soft pions vanish when the neutral-pion four-momentum vanishes (or equivalently that the divergence of the axial-vector current in the symmetry limit be independent of pion fields), a unique expression is derived for the effective Lagrangian describing the interaction of photons with odd numbers of soft pions in arbitrary charge combinations.

Recently there has been a considerable amount of literature investigating the processes $\gamma+\gamma \to \pi^+\pi^-\pi^0$. Aviv, Hari Dass, and Sawyer[1] have applied effective-Lagrangian methods to calculate the matrix elements for neutral and charged cases of $2\gamma\to 3\pi$. While there is general agreement about the result obtained by them for the neutral case, their results have been shown to be incorrect in the case of charged decays.[2,3] The discrepancy has been pointed out to be due to the fact that the innerbrehmsstrahlung graphs arising from the vertex $\gamma\to 3\pi$ were left out and also due to the fact that there should be additional terms present in the effective Lagrangian to render these pole-graph matrix elements gauge-invariant. Adler, Lee, Treiman, and Zee, and Wess and Zumino have calculated the correct effective Lagrangian describing the 3π process. In this paper we derive an effective Lagrangian for the processes $\gamma\to$ odd π's and $2\gamma\to$ odd π's using the principles of electromagnetic gauge invariance and chiral symmetry.

The starting point of our derivation is the observation, due to Adler,[4,5] that the divergence of the axial-vector current (in the presence of electromagnetism) does not vanish in the limit of chiral symmetry but is given by[6]

$$\partial_\mu j^\mu_{5,a} = C_a \vec{E}\cdot\vec{B} \qquad (a = \text{neutral}),$$

where C_a is *independent* of the pion fields. In terms of an effective Lagrangian describing the interaction of pions with photons this is equivalent to the statement that under chiral transformation about the neutral axis the variation in the Lagrangian is given by

$$\delta\mathcal{L}=C_a\vec{E}\cdot\vec{B}.$$

This has also been conjectured by Aviv and Sawyer[7] based on their loop-model calculations. C_a, however, is believed to be model-dependent. We shall show in this paper that the combined requirements of gauge invariance, restricted chiral symmetry (the particular form of $\delta\mathcal{L}$ believed to be right), and the experimental width for the $\pi^0\to 2\gamma$ decay are sufficiently restrictive as to ensure a unique expression for $\mathcal{L}$ and hence C_a.

Since the variation in the Lagrangian is quadratic in the electromagnetic field, the one-photon–n-pion part of the effective Lagrangian does not contribute to $\delta\mathcal{L}$. This at once suggests that the one-photon effective Lagrangian must be chiral-invariant. The form of the one-photon–three-pion Lagrangian as calculated by Wess and Zumino is

$$\frac{-ie}{24\pi^2 f_\pi^{\,3}}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}\pi^0\partial_\sigma\pi^+\partial_\tau\pi^-.$$

As pointed out by Weinberg,[8] the only chiral-invariant Lagrangian is the one that is constructed not out of ordinary derivatives but covariant derivatives $D_\mu\vec{\pi}$. It is not difficult to show that the only possible chiral-invariant function involving at least three pion fields and a single electromagnetic field with at the most three derivatives is

$$\epsilon_{\mu\nu\sigma\tau}A_\mu D_\nu\vec{\pi}\cdot(D_\sigma\vec{\pi}\times D_\tau\vec{\pi}).$$

The actual form of the covariant derivatives depends upon the particular gauge chosen for the pion interpolating field. In general, the structure of the covariant derivative is

$$D_\mu\vec{\pi}=A(\vec{\pi}^2)\partial_\mu\vec{\pi}+B(\vec{\pi}^2)\partial_\mu\vec{\pi}^2\vec{\pi}.$$

Thus the "one-photon–odd-number-of-pions" part of the effective Lagrangian must have the form

$$\mathcal{L}_1=\lambda\frac{e}{3f_\pi^{\,3}}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}A_\mu D_\nu\vec{\pi}\cdot(D_\sigma\vec{\pi}\times D_\tau\vec{\pi}).$$

Under the electromagnetic gauge transformation

$$A_\mu\to A_\mu+\partial_\mu\chi,\quad \pi^\pm\to e^{\pm ie\chi}\pi^\pm$$

the above Lagrangian is not invariant. To compute the change in $\mathcal{L}_1$ under this transformation, we introduce the notation

$$\pi_i\to R_{ij}\pi_j,\quad \partial_\mu\pi_i\to\partial_\mu\pi_j R_{ij}+e\partial_\mu\chi T_{ij}\pi_j,$$

where

$$R=\begin{pmatrix}\cos e\chi & -\sin e\chi & 0\\ \sin e\chi & \cos e\chi & 0\\ 0 & 0 & 0\end{pmatrix},\quad eT_{ij}\partial_\mu\chi=\partial_\mu R_{ij}.$$

Then

$$D_\mu\pi_i\to R_{ij}D_\mu\pi_j+eD_\mu\chi T_{ij}\pi_j,$$

with

$$D_\mu\chi=A(\vec{\pi}^2)\partial_\mu\chi.$$

Using the transformation property of $D_\mu\pi_i$, we find the change in $\mathcal{L}_1$ under the photon gauge transformation to be

$$\begin{aligned}\Delta\mathcal{L}_1&=\lambda\frac{e^2}{f_\pi^{\,3}}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}A_\mu D_\nu\pi^0\\&\quad\times[A(\vec{\pi}^2)+2B(\vec{\pi}^2)\vec{\pi}^2]\partial_\sigma\vec{\pi}^2D_\tau\chi\\&=\lambda\frac{e^2}{f_\pi^{\,3}}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}A_\mu\partial_\nu\pi^0(A+2B\vec{\pi}^2)\partial_\sigma\vec{\pi}^2\partial_\tau\chi A^2.\end{aligned}$$

The form of the "two-photon–pions" interaction must be such as to cancel $\Delta\mathcal{L}_1$ under the gauge transformation since the total Lagrangian must be gauge-invariant. Also, the G-parity selection rule for $2\gamma\to 3\pi$ combined with the $\Delta I<3$ selection rule for second-order electromagnetism restricts the most general form of this Lagrangian to be

$$\begin{aligned}\mathcal{L}_2&=\rho\frac{e^2}{f_\pi^{\,3}}\frac{1}{48\pi^2}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma D_\tau\pi^0G(\vec{\pi}^2)\\&\quad+\xi\frac{e^2}{f_\pi}\frac{1}{48\pi^2}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\tilde{B}(\vec{\pi}^2)\pi^0.\end{aligned}$$

We shall now show that ρ, ξ, λ, $G(\vec{\pi}^2)$, and $B(\vec{\pi}^2)$ are uniquely fixed by our requirements above.

Under the gauge transformation the change produced in $\mathcal{L}_2$ is given by

$$\begin{aligned}\Delta\mathcal{L}_2&=\rho\frac{e^2}{f_\pi^{\,3}}\frac{1}{48\pi^2}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}\partial_\sigma\chi D_\tau\pi^0G(\vec{\pi}^2)\\&=\rho\frac{e^2}{f_\pi^{\,3}}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}\partial_\mu A_\nu\\&\quad\times[A(\vec{\pi}^2)\partial_\tau\pi^0+\tilde{B}(\vec{\pi}^2)\partial_\tau\vec{\pi}^2\pi^0]G(\vec{\pi}^2)\partial_\sigma\chi.\end{aligned}$$

Transferring the ∂_μ on A_ν to the pion fields, we have

$$\Delta\mathcal{L}_2=-\frac{e^2}{f_\pi^{\,3}}\rho\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}A_\mu\partial_\tau\chi\partial_\nu\pi^0\partial_\sigma\vec{\pi}^2[\tilde{B}G-(AG)'],$$

where the prime indicates differentiation with respect to $\vec{\pi}^2$. Thus our total effective Lagrangian will be explicitly gauge-invariant if and only if

$$A^2(A+2B\vec{\pi}^2)=[\tilde{B}G-(AG)']\frac{\rho}{\lambda}.$$

For convenience we shall demand that the $\pi^0 \to 2\gamma$ part is contained entirely in the piece with $\tilde{B}(\vec{\pi}^2)$. This would impose the boundary condition on $G(\vec{\pi}^2)$, viz.,

$$G(0)=0\,.$$

So far we have not restricted ourselves to any particular gauge for the pion interpolating field. But for ease of calculation we shall adopt the "σ-model gauge." In this gauge,

$$A=1,\quad B=\frac{1}{2}\,\frac{1}{f_\pi\sigma+\sigma^2}\,,$$

where

$$\sigma=(f_\pi{}^2-\vec{\pi}^2)^{1/2}\,.$$

Thus the differential equation for G becomes

$$\frac{1}{2}\,\frac{G}{f_\pi+\sigma}+\frac{1}{2}\,\frac{dG}{d\sigma}=+f_\pi\,\frac{\lambda}{\rho}\,,$$

with $G(0)=0$. The solution is

$$G=\left(f_\pi(f_\pi+\sigma)-\frac{4f_\pi{}^3}{f_\pi+\sigma}\right)\frac{\lambda}{\rho}\,.$$

In the σ-model gauge the variations in the fields and their derivatives produced by a chiral transformation about the neutral axis are given by

$$\delta\pi_i=-i\sigma\delta_{i0},\quad \delta\sigma=i\pi^0,\quad \delta\vec{\pi}^2=-2i\sigma\pi^0,$$

$$\delta(D_\mu\pi_i)=\frac{i}{f_\pi+\sigma}\,(\delta_{i0}\vec{\pi}\cdot D_\mu\vec{\pi}-D_\mu\pi^0\pi_i)\,.$$

Using these variations, the change in $\mathcal{L}_2$ under the chiral transformation about the neutral axis can easily be computed:

$$\delta\mathcal{L}_2=\rho\,\frac{e^2}{f_\pi{}^3}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma(\delta G D_\tau\pi^0+G\,\delta D_\tau\pi^0)$$

$$+\,\xi\,\frac{e^2}{f_\pi}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}(\delta\pi^0\tilde{B}+\delta\tilde{B}\pi^0)\,.$$

We have

$$\delta G=\left(if_\pi\pi^0+\frac{4f_\pi{}^3 i}{(f_\pi+\sigma)^2}\,\pi^0\right)\frac{\lambda}{\rho}\,,$$

$$\delta D_\tau\pi^0=\frac{i}{f_\pi+\sigma}\,(\vec{\pi}\cdot D_\tau\vec{\pi}-\pi^0 D_\tau\pi^0)\,,$$

$$\delta\tilde{B}=i\,\frac{d\tilde{B}}{d\sigma}\,\pi^0.$$

The coefficient of $\pi^0 D_\tau\pi^0$ in $\delta\mathcal{L}_2$ is

$$\lambda\,\frac{ie^2}{f_\pi{}^3}\,\frac{f_\pi}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma\left(1+\frac{4f_\pi{}^2}{(f_\pi+\sigma)^2}-\frac{G}{f_\pi(f_\pi+\sigma)}\right)$$

$$=8\lambda\,\frac{ie^2}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma\,\frac{1}{(f_\pi+\sigma)^2}\,.$$

But

$$\frac{1}{(f_\pi+\sigma)^2}\,\pi^0 D_\tau\pi^0=\tfrac{1}{2}\partial_\tau\,\frac{(\pi^0)^2}{(f_\pi+\sigma)^2}\,.$$

Hence the contribution to $\delta\mathcal{L}_2$ from the above term is

$$+\tfrac{1}{2}\times\tfrac{1}{2}\times 8\lambda\,\frac{ie^2}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\,\frac{(\pi^0)^2}{(f_\pi+\sigma)^2}\,.$$

The term containing $\vec{\pi}\cdot D_\tau\vec{\pi}$ is

$$\frac{-\lambda ie^2}{f_\pi{}^3}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma$$

$$\times\,\frac{\vec{\pi}\cdot D_\tau\vec{\pi}}{f_\pi+\sigma}\left(-f_\pi(f_\pi+\sigma)+\frac{4f_\pi{}^3}{f_\pi+\sigma}\right).$$

Since $\vec{\pi}\cdot D_\tau\vec{\pi}\equiv -f_\pi\partial_\tau\sigma$, this term can be put into the form

$$\frac{\lambda ie^2}{f_\pi{}^3}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma f_\pi{}^2\partial_\tau\left(-\sigma-\frac{4f_\pi{}^2}{f_\pi+\sigma}\right)$$

$$=-\frac{\lambda ie^2}{2f_\pi}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\left(\sigma+\frac{4f_\pi{}^2}{f_\pi+\sigma}\right).$$

If we demand that the terms proportional to $(\pi^0)^2$ in $\delta\mathcal{L}_2$ be zero, as indeed we must if $\delta\mathcal{L}$ is to be independent of pion fields, we have

$$\frac{i}{4}\,8\lambda\,\frac{e^2}{48\pi^2}\,\frac{1}{(f_\pi+\sigma)^2}=-i\xi\,\frac{e^2}{f_\pi}\,\frac{1}{48\pi^2}\,\frac{d\tilde{B}}{d\sigma}$$

or

$$\frac{d\tilde{B}}{d\sigma}=-\frac{2\lambda}{\xi}\,\frac{f_\pi}{(f_\pi+\sigma)^2}\,.$$

This yields

$$\tilde{B}=\frac{2\lambda}{\xi}\,\frac{f_\pi}{f_\pi+\sigma}+D\,,$$

where D is a constant to be fixed. Thus

$$\delta\mathcal{L}_2=-\frac{\lambda ie^2}{2f_\pi}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\left(\sigma+\frac{4f_\pi{}^2}{f_\pi+\sigma}\right)-\frac{i\xi e^2}{f_\pi}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\sigma\left(D+\frac{2\lambda}{\xi}\,\frac{f_\pi}{f_\pi+\sigma}\right)$$

$$=-\frac{\lambda ie^2}{2f_\pi}\,\frac{1}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\sigma\left(1+2\frac{D\xi}{\lambda}\right)-\frac{2ie^2f_\pi\lambda}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}$$

$$=-\frac{2ie^2f_\pi\lambda}{48\pi^2}\,\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\,,$$

choosing $D=-\lambda/2\xi$.

Thus we have the desired result. The expression for $\mathcal{L}$ is therefore

$$\mathcal{L}=\lambda\left[\frac{e}{3f_\pi^{\,3}}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}A_\mu D_\nu\vec{\pi}\cdot(D_\sigma\vec{\pi}\times D_\tau\vec{\pi})+\frac{e^2}{f_\pi^{\,3}}\frac{1}{48\pi^2}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}A_\sigma D_\tau\pi^0\left(f_\pi(f_\pi+\sigma)-\frac{4f_\pi^{\,3}}{f_\pi+\sigma}\right)\right.$$
$$\left.+\frac{e^2}{f_\pi}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\pi^0\left(\frac{f_\pi}{f_\pi+\sigma}-\frac{1}{4}\right)\right].$$

λ is now fixed by demanding that $\mathcal{L}_2$ contain the correct[9] $\pi^0\to 2\gamma$ effective Lagrangian. The $\pi^0\to 2\gamma$ effective Lagrangian according to our theory is

$$\mathcal{L}_2^{\pi^0\to 2\gamma}=-\lambda\frac{e^2}{4f_\pi}\frac{1}{24\pi^2}\epsilon_{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}\pi^0$$
$$=-\frac{\lambda\alpha}{6\pi f_\pi}\pi^0\vec{\mathrm{E}}\cdot\vec{\mathrm{B}}.$$

Thus λ is fixed by the π^0 lifetime to be

$$\left(\frac{64\pi^2}{m_\pi^{\,3}\tau_0}\right)^{1/2}\frac{6\pi f_\pi}{\alpha}=\lambda,$$

where τ_0 is the lifetime of $\pi^0\to 2\gamma$ and $\alpha=e^2/4\pi$.

It is easy to check that our effective Lagrangian reproduces the structure obtained by Wess and Zumino for 3π's except for a multiplicative factor. The $2\gamma\to 3\pi^0$ is identically zero in agreement with earlier literature. Also the ratio "$F^{3\pi}/F^{\pi}$" introduced in Ref. 2 is automatically fixed in our treatment.

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[1]R. Aviv, N. D. Hari Dass, and R. F. Sawyer, Phys. Rev. Letters 26, 591 (1971).

[2]S. L. Adler, B. W. Lee, S. B. Treiman, and A. Zee, Phys. Rev. D 4, 3497 (1971).

[3]Julius Wess and Bruno Zumino, CERN Report No. CERN-Th-1398.

[4]S. L. Adler, Phys. Rev. 177, 2426 (1969).

[5]C. R. Hagen, Phys. Rev. 177, 2622 (1969).

[6]This particular form of $\partial_\mu j_\mu^{5,a}$ has been proved within particular models by Adler and is believed to be a rather general result. This has the attractive feature that it ensures Adler's consistency condition even in the presence of anomalies.

[7]R. Aviv and R. F. Sawyer, Phys. Rev. D 4, 451 (1971).

[8]S. Weinberg, Phys. Rev. 166, 1568 (1968).

[9]J. Schwinger, Phys. Rev. 82, 664 (1951).