Fields of the American Physical Society at Rochester, N. Y.

[1]S. Weinberg, Phys. Rev. 177, 2604 (1969).

[2]R. Rockmore, Phys. Rev. D 3, 3039 (1971).

[3]R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969).

[4]Our notation follows that of Ref. 3 closely.

[5]We take $\vec{t} = \frac{1}{2}\vec{\tau}$ and $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$, $\gamma_5 = \gamma^0\gamma^1\gamma^2\gamma^3$, with $\gamma^{\mu\dagger} = \gamma^\mu$ and $\gamma_5{}^2 = 1$. Note that there is a scale factor of 2 in the present definitions of $n\cdot\vec{\hat{A}}$, $n\cdot\vec{\hat{V}}$ as compared with those of Ref. 2.

[6]As in Ref. 2, expressions like $(AB\cdots Z)_{\beta\alpha}$ are to be comprehended as

$$(AB\cdots Z)_{\beta\alpha} = \sum_{\gamma\cdots\delta} A_{\beta\gamma}\cdots Z_{\delta\alpha},$$

the sums running over stable target states.

[7]Note that the requirement

$$[n\cdot\vec{\hat{A}}_j, n\cdot\hat{Q}_0^{(m)}]_{\beta\alpha} = 0$$

is also consistent with the vanishing ETCR

$$\int d^3z_1 [A_j^0(\vec{x}, x_0), n\cdot Q_0^{(k)}(\vec{z}_1, x_0)] = 0.$$

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Bound-State Problem in the Electrodynamics of Spin-Zero Particles*

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The relativistic corrections to the Coulomb binding energies of a system of two spin-zero particles of opposite charge are calculated and compared with the previous results of other authors. The decay of the bound state into two photons is discussed for various angular momentum states and a possible mechanism for production of the bound state is mentioned.

The energy levels of positronium have long ago been calculated to a very high accuracy in an approach based on the Bethe-Salpeter equation of quantum electrodynamics,[1] and the results agree with experiments. Nevertheless various other methods have been proposed recently for calculating bound-state energies in relativistic systems with long-range forces arising from massless scalar or vector-meson exchange.[2-4] In part these efforts are motivated by a suspicion that the Bethe-Salpeter approach may be incomplete or incorrect (as suggested by the existence of abnormal solutions in soluble cases), but the main motivation is probably to test new methods on problems that have already been worked out by means of old ones.

One particular calculation which has been done by several authors is that of the binding energies of an electrodynamically bound system of two spin-zero oppositely charged particles in the theory of pure quantum electrodynamics. These calculations center on evaluating the leading relativistic corrections (of order α^4) to the binding energies. Speculation as to these terms have been made on the basis of the eikonal approximation,[2] from a pseudopotential approach,[5] and from an examination of the infrared factor.[6] But to our knowledge no straightforward calculation based on the Bethe-Salpeter equation has ever been presented for the sake of comparison. An estimate of the α^4 terms on the basis of a classical semirelativistic Hamiltonian seems to us to be ambiguous and there is not any good reason to get an answer by throwing out the spin-dependent terms in the known positronium result.

In the present paper we report the result of the calculation on the basis of the Bethe-Salpeter equation and compare our results with the speculation mentioned above. The problem which has been solved is the following: given a positively charged spinless boson of mass m_1 and a negatively charged one of mass m_2, interacting via a photon exchange, to find the binding energies to order α^4. From there we go to the special case of equal masses, which would describe a bound state of π^+, π^- mesons; this bound state we call pionium, like positronium in the parallel case of fermions.

In Sec. I we present an outline of the calculation and comparison with other results mentioned above. In Sec. II we discuss the decay widths of pionium and in Sec. III consider a possible production mechanism.

I. CALCULATION OF THE BINDING ENERGY OF PIONIUM

Here we first mention the Feynman rules we use, establish the Bethe-Salpeter equation for our case, and then proceed to solve for its wave function and energies to various orders. The Feynman

rules were derived from the Coulomb-gauge Hamiltonian

$$H_{\rm em} = H_{\rm transverse\ photons} + \frac{1}{8\pi}\int d^3x d^3x' \frac{\rho(\vec{x},t)\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|} \tag{1}$$

in which ρ is the boson charge density, given in the interaction picture by

$$\rho(x) = ie[\varphi_1^*(x)\overleftrightarrow{\partial}_0\varphi_1(x) - \varphi_2^*(x)\overleftrightarrow{\partial}_0\varphi_2(x)], \tag{2}$$

where φ_1 is the field of mass m_1, charge e and φ_2 that of mass m_2 and charge $-e$. The Feynman rules turn out to be equivalent to using the usual vertex factors, $\pm ie(p+p')_\mu$ for a single-photon vertex, $2ie^2 g_{\mu\nu}$ for a two-photon vertex, and a propagator for the electromagnetic field, $D_{\mu\nu}(k)$,

$$\begin{aligned} &D_{00}(k) = 1/|\vec{k}|^2, \\ &D_{0i}(k) = D_{i0}(k) = 0, \\ &D_{ij}(k) = \frac{1}{k_\mu k^\mu}\left(\delta_{ij} - \frac{k_i k_j}{|\vec{k}|^2}\right). \end{aligned} \tag{3}$$

Thus diagrams such as those of Figs. 1(b), 1(c), and 1(d) have meaning for the exchange of Coulomb photons, even though the Hamiltonian (1) has no part recognizable as an $A_0 A_0 \varphi^* \varphi$ interaction.

We begin from the Bethe-Salpeter (B-S) equation[7] for single-ladder graphs, given in the center-of-mass frame by

$$G_0^{-1}(P,p)\psi_B(p) = \int d^4p' V(P,p,p')\psi_B(p'), \tag{4}$$

where $P = (E, \vec{0})$ is the momentum associated with the center of mass and p and p' are the relative momenta; E is the total energy of the system. P is further subdivided into P_1 and P_2 for convenience:

$$P_1 = (E_1, \vec{0}), \qquad P_2 = (E_2, \vec{0}), \tag{5}$$

$$E_1 = m_1 + \tfrac{1}{2}E_B, \quad E_2 = m_2 + \tfrac{1}{2}E_B, \tag{6}$$

$$E = E_1 + E_2; \tag{7}$$

here E_B is the binding energy of the system. In Eq. (4), $\psi_B(p)$ is the B-S wave function and G_0^{-1} and V are given by

$$G_0^{-1}(P,p) = [(P_1+p)^2 - m_1^2 + i\epsilon][(P_2-p)^2 - m_2^2 + i\epsilon], \tag{8}$$

$$V(P,p,p') = -\frac{2i\alpha}{(2\pi)^3}(2P_1 + p + p')_\mu \times D^{\mu\nu}(p-p')(2P_2 - p - p')_\nu . \tag{9}$$

The "potential" V is now split into $V^0 + \delta V$, where V^0 is the instantaneous Coulomb interaction obtained from (9) by keeping only terms of lowest order in α [see discussion following Eq. (21)], i.e.,

$$V^0(P,p,p') = -\frac{2i\alpha}{(2\pi)^3}\frac{4m_1 m_2}{|\vec{p}-\vec{p}'|^2}, \tag{10}$$

and δV contains the difference between the true Coulomb potential and (10), as well as the exchange of transverse photons.

The problem with V^0 can be solved to give the lowest-order B-S wave function ψ^0. Defining a three-dimensional wave function

$$\varphi(\vec{p}) = (\text{const})\int dp_0 \psi_B(p_0, \vec{p}), \tag{11}$$

Eq. (4) reduces to

$$\psi_B(p) = (\text{const})\, G_0(P,p)\int d^3p\, V^0(P,p,p')\varphi(\vec{p}'). \tag{12}$$

Integrating Eq. (12) over p_0 we have

$$\varphi(\vec{p}) = g_0(\vec{p})\left(-\frac{8i\alpha m_1 m_2}{(2\pi)^3}\right)\int d^3p' \frac{1}{|\vec{p}-\vec{p}'|^2}\varphi(\vec{p}'), \tag{13}$$

where

$$g_0(\vec{p}) \equiv \int dp_0 G_0(P,p) = -2\pi i \frac{Q_1(E_1+E_2+Q_1) + Q_2(E_1+E_2-Q_2)}{2Q_1Q_2(E_1+E_2+Q_1+Q_2)(E_1+E_2+Q_1-Q_2)(E_1+E_2-Q_1-Q_2)}, \tag{14}$$

and we have defined

$$Q_i \equiv (m_i^2 + \vec{p}^2)^{1/2} = m_i + \frac{\vec{p}^2}{2m_i} - \frac{\vec{p}^4}{8m_i^3} + \cdots . \tag{15}$$

It will be shown later in our calculations that the three-dimensional momentum $\vec{p}$ occurring as function of the φ's is effectively an order α smaller in magnitude compared with the mass m_i. If then in (14) we keep only the lowest-order terms, Eq. (13) reduces to the Schrödinger equation for Coulomb potential in momentum space,

$$\left(E_B - \frac{\vec{p}^2}{2\mu}\right)\varphi(\vec{p}) = -\frac{\alpha}{2\pi^2}\int d^3p' \frac{1}{|\vec{p}-\vec{p}'|^2}\varphi(\vec{p}'), \tag{16}$$

where μ is the reduced mass of the system. The solutions for φ and E_B are well known[8]; the value of the binding energy obtained from this equation we call $E_B^{(0)}$:

$$E_B^{(0)} = -\frac{\mu\alpha^2}{2n^2}. \tag{17}$$

Since we know $\varphi(\vec{p})$, Eq. (16) can be used to obtain

a first approximation to the B-S wave function $\psi_B(p)$; from Eqs. (12) and (16) we have

$$\psi_B^{(0)}(p) = NG_0(P,p)(E_B - \vec{p}^2/2\mu)\varphi(\vec{p}), \tag{18}$$

where N is a constant to be determined. To evaluate N one resorts to the normalization condition. Starting from the two-body Green's function in momentum space[9] near the bound state,

$$G(P,p,p') = i\,\frac{\psi_B(p)\psi_B(p')}{E - E_B} + \cdots, \tag{19}$$

and the equation for G,

$$G = G_0 + G_0 V G, \tag{20}$$

one can obtain the normalization condition

$$i\int d^4p\, d^4p'\,\bar{\psi}_B(p)\frac{\partial G^{-1}(P,p,p')}{\partial E}\psi_B(p') = 1. \tag{21}$$

We shall evaluate this integral in some detail because important aspects of approximations we make throughout this paper are aptly exhibited here. We are interested in the lowest order for $\psi^{(0)}$ and so approximate G^{-1} by G_0^{-1} in (21). The relative energy integrals can be performed trivially using a contour in the upper half planes.

All quantities in the remaining integral are functions of $|\vec{p}|^2$ and the wave functions φ go like[10] at least $(\vec{p}^2 + m^2\alpha^2)^{-2}$. In all integrations with $|\varphi(\vec{p})|^2$, therefore, there will be very strong singularities at $|\vec{p}| = \pm i m\alpha$ and these will generally dominate over all other poles. In the present case and in all the cases we encounter, it is correct to estimate the leading order in α of an integral by replacing all such $|\vec{p}|$ by $m\alpha$. This explains why we were able to neglect the higher-order terms in obtaining the Schrödinger equation and, partially, why we can approximate $V(P,p,p')$ by $V^0(P,p,p')$. The integral can now be approximated to the lowest order in α. It is of some interest to note that the terms contributing to this lowest order come from the residues of the relative energy pole nearest the origin, $E_2 - Q_2 + i\epsilon$; in fact, it will be found that, generally, the nearby poles contribute the lowest orders in α, while the far-away poles contribute to higher orders. One obtains finally the normalized, lowest-order B-S wave function,

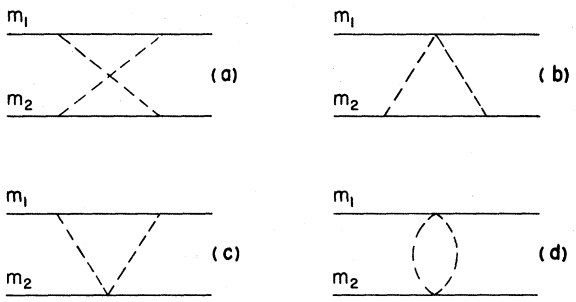

FIG. 1. The fourth-order nonladder graphs, contributing V_a, V_b, V_c, and V_d to the potential.

$$\psi_B^{(0)}(p) = (2m_1m_2/\pi)^{1/2} \times \frac{(E_B^{(0)} - \vec{p}^2/2\mu)\varphi(\vec{p})}{[(P_1+p)^2 - m_1^2 + i\epsilon][(P_2-p)^2 - m_2^2 + i\epsilon]}, \tag{22}$$

where $\varphi(\vec{p})$ is defined by

$$\varphi(\vec{p}) = i(2m_1m_2/\pi)^{1/2}\int dp_0\,\psi_B^{(0)}(p). \tag{23}$$

For the adjoint wave functions, one replaces ψ by $\bar{\psi}$ and φ by φ^*; the $i\epsilon$ prescription remains the same because it has the same source in both forms of the B-S equation.

With Eqs. (4), (13), (16), (22), and (23) all the necessary groundwork has been done and the calculations for corrections of order α^4 to the binding energy can begin. These corrections come essentially from three sources: the part left out from (14) in deducing the Schrödinger equation, the contributions coming from δV, and the contributions from those second-order graphs (Fig. 1) not included in the definition of V.

Corrections from Eq. (14). By expanding Eq. (13) to the next higher order in α one can easily pick up the terms contributing to α^4. In terms of expectation values in the three-dimensional domain, the contribution to δE_B, here called $\delta E_{B,G_0}$, is given by

$$4m_1^2m_2(m_1+m_2)\delta E_{B,G_0} = -2m_1^2m_2(E_B^{(0)})^2 - \frac{2(m_1+m_2)}{m_2}(m_1^2 - m_1m_2 + m_2^2)E_B^{(0)}\langle\vec{p}^2\rangle + \frac{(m_1+m_2)^2(m_1^2+m_2^2)}{2m_1m_2^2}\langle\vec{p}^4\rangle. \tag{24}$$

Using Eq. (16) and taking Fourier transforms, these expectation values can be transformed into those over r^{-n}, which are well known and tabulated[11]:

$$\delta E_{B,G_0} = -\frac{m_1^2 - m_1m_2 + m_2^2}{2m_1m_2(m_1+m_2)}(E_B^{(0)})^2 + \frac{1}{m_1+m_2}\,\alpha E_B^{(0)}\left\langle\frac{1}{r}\right\rangle + \frac{m_1^2+m_2^2}{2m_1m_2(m_1+m_2)}\,\alpha^2\left\langle\frac{1}{r^2}\right\rangle. \tag{25}$$

Corrections from δV. To find the second group of corrections we first set up the formalism. By introducing a variation δ on the B-S equation (4), and multiplying the resultant equation from the left by $\bar{\psi}_B(p)$, we have on integrating over all momenta,

$$\int d^4p\,\bar{\psi}_B(p)\delta G_0^{-1}(P,p)\psi_B(p) - \int d^4p\,d^4p'\bar{\psi}_B(p)\delta V(P,p,p')\psi_B(p')$$

$$= -\int d^4p\left(\bar{\psi}_B(p)G_0^{-1}(P,p) - \int d^4p'\,\bar{\psi}_B(p')V(P,p',p)\right)\delta\psi_B(p). \tag{26}$$

By virtue of the adjoint equation to (4), the right-hand side is zero. Taking the variation of G_0^{-1} to be with respect to E_B, and using the normalization condition, Eq. (21), we find that

$$\delta E_{B,\delta V} = i\int d^4p\,d^4p'\bar{\psi}_B(p)\delta V(P,p,p')\psi_B(p'). \tag{27}$$

From the definition of δV we have

$$\delta V(P,p,p') = -\frac{2i\alpha}{(2\pi)^3}\left(\frac{2(m_1+m_2)E_B^{(0)}}{|\vec{p}-\vec{p}'|^2} + 2(m_2-m_1)\frac{(p_0+p_0')}{|\vec{p}-\vec{p}'|^2} - \frac{(p_0+p_0')^2}{|\vec{p}-\vec{p}'|^2} - \frac{[(\vec{p}+\vec{p}')^2 - (\vec{p}^{\,2}-\vec{p}'^{2})/|\vec{p}-\vec{p}'|^2]}{(p-p')_\mu(p-p')^\mu + i\epsilon}\right). \tag{28}$$

All these terms contribute in various ways. Using the approximation schemes mentioned before, expression (27) can be calculated without too much trouble; it yields

$$\delta E_{B,\delta V} = -\frac{1}{2m_1m_2}\,\alpha\left\langle\frac{1}{r^3}\,\vec{r}\cdot\vec{\nabla}\right\rangle - \frac{m_1+m_2}{m_1m_2}\,\alpha E_B^{(0)}\left\langle\frac{1}{r}\right\rangle - \frac{m_1^2+m_1m_2+m_2^2}{m_1m_2(m_1+m_2)}\alpha^2\left\langle\frac{1}{r^2}\right\rangle + \frac{1}{2m_1m_2}\,l(l+1)\alpha\left\langle\frac{1}{r^3}\right\rangle. \tag{29}$$

Corrections from two-photon exchanges. These corrections come from essentially two sources: First, those second-order graphs (Fig. 1) that have not been included in V; since these are higher order in interaction and presumably have smaller matrix elements, we consider them as perturbations to V; we call this contribution $X(P,p,p')$. Second, in the perturbation calculation of the effects of δV there will be second- and higher-order terms like $\delta V G'\delta V$, $\delta V G'\delta V G'\delta V$, etc., where G' is the full Green's function for the "potential" V^0. It turns out that a part of $\delta V G'\delta V$ contributes to α^4 order, so that the correction to δE_B from these two-photon-exchange terms can be written as a sum

$$\delta E_{B,X} = \langle\delta V G'\delta V\rangle_\psi + \langle X\rangle_\psi, \tag{30}$$

where the expectation values here are with respect to the B-S wave functions ψ.

To extract the α^4 terms from (30) we make approximations in our calculations as before. Each of these terms has a coefficient of, at least, α^2 (an α for each of the δV in the first term, an e^4 from various vertices in the second) so that the integrals in these expectation values need contribute only another factor α^2 for our accuracy. This means that for each internal photon line, we need retain only the D_{00} part of the propagator; the rest will contribute to order α^5 or higher.

Consider now the graph in Fig. 1(c). The contribution X_Δ to X of this graph is, to our approximation,

$$X_\Delta = -\frac{4\alpha^2}{(2\pi)^6}\int\frac{d^3k}{|\vec{p}-\vec{k}|^2|\vec{k}-\vec{p}'|^2} \times(-2)\int\frac{dk_0(2P_1+p+k)_0(2P_1+p'+k)_0}{(P_1+k)^2-m_1^2+i\epsilon}, \tag{31}$$

where k is the loop momentum. The first factor

$$-\frac{4\alpha^2}{(2\pi)^6}\int\frac{d^3k}{|\vec{p}-\vec{k}|^2|\vec{k}-\vec{p}'|^2} \equiv C, \tag{32}$$

say, will be common to all the second-order graphs and, for the moment, we look at the second factor only. Since

$$\langle X_\Delta\rangle = i\int d^4p\,d^4p'\bar{\psi}_B(p)X_\Delta(P,p,p')\psi_B(p'), \tag{33}$$

the lowest order from the p, p' integrals come from the nearby poles as already mentioned. An estimate of α dependence at this stage shows that

$$\langle X_\Delta\rangle \sim \alpha^5\times(k \text{ integration}) \tag{34}$$

so that all we need is that the k integral give a factor α^{-1}. At this point we need to add the other graphs in Fig. 1 and the $\delta V G'\delta V$ term represented by Fig. 2 to get rid of the divergences in the loop relative-energy integrations. It turns out that

$$X \sim \alpha^2\int\frac{d^3k}{|\vec{p}-\vec{k}|^2|\vec{k}-\vec{p}'|^2}f(\vec{k}), \tag{35}$$

where $f(\vec{k})$ is of order 1 or more in α. This integral contributes the most when $|\vec{k}| \sim |\vec{p}|, |\vec{p}'|$. For,

consider the following:

$$|\vec{k}| \gg |\vec{p}|, |\vec{p}'| : \; X \sim \alpha^2 \int \frac{dk}{k} f(k),$$

which gives a factor of α^5 for $\langle X \rangle$;

$$|\vec{k}| \ll |\vec{p}|, |\vec{p}'| : \; X \sim \alpha^2 \frac{1}{|\vec{p}|^2 |\vec{p}'|^2} \int d^3k f(\vec{k}).$$

Let $|\vec{k}| \sim \alpha^n$, where $n > 1$. Then $X \sim \alpha^{3n+1} f(\alpha^n) = \alpha^m$ with $m > 4$ for $f(k) \sim 1$. Thus both these cases are excluded. Further, since $|\vec{p}| \sim |\vec{p}'|$ in order of α, the only other case left is when $|\vec{k}| \sim |\vec{p}| \sim |\vec{p}'|$, which gives us just the factor we need. All integrals involved in evaluating the expectation values of Eq. (30) can now be done rather trivially and we get the simple result

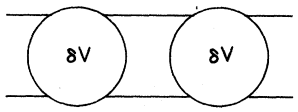

FIG. 2. The second-order term in the perturbation δV. In principle the two intermediate lines should stand for the zeroth-order Bethe-Salpeter Green's function in the potential V_0, but in the present calculation they can be replaced by the product of free propagators.

$$\delta E_{B,X} = -\frac{1}{2(m_1+m_2)} \alpha^2 \left\langle \frac{1}{r^2} \right\rangle . \tag{36}$$

Total energy of the bound system. Summing up the various contributions to δE_B [Eqs. (25), (29), and (36)] and knowing the expectation values from Ref. 11 we can now write the total energy of the bound system correct to order α^4:

$$E = m_1 + m_2 + E_B^{(0)} + \delta E_B$$
$$= m_1 + m_2 - \frac{\alpha^2}{2n^2} \frac{m_1 m_2}{m_1+m_2} + \alpha^4 \left[\frac{\delta_{l0}}{n^3} \frac{m_1^2 m_2^2}{(m_1+m_2)^3} + \frac{1}{8n^4} \frac{m_1 m_2}{m_1+m_2} \left(3 - \frac{m_1 m_2}{(m_1+m_2)^2} \right) - \frac{1}{n^3} \frac{m_1 m_2}{m_1+m_2} \frac{1}{2l+1} \right], \tag{37}$$

or, for the equal-mass case which would describe pionium (π^+-π^- bound state),

$$E = m \left[2 - \frac{\alpha^2}{4n^2} + \alpha^4 \left(\frac{\delta_{l0}}{8n^3} + \frac{11}{64 n^4} - \frac{1}{2n^3(2l+1)} \right) \right] . \tag{38}$$

Except for the δ_{l0} term, this result is the same as the energy of positronium and agrees with the calculations of Brezin, Itzykson, and Zinn-Justin[2] and with an earlier formula due to Lévy.[6] The origin of the δ_{l0} term is in the Darwin interaction; in the positronium calculation there are other s-wave terms that all finally add up to cancel this quantity. That such purely s-wave interaction does not occur in the eikonal calculations is not surprising because these approximations do not take into account singular interactions like δ functions in s waves. Why the rest of the terms agree with the calculation of Brezin *et al.* is not readily understood. One may also note that with all the differences between boson and fermion electrodynamics, the fact that we get almost identical results in the case examined is no less surprising either.

As a final remark to this section, we note that the limiting form of (37) when one mass (say m_2) approaches infinity agrees with the energy-level formula for a Klein-Gordon particle in a Coulomb field:

$$E_B(m_1, m_2 \to \infty) = m_1 \left[-\frac{\alpha^2}{2n^2} + \alpha^4 \left(\frac{3}{8n^4} - \frac{1}{n^3(2l+1)} \right) \right] . \tag{39}$$

This is reassuring, for we would have expected this to happen by comparing with the positronium case, and we know that, in either case, the B-S equation goes over to the corresponding equation for a single particle in the Coulomb field.

II. WIDTHS OF THE BOUND STATES

The widths of the energy levels of the bound system in various (n, l) states can be calculated by considering the pionium as a resonance state in photon-photon scattering: One calculates the T matrix of the process and then the width can be identified using the Breit-Wigner resonance formula. Because of the two types of vertices that are permissible in photon-pion interactions, there will be essentially two calculations for the two processes. We consider first the process in which each photon-pion vertex has only one photon in it.

The S-matrix element corresponding to the graph in Fig. 3 can be calculated by expressing the bound state B by its Green's function $G(P, p, p')$, the rest of the factors being as usual. Since at or near the bound state G has the form given by (19), the S matrix comes out in the simple form

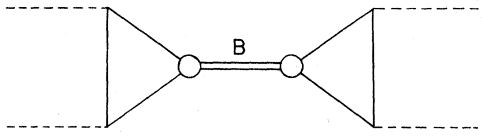

FIG. 3. The π^+-π^- bound state as a resonance in photon-photon scattering; the one-photon vertices.

$$S_{fi} = \delta^4(p_f - p_i)X(k)\frac{1}{E - E_B}\bar{X}(k'), \tag{40}$$

where k and k' are the relative momenta in the incoming and outgoing channels, respectively, and X contains the various factors for each half of the diagram. Using the T-matrix formalism and the Breit-Wigner formula, one obtains the width of the bound states,

$$\Gamma = -\tfrac{1}{2}\sum_{\text{polar}} \int d^3k\, d^3k'\, \delta^4(p_f - p_i)X(k)\bar{X}(k'), \tag{41}$$

where the minus sign occurs because of the way we defined G and the factor $\frac{1}{2}$ because of identical particles in the final state.

The polarization sums in the Coulomb gauge are easy to evaluate. For the rest of the calculation of Γ one has to specify the angular momentum state one is interested in. For s waves for the first three values of n we found the following result:

$$\begin{aligned} &\Gamma_{1s} = \tfrac{1}{2}m\alpha^5, \\ &\Gamma_{2s} = \tfrac{1}{16}m\alpha^5, \\ &\Gamma_{3s} = \tfrac{1}{54}m\alpha^5, \\ &\Gamma_{1s} : \Gamma_{2s} : \Gamma_{3s} = 1 : \tfrac{1}{8} : \tfrac{1}{27}, \end{aligned} \tag{42}$$

i.e.,

$$\Gamma_{ns} = \frac{1}{2n^3}m\alpha^5, \quad n = 1, 2, 3.$$

Once again we find a remarkable result: The Γ for the (1s) wave of pionium is identical in form to the result for the singlet positronium state.

The higher partial-wave calculations can also be done in a similar manner. It turns out that for p waves

$$\Gamma_{2p} = \frac{25}{27}\frac{m\alpha^9}{\pi}. \tag{43}$$

An order of magnitude calculation of the width for d waves shows that it is several orders of α smaller than for p waves. Thus the s waves have the largest width and are most likely to decay into the 2γ mode.

For the width from the contact interaction, consider Fig. 4. In this case also, the S matrix can be put in the form of Eq. (40) and Γ as in (41). Not surprisingly Γ in this case has a nonzero value only in the s wave; we obtain

$$\Gamma_{\text{contact}} = 2m\alpha^5. \tag{44}$$

Pionium then will decay more readily by annihilation into 2γ than passing through an intermediate state. The width for decay into 2γ for all states is smaller than the separation of energy levels.

Considering the probable decay of pionium via weak and strong interactions, we find that weak decays into muonic pairs, say, will be considerably slow while strong decays into, say, $2\pi^0$, can be competitive or enhanced in certain states of the bound system.

III. PRODUCTION MECHANISM

Among the many possible ways of producing pionium, one of rather recent interest is via a two-photon annihilation process in colliding electron-positron beams. In such processes (Fig. 5) the colliding e^- and e^+ emit photons that can annihilate into $\pi^+\pi^-$, π^0, $\mu^+\mu^-$, etc. Brodsky, Kinoshita, and Terazawa[12] have shown that at energies $E_{\text{c.m.}} = (\frac{1}{4}s_{\text{c.m.}})^{1/2}$ above 1 GeV the most frequent events occur in this manner and have derived a formula for the total cross section for a process $e + e \to e + e + N$ where N is a neutral $C = +1$ state:

$$\sigma^N = 2\left(\frac{\alpha}{\pi}\right)^2\left(\ln\frac{E}{m_e}\right)^2\int_{s_{\text{th}}}^{4E^2}\frac{ds}{s}f((s/4E^2)^{1/2})\sigma^N_{\gamma\gamma}(s), \tag{45}$$

where

$$f(x) = (2 + x^2)^2 \ln x^{-1} - (1 - x^2)(3 + x^2) \tag{46}$$

and $\sigma^N_{\gamma\gamma}(s)$ is the two-photon annihilation cross section for two photons of c.m. energy squared s. In this formula E is the c.m. energy of the electron; for the total energy of pionium, below, we use the symbol M_B.

The calculation of $\sigma^B_{\gamma\gamma}(s)$ can be carried out rather easily by using the formulas derived in Sec. II. It turns out that

$$\sigma^B_{\gamma\gamma}(s) = 8\pi\alpha^5\frac{m}{M_B}\delta(s - M_B{}^2), \tag{47}$$

from where we can easily derive the cross section for producing pionium by using Eq. (45). We get, to lowest order in α,

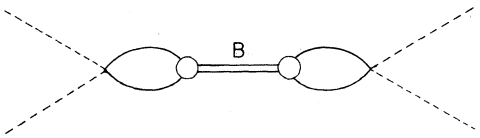

FIG. 4. The π^+-π^- bound state as a resonance in photon-photon scattering; the two-photon vertices.

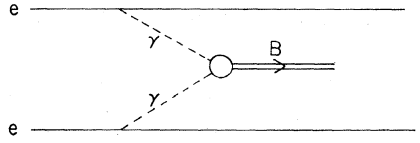

FIG. 5. Production of π^+-π^- bound state by colliding beams.

$$\sigma^B = \frac{4\alpha^2}{\pi}\left(\ln\frac{E}{m_e}\right)^2 \frac{1}{m^3\tau} f(m/E), \tag{48}$$

where m is the pion mass and τ the lifetime of the pionium bound state (s wave). This is almost the same form as the cross section for the colliding-beam production of π^0 at large E. At those energies where the f function is indistinguishable for the two cases, the cross section for π^0, however, will still be about 10^4 times greater than that for pionium.

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