[1] K. Wilson, Phys. Rev. 129, 1499 (1969).

[2] R. A. Brandt, Phys. Rev. Letters 23, 1260 (1969); Phys. Rev. D 1, 2808 (1970).

[3] L. S. Brown, in *Lectures in Theoretical Physics*, edited by K. T. Mahanthappa and W. E. Brittin (Gordon and Breach, New York, to be published).

[4] R. Jackiw, R. Van Royen, and G. B. West, Phys. Rev. D 2, 2473 (1970).

[5] R. A. Brandt and G. Preparata, Nucl. Phys. B27, 541 (1971).

[6] M. A. B. Bég, J. Bernstein, D. J. Gross, R. Jackiw, and A. Sirlin, Phys. Rev. Letters 25, 1231 (1970).

[7] Khalil M. Bitar and N. N. Khuri, Phys. Rev. D 3, 462 (1971).

[8] Note that we are basically looking for a term of the form C/q_0^2 in M_1. Such a term contributes equally to a tensor and scalar Schwinger term. By putting $\vec{q}\cdot\vec{P}$ we suppress the tensor contribution temporarily and determine C by its contribution to the scalar part.

[9] R. Rajaraman and R. Rajasekaran, Phys. Rev. D 3, 266 (1971); M. Damashek and F. Gilman, *ibid.* 1, 1319 (1970).

[10] T. P. Cheng and Wu-Ki Tung, Phys. Rev. Letters 24, 851 (1970).

[11] Consider, for example, an expression of the form $F(\omega,q^2)=[\beta(q^2)/\omega^2]e^{q^2/\omega}$. For $-q^2\rightarrow\infty$ at any ω fixed and finite, $F(\omega,q^2)\rightarrow 0$. However, for large ω *and* large $-q^2$ such that $-q^2=\omega$, $F(\omega,q^2)$ would grow indefinitely if $\beta(q^2)\sim q^6$.

[12] G. Mack, Phys. Letters 35B, 234 (1971).

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Breaking the Symmetry of the Baryons and the Mesons*

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We present an approach to the breaking of SU(3) × SU(3) symmetry which emphasizes the Nambu-Goldstone realization of this symmetry. Here the octet π, K, η are the ground-state mesons. For matrix elements which are not analytic in the symmetry-breaking parameters, we can establish exactly the leading behavior in symmetry breaking. An example of this is a chiral-limit theorem on the meson decay constants $f_K/f_+(0)f_\pi - 1 = [3(m_K^2 - m_\pi^2)/64\pi^2 f_\pi^2] \times \ln[64\pi^2 f_\pi^2/3(m_K+m_\pi)^2] + O(\lambda)$. For matrix elements which are analytic to leading order in symmetry breaking, we advance the hypothesis of threshold dominance of the Goldstone-boson-pair states. When this hypothesis is applied to the mass splittings of the ground-state mesons there results an eigenvalue problem to which the unique nontrivial solution corresponds to octet enhancement. This is independent of any assumption about the Hamiltonian symmetry breaking. When we apply these ideas to the baryon mass splittings we again obtain octet solutions corresponding to tadpole-model results and a new result $\frac{3}{10}(f/d)_B = (f/d)_A/[3(f/d)_A^2-1]$ relating the baryon mass f/d ratio to the axial-vector–baryon f/d ratio, in good agreement with experiment. We also discuss electromagnetic mass shifts in this context and advocate that for $\Delta I = 1$ mass shifts the Cottingham formula diverges (and should be abandoned). If the Cottingham formula diverges for $\Delta I = 1$ mass shifts, then we no longer have Dashen's sum rule $\mu_{K^+}^2 - \mu_{K^0}^2 = \mu_{\pi^+}^2 - \mu_{\pi^0}^2$. An alternative, finite approach for $\Delta I = 1$ mass shifts is suggested and developed.

I. INTRODUCTION

This paper is devoted to a study of the breaking of the SU(3) symmetry of the strong interactions. This is undertaken with the recognition that it is low-energy Goldstone-boson-pair states that dominate symmetry-breaking matrix elements.

The primary assumption on which we base this work is that in the absence of symmetry breaking the Hamiltonian for the strong interactions is SU(3)×SU(3)-invariant,[1] but the vacuum state is just SU(3)-symmetric. Coleman's theorem[2] then requires that the SU(3) symmetry of the vacuum state be manifest for all physical states so that they may be classified according to the irreducible representations of SU(3). But the vacuum symmetry, SU(3), is not the same as the Hamiltonian symmetry, SU(3)×SU(3). In this instance the Goldstone theorem[3] requires the existence of an octet of massless pseudoscalar ground-state mesons. These are identified with the octet π, K, and η. That the physical pseudoscalar mesons are massive is to be accounted for by the presence of symmetry-breaking terms in the Hamiltonian. Explicit symmetry breaking is also responsible for removing the SU(3) degeneracy of other states. In the absence of such symmetry-breaking terms, however, the ground-state mesons are strictly massless. The attractiveness of this picture lies in the fact that the pseudoscalar octet π, K, η is indeed the lowest-lying octet of hadrons.

We now consider what happens if we add to the SU(3)$\times$SU(3)-symmetric Hamiltonian H_0 a term H' which breaks this symmetry

$$H=H_0+\lambda H'. \tag{1.1}$$

The term H' is responsible for removing the SU(3) degeneracy of the states and also gives the ground-state Goldstone bosons a mass. These masses vanish as $\lambda\to 0$ and we recover the symmetry.[4] If the SU(3)$\times$SU(3) symmetry of H were realized on the states so that there would be parity doubling, we might suppose that it is possible to do power-series perturbation theory in λ for computing matrix elements. However, in the Nambu-Goldstone realization of chiral symmetry which we consider here, such power-series perturbation theory in λ is not possible since in general the S matrix and matrix elements of currents are not analytic in λ near $\lambda=0$.[5] The reason for this nonanalytic behavior is that as $\lambda\to 0$ the ground-state mesons become massless, and the strong interactions acquire a long-range component. The contribution from meson loops then can give rise to factors like $\lambda\ln\lambda$ or $\lambda^{1/2}$. In the chiral-symmetry limit $\lambda\to 0$ there are no infrared divergences since the pseudoscalar mesons have P-wave coupling. Still the symmetry limit is approached in a nonanalytic fashion.

Whether or not a particular matrix element is nonanalytic to leading order in λ must be examined in each particular instance. We will find it necessary in this work to distinguish between matrix elements which are analytic to leading order in λ and those which are nonanalytic to leading order.

An essential assumption in our work on symmetry breaking will be that to leading order in λ, whether it is nonanalytic or not specifies the dominant correction to the symmetric value of a matrix element so that we may ignore, as a first approximation, higher-order terms in λ. As a second assumption we will suppose that the divergence of vector and axial-vector currents $\partial_\mu V_\mu^a(x)$ and $\partial_\mu A_\mu^a(x)$ are gentle operators so that their matrix elements between physical states obey unsubtracted dispersion relations. This assumption enables us to calculate the matrix elements of the symmetry-breaking Hamiltonian from unitarity.

As a consequence of these two assumptions we discover that matrix elements which are nonanalytic to leading order in λ can, to this leading order, be computed exactly.[6] Hence we establish the existence of chiral-limit theorems. More explicitly, the nonanalytic behavior can be viewed as a consequence of the threshold divergence of dispersion relations for the matrix element as the threshold approaches the chiral limit. The absorptive part in the dispersion integral for the production of the ground-state mesons is controlled at the threshold by current-algebra low-energy theorems. Hence, the coefficient of the nonanalytic term is determined from the threshold behavior of the absorptive part.

As an example of such a chiral-limit theorem we will establish the behavior of the pion and kaon decay constants as $\lambda\to 0$ according to

$$\frac{f_K}{f_+(0)f_\pi}-1=\frac{3(m_K{}^2-m_\pi{}^2)}{64\pi^2 f_\pi{}^2}\ln\left(\frac{64\pi^2 f_\pi{}^2}{3(m_K+m_\pi)^2}\right)+O(\lambda). \tag{1.2}$$

Here the left-hand side, which vanishes in the chiral limit, is determined from Cabibbo theory to be 0.28, while the right-hand side, behaving like $\lambda\ln\lambda$, is 0.19.

Some matrix elements are analytic to leading order in λ and for such matrix elements we can not establish chiral-limit theorems. However, the absorptive parts in dispersion relations for such matrix elements can be computed exactly in the symmetry limit at the thresholds for the production of the ground-state mesons. The characteristic of the analytic matrix element is that the dispersion relations do not diverge at the thresholds, and hence one can not prove threshold dominance as in the nonanalytic case. In order to estimate such a matrix element, we will introduce as a third hypothesis the assumption of threshold dominance of the relevant dispersion integrals. This assumption will be given a precise formulation in Sec. II in terms of the domination of an analytic matrix element by the leading pole in the threshold plane.

This hypothesis of threshold dominance has already been applied to the calculation of the anomalous magnetic moments of the nucleons with success.[7] In this application we learn that the two-pion intermediate state contributes a large isovector term at threshold and a smaller isoscalar piece. The smallness of the isoscalar anomalous moment relative to the isovector anomalous moment can then be understood from this threshold-dominance picture.

The principal physical idea in our program of examining symmetry breaking is that it is the long-range components of the strong interactions in the chiral-symmetry limit that dominate symmetry-breaking effects. For nonanalytic matrix elements one may prove this is true, while for those analytic to leading order in λ we advance this as a hypothesis. A consequence of this idea is the identification of the primary component of the tadpole as pseudoscalar Goldstone-boson-pair states. This identification of the tadpole was originally suggested as a possibility by Coleman and Glashow in their work on the tadpole model of symmetry breaking.[8]

With this identification of the tadpole we can actually go beyond the tadpole model which implements the experimentally observed octet enhancement of symmetry breaking. Working within the context of our three assumptions and making no assumption about the transformation properties of the symmetry-breaking Hamiltonian, we will prove that matrix elements are octet enhanced. In the tadpole model this was an additional assumption. In the context of our program this experimentally observed octet enhancement emerges as the solution to an eigenvalue problem for the ground-state meson mass splittings. As will be discussed in Sec. III the eigenvalue is positive as a direct consequence of the attractive nature of the ground-state meson-meson scattering amplitude at threshold in the chiral limit. This requirement uniquely fixes the eigenvalue corresponding to the octet solution to the eigenvalue problem.

This program of calculating symmetry breaking is reminiscent of the Dashen-Frautschi S-matrix perturbation theory.[9] What is new in our program is the recognition of the importance of Goldstone-boson-pair states and the implementation of current-algebra constraints.

Once octet enhancement is established for the ground-state meson mass splittings, they serve to establish octet enhancement for other matrix elements. In particular, we examine in Sec. IV the splittings of the baryon octet[6] which again exhibit octet behavior. Our identification of the tadpole as meson pairs takes us beyond the tadpole-model results in establishing the relation

$$\tfrac{3}{10}(f/d)_B = (f/d)_A/[3(f/d)_A{}^2 - 1], \qquad (1.3)$$

where $(f/d)_B$ is the baryon mass splitting f/d ratio, and $(f/d)_A$ is the same ratio for axial-vector–baryon couplings. This relation is in excellent accord with the experimental determination of these ratios and provides additional evidence that we have identified the major component of the tadpole.

In Sec. V we will discuss in detail the role of electromagnetism in our approach to SU(2) violations in hadronic interactions. Besides pseudoscalar-meson pairs dominating the matrix elements of the divergence of the isospin current, we also consider the two-meson–one-photon state as a "driving term."

Our approach to isospin violations in the strong interactions is compared to the Cottingham approach.[10] It is suggested that for $\Delta I = 1$ mass shifts the Cottingham formula diverges, while for $\Delta I = 2$ mass shifts it is finite. Further, the original hope of relating the nucleon mass shift to experimental electroproduction data via the Cottingham formula is not practical since there are contributions to this integral which are all but impossible to extract from data. So we abandon the Cottingham formula for $\Delta I = 1$ mass shifts. As an alternative we suggest the use of crossed-channel dispersion relations which in the context of our assumptions enable us to relate isospin violations in the baryon sector to those in the meson sector. The isospin violations in the meson sector are then determined by perturbation theory in α, the fine-structure constant, about an eigenvalue problem. A preliminary attempt to compute the driving terms is made, and this suggests how the correct sign for the $\Delta I = 1$ meson mass splittings might be obtained.

One problem which will not be dealt with in this paper is the question of the symmetry and transformation properties of the breaking term in the Hamiltonian. It is a virtue of the present program that results can be obtained without such assumptions. What is remarkable is that, although SU(3)×SU(3) as a symmetry of the strong interactions was proposed over ten years ago by Gell-Mann, so little is really known about the symmetry of the breaking term relative to the invariant term. This is the question of the relative strength of SU(3) violations versus SU(2)×SU(2) violations. What makes this question so difficult to answer is that with the Nambu-Goldstone realization the symmetry breaking of the Hamiltonian and the vacuum become entangled.

To get an idea of the relative strength of these violations we must find experimental parameters which vanish if SU(2)×SU(2) is exact or if SU(3) is exact. If SU(2)×SU(2) is an exact symmetry of the Hamiltonian, then the pion mass vanishes. If we choose to compare this with the kaon mass then a measure of chiral SU(2)×SU(2) breaking is

$$m_\pi{}^2/m_K{}^2 = 0.075. \qquad (1.4)$$

Further, if SU(2)×SU(2) is exact then the corrections to the Goldberger-Treiman relation vanish. They are observed to be

$$1 - Mg_A/f_\pi g = 0.08 \pm 0.02. \qquad (1.5)$$

Finally, the σ term in pion-nucleon scattering which essentially measures the amount the nucleon mass is shifted when we turn on SU(2)×SU(2)-violating forces has been estimated by Cheng and Dashen[11] to be about 110 MeV, although it could be smaller by a factor of 2 or 3.[12] Hence, we have as a third measure of SU(2)×SU(2) violations

$$\sigma/M = 0.12 \text{ to } 0.04, \qquad (1.6)$$

which is compatible with the estimates (1.4) and (1.5). There is nothing particularly large about this σ-term estimate relative to other measurements of $SU(2)\times SU(2)$ violations.

It would, of course, be desirable to have more such experimental parameters since (1.4) and (1.6) are not free of theoretical and experimental criticism. One would conclude from such numbers that $SU(2)\times SU(2)$ is a good symmetry to about 8% to 5%.

This is to be contrasted with measurable violations of SU(3) symmetry of which we have an abundance in the mass spectrum. Choosing multiplets free of f and d coupling problems, we have from the baryon $\underline{10}$ and vector meson $\underline{8}$

$$(M_{\Sigma^*}-M_{N^*})/M_{N^*}\sim 0.1, \tag{1.7}$$

$$(M_{K^*}{}^2-M_\rho{}^2)/M_\rho{}^2\sim 0.3. \tag{1.8}$$

From such numbers one would conclude that SU(3) is violated to 10–30% which appears on the face of it larger than our estimates of $SU(2)\times SU(2)$ violations. About how such experimental numbers can be fitted with parameters in models for the symmetry-breaking Hamiltonian we have nothing to say.

II. THE THRESHOLD PLANE

In our introduction we remarked that matrix elements which are analytic to leading order in λ can be estimated if we make the hypothesis of threshold dominance. We now will make this hypothesis precise.

Our interest in studying SU(3) symmetry breaking centers on establishing matrix elements of the divergence of the vector current $i\partial_\mu V_\mu^a(x)$ for $a=K^-,\pi^-$. We denote these matrix elements by

$$D^a_{\alpha\beta}(t)=\langle\alpha|i\partial_\mu V^a_\mu(0)|\beta\rangle,\quad t=(p_\alpha-p_\beta)^2 \tag{2.1}$$

where α and β are arbitrary states. Since we have assumed that $i\partial_\mu V^a_\mu(x)$ is a gentle operator, we have for $D^a_{\alpha\beta}(t)$ an unsubtracted dispersion relation of the form

$$D^a_{\alpha\beta}(t)=\frac{1}{\pi}\int_{t_0}^\infty\frac{dt'}{t'-t}\,\mathrm{Im}D^a_{\alpha\beta}(t'). \tag{2.2}$$

From the unitarity condition on $\mathrm{Im}D^a_{\alpha\beta}(t)$ we have that this absorptive part is given by (see Fig. 1)

$$\mathrm{Im}D^a_{\alpha\beta}(t)=-\tfrac{1}{2}i\sum_n(2\pi)^4\delta^4(p_{\bar\beta}+p_\alpha-p_n)\times\langle\alpha\bar\beta|n\rangle\langle n|i\partial_\mu V^a_\mu|0\rangle. \tag{2.3}$$

In the chiral limit the threshold $t_0=0$ corresponds to the production of 2, 4, 6, ... ground-state mesons in the state $|n\rangle$. In particular, we will focus our attention on $D^a_{\alpha\beta}(0)$ since this quantity specifies the matrix element of the symmetry-breaking Hamiltonian. In the chiral limit we have to leading order

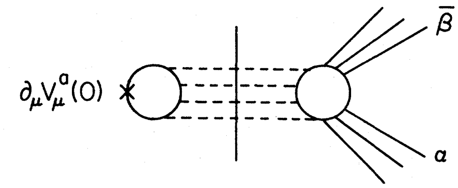

FIG. 1. Unitarity condition on $\mathrm{Im}\,D^a_{\alpha\beta}(t)$.

$$D^a_{\alpha\beta}(0)=\frac{1}{\pi}\int_0^\infty\frac{dt}{t}\,\mathrm{Im}D^a_{\alpha\beta}(t), \tag{2.4}$$

where we have assumed that the limit $t_0\to 0$ exists as is the case for matrix elements analytic in λ to leading order. We further observe that as $t\to 0$ the leading behavior of the absorptive part is specified by current algebra according to

$$\mathrm{Im}D^a_{\alpha\beta}(t)\underset{t\to 0}{\sim}C_2t^r. \tag{2.5}$$

Here $r>0$ and the constant C_2 is determined from the current-algebra low-energy theorem for the intermediate states containing a pair of Goldstone bosons and phase space. Higher multiparticle states such as four or more bosons will contribute to $\mathrm{Im}D^a_{\alpha\beta}(t)$ terms less singular than t^r, the two-particle state contribution, purely from phase-space considerations.

Next we introduce the dimensionless variable $x=t/4M^2$, where $M>0$ is some scaling mass and define $\mathrm{Im}D^a_{\alpha\beta}(x)=(4M^2)^{-r}\,\mathrm{Im}D^a_{\alpha\beta}(t)$, so that

$$(4M^2)^{-r}D^a_{\alpha\beta}(0)=\frac{1}{\pi}\int_0^\infty\frac{dx}{x}\,\mathrm{Im}D^a_{\alpha\beta}(x). \tag{2.6}$$

Instead of this integral we consider

$$F^a_{\alpha\beta}(\gamma)=\frac{1}{\pi}\int_0^\infty\frac{dx}{x^{1+\gamma}}\,\mathrm{Im}D^a_{\alpha\beta}(x), \tag{2.7}$$

so that

$$(4M^2)^{-r}D^a_{\alpha\beta}(0)=F^a_{\alpha\beta}(0). \tag{2.8}$$

It is clear in this development that if we know $F^a_{\alpha\beta}(\gamma)$ in the γ plane (the threshold plane, see Fig. 2), we can reconstruct $\mathrm{Im}D^a_{\alpha\beta}(t)$ by inverting the Mellin transform (2.7), and hence obtain $D^a_{\alpha\beta}(t)$ from the dispersion relation (2.2).

$F^a_{\alpha\beta}(\gamma)$ contains all the information in $D^a_{\alpha\beta}(t)$. It has the virtue of displaying the threshold behavior of Goldstone-boson pairs as poles in the right-half γ plane – the threshold poles – which are from the Goldstone-pair states of 2, 4, 6, ... ground-state mesons, and the residue of the leading pole at $\gamma=r$ is completely specified by current algebra. The singularities in the left-half γ plane are determined by the high-energy behavior of $\mathrm{Im}D^a_{\alpha\beta}(x)$ as $x\to\infty$. If, for example, $\mathrm{Im}D^a_{\alpha\beta}(x)\sim Ax^{-s}\ln^m x$ as $x\to\infty$, then the leading singularity of $F^a_{\alpha\beta}(\gamma)$ in the

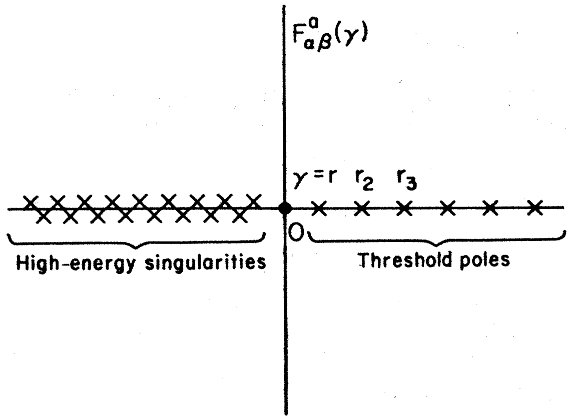

FIG. 2. Threshold plane.

left-half γ plane is $(A/\pi)/(\gamma + s)^m$.

If we assume that the behavior of $\mathrm{Im}D^a_{\alpha\beta}(x)$ is such that $F^a_{\alpha\beta}(\gamma) \to 0$, $\gamma \to \infty$ for $\mathrm{Re}\gamma > \gamma_0$ with $\gamma_0 < 0$, we may represent $F^a_{\alpha\beta}(\gamma)$ by

$$F^a_{\alpha\beta}(\gamma) = \frac{C_2/\pi}{r-\gamma} + \sum_{N>2} \frac{C_N/\pi}{r_N - \gamma} + B_{\alpha\beta}(\gamma), \tag{2.9}$$

where the r_N are determined from threshold behavior of four or more Goldstone pairs, and $B_{\alpha\beta}(\gamma)$ is a background integral from $\gamma_0 - i\infty$ to $\gamma_0 + i\infty$ and incorporates the contribution from the high-energy region. This result, Eq. (2.9), is derived in Appendix A. This background integral can be pushed back depending on the leading behavior of $\mathrm{Im}D^a_{\alpha\beta}(x)$ as $x \to \infty$. Our assumption can now be stated that $F_{\alpha\beta}(0)$ is dominated by the leading threshold pole so that

$$F^a_{\alpha\beta}(0) = (4M^2)^r D^a_{\alpha\beta}(0) \simeq C_2/r\pi. \tag{2.10}$$

The case for matrix elements $D^a_{\alpha\beta}(0)$ nonanalytic to leading order in λ is characterized by $r<0$, and such matrix elements must be treated separately. For these matrix elements no assumption as the above need be made, as was emphasized in the Introduction.

The assumption of retaining just the leading threshold pole is equivalent to writing

$$D^a_{\alpha\beta}(0) \simeq \frac{1}{\pi}\int_0^{4M^2} \frac{dt}{t}\, \mathrm{Im}D^a_{\alpha\beta}(t), \tag{2.11}$$

using the theorem on the absorptive part,

$$\mathrm{Im}D^a_{\alpha\beta}(t) \underset{t\to 0}{\sim} C_2 t^r,$$

and substituting this result into (2.11),

$$D^a_{\alpha\beta}(0) \simeq (4M^2)^r C_2/r\pi. \tag{2.12}$$

This establishes the connection between the threshold plane and threshold dominance.

For a single pair of states α and β we learn nothing from this procedure since the matrix element is determined only up to the positive, but otherwise arbitrary constant $(4M^2)^r$. But for α and β considered as varying over the members of the same SU(3) representation, the multiplicative constant $(4M^2)^r$ will be independent of α and β to leading order in chiral breaking. Hence, the ratios of all such matrix elements are specified independent of M within the context of our assumptions. An exception to this remark is the application of these ideas to the ground-state meson mass splittings, where the constant M is determined from an eigenvalue problem.

III. BREAKING THE SYMMETRY OF THE GOLDSTONE BOSONS

Of particular importance in our approach to symmetry breaking is to establish the character of symmetry breaking in the Goldstone bosons themselves since these matrix elements, using unitarity, will by our hypothesis dominate other matrix elements. We now will establish an eigenvalue problem for the meson mass differences $\mu_K{}^2 - \mu_\pi{}^2$, $\mu_K{}^2 - \mu_\eta{}^2$, $\mu_{K^+}{}^2 - \mu_{K^0}{}^2$, $\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2$ and the transition mass $\Delta_{\pi^0\eta} = \langle \pi^+ | -i\partial_\mu V_\mu^{\pi^-}(0) | \eta\rangle$.

The matrix element of the divergence of the vector current between these meson states is specified according to

$$\begin{aligned} &\langle M^a(p_2)| -i\partial_\mu V^b_\mu(0) | M^c(p_1)\rangle = d^{abc}(t), \\ &d^{abc}(t) = i f^{abc}[(\mu_a{}^2 - \mu_c{}^2) f_+(t) + t f_-(t)], \end{aligned} \tag{3.1}$$

where $f_\pm(t)$ are the usual vector form factors. The nonrenormalization theorem implies $f_+(0) = 1 + O(\lambda^2 \ln\lambda)$, so that

$$d^{abc}(0) = i f^{abc}(\mu_a{}^2 - \mu_c{}^2) + O(\lambda^3 \ln\lambda). \tag{3.2}$$

In accord with our assumption that we will keep only the leading order in λ, we will drop second- and higher-order terms in what follows. We assume that $d^{abc}(t)$ obeys an unsubtracted dispersion relation, so that

$$d^{abc}(0) = \frac{1}{\pi}\int_{t_0}^{\infty} \frac{dt}{t}\, \mathrm{Im}d^{abc}(t), \tag{3.3}$$

where unitarity specifies the absorptive part according to

$$\begin{aligned} \mathrm{Im}d^{abc}(t) = -\tfrac{1}{2} i (2\pi)^4 \sum_n \langle M^a(p_2) | M^c(p_1) n\rangle \\ \times \langle n | -i\partial^\mu V^b_\mu(0) | 0\rangle \delta^4(P_n - q), \end{aligned} \tag{3.4}$$

with $q = p_2 - p_1$.

As remarked in the Introduction, the absorptive part in the chiral limit and at threshold is specified exactly by current-algebra low-energy theorems. To establish this theorem we need retain only the two-meson state in the intermediate-state sum since the contribution of states containing four or more mesons will vanish relative to the two-meson state at threshold. As we show below the two-meson state contributes to the absorptive part a term proportional to t as $t \to 0$. Hence the integral (3.3) is finite as $t_0 \to 0$, and we have the case of an analytic matrix element. In the language of the threshold plane, the two-meson state gives rise to the leading threshold pole at $\gamma = 1$.[13]

The two-meson states contribute to the absorptive part according to

$$\mathrm{Im}\, d^{abc}(t) = -\tfrac{1}{2} i d^{bef}(t) \Phi_2(t) M^{ef}_{ac}(t), \tag{3.5}$$

where $M^{ef}_{ac}(t)$ is the S-wave meson-meson scattering amplitude, and

$$\Phi_2(t) = \frac{1}{8\pi}\left(\frac{(t-t_0)(t-t_1)}{t^2}\right)^{1/2} \tag{3.6}$$

is two-body phase space with the threshold $t_0 = (\mu_e + \mu_f)^2$ and pseudothreshold $t_1 = (\mu_e - \mu_f)^2$. In the chiral limit $t_{0,1} \to 0$ and $\Phi_2(t) = 1/8\pi$, while $d^{bef}(0) = i f^{ebf}(\mu_e^2 - \mu_f^2)$.

Applying Weinberg's treatment[14] of meson-meson scattering in the chiral SU(3)×SU(3) limit, we find for this amplitude symmetrized with respect to e and f as is required in (3.5),

$$M^{ef}_{ac}(t) \underset{t\to 0}{\sim} \frac{it}{f^2}\left[T_{ac;ef} - \tfrac{1}{2}(T_{ec;af} + T_{fc;ae})\right], \tag{3.7}$$

where $f = f_\pi = f_K = f_\eta \sim \mu_\pi/\sqrt{2}$ is the meson decay constant, and

$$T_{ac;ef} = \tfrac{2}{3}\delta_{ac}\delta_{ef} + d_{acj}d_{efj}, \tag{3.8}$$

where d_{acj} is Gell-Mann's totally symmetric tensor. Putting these results together we obtain a theorem on the behavior of the absorptive part in the chiral limit:

$$\mathrm{Im}\, d^{acb}(t) \underset{t\to 0}{\sim} \frac{i f^{ebf}(\mu_e^2 - \mu_f^2)}{16\pi} \times\left[T_{ac;ef} - \tfrac{1}{2}(T_{ec;af} + T_{fc;ae})\right](t/f^2). \tag{3.9}$$

What remains to be done is to use this result to extract the residue of the leading threshold pole or equivalently insert (3.9) in the dispersion integral and integrate to $t = 4M^2$, where M is the scaling mass. Finally we will set $b = K^+$ since this gives the medium-strong mass differences. There result the following relations:

$$\begin{aligned}(\mu_K^2 - \mu_\pi^2) &= \tfrac{5}{8} a(\mu_K^2 - \mu_\pi^2) - \tfrac{3}{8} a(\mu_K^2 - \mu_\eta^2),\\ (\mu_K^2 - \mu_\eta^2) &= -\tfrac{3}{8} a(\mu_K^2 - \mu_\eta^2) - \tfrac{3}{8} a(\mu_K^2 - \mu_\pi^2),\end{aligned} \tag{3.10}$$

$$a \equiv \frac{1}{4\pi^2}\frac{M^2}{f^2} > 0. \tag{3.11}$$

If we set $b = \pi^-$ corresponding to mass difference in an isomultiplet, there result the relations

$$\begin{aligned}&\mu_{\pi^+}^2 - \mu_{\pi^0}^2 = -\tfrac{1}{2} a(\mu_{\pi^+}^2 - \mu_{\pi^0}^2),\\ &\mu_{K^+}^2 - \mu_{K^0}^2 = \tfrac{1}{4} a(\mu_{K^+}^2 - \mu_{K^0}^2) + \tfrac{1}{2}\sqrt{3}\, a\Delta_{\pi^0\eta},\\ &\Delta_{\pi^0\eta} = \tfrac{1}{4}\sqrt{3}\, a(\mu_{K^+}^2 - \mu_{K^0}^2).\end{aligned} \tag{3.12}$$

There are two solutions to these equations one of which is the trivial solution in which all the mass differences vanish. The nontrivial solution requires that the determinant for (3.10) and (3.12) vanishes or

$$\det\begin{vmatrix} 8-5a & 3a \\ 3a & 8+3a\end{vmatrix} = 0, \tag{3.13}$$

which has for the eigenvalue

$$a = \tfrac{4}{3} \text{ or } a = -2. \tag{3.14}$$

The negative eigenvalue is ruled out by the requirement (3.11) $a > 0$ which reflects the attractive character of meson-meson scattering in the chiral limit at threshold. Hence the nontrivial solution is uniquely fixed, and with $a = \frac{4}{3}$ we have from (3.10) and (3.12)

$$\begin{aligned}&4\mu_K^2 = 3\mu_\eta^2 + \mu_\pi^2,\\ &\mu_{\pi^+}^2 - \mu_{\pi^0}^2 = 0,\\ &\Delta_{\pi^0\eta} = (\tfrac{1}{3})^{1/2}(\mu_{K^+}^2 - \mu_{K^0}^2),\quad a = \tfrac{4}{3},\end{aligned} \tag{3.15}$$

the octet mass formulas. The unphysical eigenvalue $a = -2$ we note corresponds to a pure $\underline{27}$-type splitting, but this is ruled out. We conclude that our assumptions imply octet enhancement in the meson sector without any assumption about the transformation properties of the symmetry-breaking Hamiltonian. In this instance the otherwise arbitrary scaling mass is fixed by the eigenvalue condition to be

$$M^2 = \tfrac{16}{3}\pi^2 f^2 \approx 0.46\ \mathrm{GeV}^2. \tag{3.16}$$

In this treatment we have assumed that Goldstone-boson-pair states are the dominant contribution to these matrix elements. If in addition there is a scalar κ^- meson which couples strongly to $\partial_\mu V_\mu^{K^-}(x)$, our symmetry-breaking eigenvalue equations are modified by an inhomogeneous term on the right-hand sides of (3.10) or (3.12). If the scalar mesons are an octet, then we still obtain the octet solutions (3.15) to these inhomogeneous equations, but we lose the eigenvalue conditions (3.14) on a. If there are scalar mesons which couple strongly, then our assumption of pure threshold dominance is wrong and the tadpole is then a combination of two mechanisms.[15] In the experimental absence of such scalar mesons, we will continue to assume pure threshold dominance.

Nowhere in this development have we appealed to any information regarding the forces responsible for violating the conservation of the strangeness-changing current $V_\mu^{K^-}(x)$ and the isospin current $V_\mu^{\pi^-}(x)$. We have shown in our approach to symmetry breaking that there are self-consistent solutions for these currents not conserved. While we have no knowledge of the forces which violate the conservation of $V_\mu^{K^-}(x)$, we do know that electromagnetism violates isospin conservation. Perhaps this force is responsible for all of isospin nonconservation. If this is the case, then, if we shut off the electromagnetic interactions, the mass differences of isomultiplets should vanish. The approach we have developed does not in any way rule this possiblility out. We see however that our solution (3.15) obtained by retaining only the leading threshold pole does not set the scale of the meson mass splittings. There is nothing that tells us that the splittings in an isospin multiplet are small relative to splittings in a U-spin multiplet.

If we now incorporate the photon as a zero-mass particle into our approach, we find that besides the two-meson intermediate states there is a contribution to $\mathrm{Im}d^{a\pi^-c}(t)$ from the two-meson–one-photon state. We will discuss this contribution from this state in more detail in Sec. V where we develop the role of electromagnetism in our approach. We remark here that this state does not contribute to the leading threshold pole at $\gamma=1$ but contributes to $\mathrm{Im}d(t)$ terms that vanish like $e^2t^2\ln t$ and e^2t^2 as $t\to 0$ corresponding to double and single poles at $\gamma=2$ in the threshold plane. So that within the assumption of keeping only the leading threshold pole we have the pure octet formulas (3.15).

It is also worth remarking in this context that if we attempt to retain higher-order terms in chiral breaking by putting such terms into the meson-meson scattering amplitude and also examine the t^2 term in the absorptive part as $t\to 0$ from the two-meson intermediate state (this is necessarily model-dependent), we then obtain transcendental equations for the symmetry-breaking parameters. The discussion of such equations takes us beyond the intended scope of this paper.

An interesting by-product of our discussion of these matrix elements is an exact chiral-limit theorem[16] for the meson decay constants which is

$$\frac{f_K}{f_+(0)f_\pi}-1=\frac{3(\mu_K{}^2-\mu_\pi{}^2)}{64\pi^2 f_\pi{}^2}\ln\frac{64\pi^2 f_\pi{}^2}{3(\mu_K+\mu_\pi)^2}+O(\lambda). \tag{3.17}$$

This result is easily obtained by utilizing the Dashen-Weinstein[17] theorem for

$$D(t)=(\mu_K{}^2-\mu_\pi{}^2)f_+(t)+tf_-(t),$$

which is the matrix element for the divergence of the strangeness-changing current between π^0 and K^- states. Their theorem can be written in the form

$$D'(0)=\frac{f_K}{f_+(0)f_\pi}-1+O(\lambda). \tag{3.18}$$

From the dispersion relation for $D(t)$ we obtain

$$D'(0)=\frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt}{t^2}\,\mathrm{Im}D(t), \tag{3.19}$$

and our theorem on the absorptive part is

$$\mathrm{Im}D(t)\underset{t\to 0}{\sim}\frac{t}{4\pi f^2}\left[\tfrac{5}{8}(\mu_K{}^2-\mu_\pi{}^2)-\tfrac{3}{8}(\mu_K{}^2-\mu_\eta{}^2)\right]. \tag{3.20}$$

Ignoring the dependence of the threshold on mass splitting and setting $t_0=(\mu_K+\mu_\pi)^2$ and $\mu_K{}^2-\mu_\pi{}^2=-3(\mu_K{}^2-\mu_\eta{}^2)$ we obtain our chiral-limit theorem (3.17) from

$$D'(0)=\frac{1}{\pi}\int_{t_0}^{4M^2}\frac{dt}{t^2}\,\mathrm{Im}D(t)+O(\lambda). \tag{3.21}$$

We establish the scale parameter M^2 from the solution to the eigenvalue problem, $M^2=\frac{16}{3}\pi^2 f^2$. Hence, we have that $f_K/f_\pi=1+O(\lambda\ln\lambda)$. Of course, one may change the scale of the logarithm and affect only the correction $O(\lambda)$.

Applying these same techniques we can obtain a similar theorem for $f_K/f_+(0)f_\eta-1$, where $f_+(0)$ is the analogous form factor for η decay. Using this result we obtain an octet-breaking-type formula for the decay constants

$$f_K-f_\pi=-3(f_K-f_\eta)+O(\lambda), \tag{3.22}$$

where the difference of decay constants is $O(\lambda\ln\lambda)$.

IV. BREAKING THE SYMMETRY OF THE BARYON OCTET

It is clear that once one establishes that the matrix elements of the current divergences into Goldstone-boson-pair states are octet-enhanced, then this implies from our identification of the tadpole that all symmetry breaking is octet-enhanced. Within an SU(3) multiplet one can calculate the symmetry breaking up to a proportionality constant, the scaling mass, the sign of which is known, and coupling constants of the bosons to the other states.

As an illustration of this idea we have computed the baryon $\underline{8}$ mass splittings.[6] The input to this calculation is the three assumptions specified in the Introduction. We established the chiral-limit threshold behavior of the absorptive part illustrated in Fig. 3 with a Goldstone pair in the intermediate state. The results of this calculation are

$$\frac{M_N - M_\Sigma}{M} = \frac{g_A^2}{96\pi f^2}\left[(\mu_K^2 - \mu_\pi^2)(15 - 48\alpha + 28\alpha^2) + 3(\mu_K^2 - \mu_\eta^2)(-3 + 8\alpha - 4\alpha^2)\right],$$

$$\frac{M_\Sigma - M_\Xi}{M} = \frac{g_A^2}{96\pi f^2}\left[(\mu_K^2 - \mu_\pi^2)(-15 + 12\alpha + 8\alpha^2) + 3(\mu_K^2 - \mu_\eta^2)(3 - 4\alpha)\right],$$

$$\frac{M_\Lambda - M_\Xi}{M} = \frac{g_A^2}{96\pi f^2}\left[(\mu_K^2 - \mu_\pi^2)(9 - 36\alpha + 24\alpha^2) + 3(\mu_K^2 - \mu_\eta^2)(3 - 4\alpha)\right],$$

$$\frac{M_p - M_n}{M} = \frac{g_A^2}{8\pi f^2}\left[(\mu_{K^+}^2 - \mu_{K^0}^2)\tfrac{1}{6}(3 - 4\alpha^2) - \frac{\Delta_{\pi^0\eta}}{\sqrt{3}}(3 - 4\alpha)\right], \tag{4.1}$$

$$\frac{M_{\Sigma^+} - M_{\Sigma^-}}{M} = \frac{g_A^2}{4\pi f^2}\left[(\mu_{K^+}^2 - \mu_{K^0}^2)\alpha(1 - \alpha) - \frac{2\Delta_{\pi^0\eta}}{\sqrt{3}}\alpha(1 - \alpha)\right],$$

$$\frac{M_{\Xi^0} - M_{\Xi^-}}{M} = \frac{g_A^2}{8\pi f^2}\left[(\mu_{K^+}^2 - \mu_{K^0}^2)\tfrac{1}{6}(-3 + 12\alpha - 8\alpha^2) - \frac{\Delta_{\pi^0\eta}}{\sqrt{3}}(2\alpha - 1)(3 - 2\alpha)\right],$$

$$\frac{M_{\Sigma^+} + M_{\Sigma^-} - 2M_{\Sigma^0}}{M} = \frac{-g_A^2}{4\pi f^2}(\mu_{\pi^+}^2 - \mu_{\pi^0}^2)(1 - \alpha)^2,$$

where $f \simeq \mu_\pi/\sqrt{2} \simeq 90$ MeV is the decay constant, $g_A = 1.24$, and $(f/d)_A = (1-\alpha)/\alpha$ is the f/d ratio for axial-vector–baryon couplings, $\alpha^{\text{exp}} = 0.66 \pm 0.02$. Here M is the scaling mass.

Within the same scheme in which we have established these relations, the mesons obey the octet formulas (3.15). Using these relations on the meson masses we obtain from (4.1) the tadpole-model results[8]

$$2(M_N + M_\Xi) = 3M_\Lambda + M_\Sigma,$$

$$M_{\Xi^-} - M_{\Xi^0} = M_{\Sigma^-} - M_{\Sigma^+} + M_p - M_n$$

$$-(f/d)_B = \frac{2}{3}\frac{M_N - M_\Xi}{M_\Lambda - M_\Sigma},$$

$$-(f/d)_{\text{em}} = \frac{M_{\Sigma^+} - M_{\Sigma^-}}{(M_p - M_n) + (M_{\Xi^-} - M_{\Xi^0})}, \tag{4.2}$$

$$(f/d)_B = (f/d)_{\text{em}},$$

$$\frac{\mu_{K^0}^2 - \mu_{K^+}^2}{\mu_K^2 - \mu_\pi^2} = \frac{\frac{1}{2}(M_n + M_{\Xi^0}) - \frac{1}{2}(M_p + M_{\Xi^-})}{\frac{1}{2}(M_\Xi + M_N) - M_\Sigma},$$

$$M_{\Sigma^+} + M_{\Sigma^-} - 2M_{\Sigma^0} = 0,$$

and a new relation which goes beyond the tadpole model,

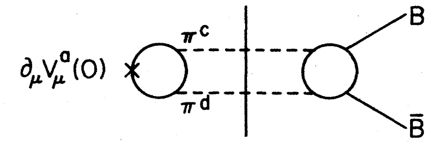

FIG. 3. Leading threshold singularity.

$$\tfrac{3}{10}(f/d)_B = (f/d)_A/[3(f/d)_A^2 - 1]. \tag{4.3}$$

Using $(f/d)_B^{\text{exp}} = -3.3$ we obtain from (4.3) $\alpha = 0.69$ which is in excellent agreement with the experimental number. We also go beyond the simple tadpole model in the qualitative observation that since $M > 0$, (4.1) has the correct sign for all the mass shifts if we use the octet formulas (3.15) for the mesons. From (4.1), using as input the experimental parameters, we obtain $M \simeq 370$ MeV, which is a factor of 2 smaller than the scale mass we found as a solution to the eigenvalue problem (3.16).

Our treatment of the electromagnetic mass shifts is not complete since we must also include a "nontadpole piece." It is encouraging to note that the $\Delta I = 1$ electromagnetic mass shifts as given just by (4.1) all have the correct signs if we use the experimental meson electromagnetic mass shifts. However, the $\Delta I = 2$ mass-shift relation given by (4.1) has the wrong relative sign. This indicates the need to consider nontadpole driving terms, which are well known to be the whole contribution to $\Delta I = 2$ mass shifts. We will discuss these problems relating to electromagnetism in the following section.

V. ELECTROMAGNETIC VIOLATION OF SU(2) SYMMETRY

So far we have developed our approach without making explicit assumptions about the dynamics of the symmetry-breaking forces. We now turn to examining the role of electromagnetism in breaking the SU(2) symmetry of the strong interactions. We will compare our t-channel approach with the Cottingham approach[10] with which we assume the reader is familiar. However, we will review a few features of the Cottingham formula relevant to our discussion.

The original point of Cottingham's approach to the mass-shift problem was to relate the nucleon mass shift to inelastic electron-nucleon scattering cross sections which could be measured experimentally. This was possible providing that the virtual forward Compton amplitudes obeyed unsubtracted dispersion relations. Subsequently, Harari pointed out that Regge theory implied that one of the Compton amplitudes required a subtraction for $\Delta I = 1$ mass shifts, while no subtraction was implied for $\Delta I = 2$ mass shifts. It is well known that the Cottingham formula gives good experimental agreement for $\Delta I=2$ mass shifts like $\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2$ from just a few low-lying states, while for $\Delta I=1$ mass shifts the sign is not even understood. So there is a problem for the $\Delta I=1$ mass shifts.

The experimental evidence that there was only a small longitudinal cross section for virtual Compton scattering at high energies suggested that one assume that the longitudinal amplitude in the Cottingham formula be unsubtracted. This assumption, compatible with data, does away with the subtraction problem[18] and one can again express the nucleon mass shift in terms of the scattering data. However, a new problem arises if we now assume that the electroproduction data have scaling behavior, that is, the inelastic structure function satisfies[19]

$$\nu W_2(q^2,\nu)\to F_2(\omega),$$
$$W_L(q^2,\nu)\to F_L(\omega)+H(\omega)/q^2, \quad \omega=-q^2/\nu$$

as $-q^2\to\infty$ and $\nu\to\infty$ with their ratio fixed. Since we have assumed no longitudinal cross section $F_L(\omega)=0$, we find on these assumptions that the mass shift is divergent[20]:

$$\delta m \sim \frac{e^2}{4\pi}(\ln\Lambda^2)\int_0^2 d\omega\,[F_2(\omega)+2H(\omega)/\omega] \quad \text{as } \Lambda\to\infty. \tag{5.1}$$

This is also what one would expect on the grounds that the nucleon has point-like constituents, and (5.1) just reflects the standard logarithmic divergence from point Fermi particles. The structure function $F_2(\omega)$ for the proton-neutron difference has been measured experimentally[21] and is not zero. The function $H(\omega)$ is all but impossible to measure because (i) it is a part of the longitudinal amplitudes which are difficult to obtain, (ii) it is a nonscaling part of such a longitudinal cross section, and (iii) what is relevant is the difference between proton scattering and neutron scattering. Hence, for all practical purposes the Cottingham formula has lost contact with experimental data.

There are various attitudes one may take to the presence of such a divergence. First is that the assumptions gone into obtaining this result are false. Alternatively, one may assume that the cutoff Λ gets replaced by some finite quantity. An example of this is Lee-Wick electrodynamics.[22] One could find that some dynamical effect makes the integral (5.1) vanish so there is no divergence. In that case the remaining finite terms in the Cottingham formula are presumably dominated by the Born terms[23] and this is known to fail to give the correct sign for $\Delta I=1$ mass shifts.

We will assume here that the Cottingham formula diverges for $\Delta I=1$ mass shifts, while it is finite for $\Delta I=2$ mass shifts. If the Cottingham formula diverges, it is, of course, not the correct expression for the physical mass shift, which is finite. The evidence we present in favor of this point of view is (i) the SLAC data are consistent with the scaling hypothesis and this strongly suggests (but does not prove) that the Cottingham formula diverges for the nucleon mass difference, (ii) if the Cottingham formula is finite and is the correct expression for the mass shift, then Dashen[24] has shown that to leading order in chiral breaking we have the sum rule $\mu_{\pi^+}{}^2-\mu_{\pi^0}{}^2=\mu_{K^+}{}^2-\mu_{K^0}{}^2$, which is in violent disagreement with experiment.

So we will drop the Cottingham approach for $\Delta I=1$ mass shifts. To see what might be going on in this problem let us consider the SU(3)×SU(3)-symmetric gluon model with a renormalizable gluon interaction and a triplet of massless quarks. We presume that the physical quarks acquire a mass through the mechanism described by Nambu and Jona-Lasinio[25] so this is a nonperturbative solution. Now we introduce the electromagnetic interaction via the standard minimal interaction. The isospin current is no longer conserved and the divergence of the renormalized isospin current is given by[26]

$$\partial_\mu V_\mu^{\pi^+}(x)=-ie:A_\mu(x)V_\mu^{\pi^+}(x):-\delta M\,\bar q(x)\lambda^{\pi^+}q(x), \tag{5.2}$$

where $A_\mu(x)$ is the electromagnetic field and δM is the infinite counterterm required to renormalize the electromagnetic interaction. This procedure renders the matrix elements of $\partial_\mu V_\mu^{\pi^+}(x)$ finite. If we shut off the electromagnetic interaction, then $\partial_\mu V_\mu^{\pi^+}(x)=0$, so isospin violations are purely electromagnetic.

For example, if we take the matrix element of $i\partial_\mu V_\mu^{\pi^+}(0)$ between the neutron and proton, we obtain for

$$\bar u(p')D(t)u(p)=\langle p(p')|i\partial_\mu V_\mu^{\pi^+}(0)|n(p)\rangle$$

the relation

$$D(t)=C(t)+\delta M(t), \tag{5.3}$$

and to lowest order in e^2 (which we presume suffices) $D(0)=M_p-M_n$, so that

$$M_p-M_n=C(0)+\delta M(0). \tag{5.4}$$

Here $C(t)$ is the matrix element of the first term in (5.2), which is written in an effective form

$$\bar u(p')u(p)C(t)=\frac{1}{2}\frac{e^2}{(2\pi)^4}\int\frac{d^4q}{q^2}\,T(q,p',p),$$
$$T(q,p',p)=\int d^4x\,e^{-iq\cdot x}\langle p(p')|(J_\mu(x)V_\mu^{\pi^+}(0))_+|n(p)\rangle, \tag{5.5}$$

and $\delta M(t)$ is the matrix element of the counterterm in (5.2).

It is straightforward to show by an isospin rotation that $C(0)$ corresponds to the Cottingham formula. In this model with a pointlike quark triplet we would expect $C(0)$ to diverge for $\Delta I = 1$ mass shifts and $C(0)$ is finite for $\Delta I = 2$ mass shifts, so no counterterm is required in this case. For $\Delta I = 1$ mass shifts one could hope to compute from (5.4) in terms of the physical $\mathfrak{N}$-$\mathcal{P}$ quark mass difference and the Coulomb or Born term. This would be similar to the calculation of nuclear mass splittings in an isomultiplet. But then one is left with the completely unknown quark mass difference. However, one could do the same calculation for another $\Delta I = 1$ mass difference and parametrize this in terms of the quark mass difference. We can thus obtain relations between, say, baryon and meson mass shifts or between baryon mass shifts themselves without reference to the quark masses. This is the essential idea of the tadpole model results which we have reproduced in Secs. III and IV. However, here we have not started with a formula like (5.4) but instead have chosen to compute mass shifts from the crossed-channel dispersion relation

$$D(0) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t}\,\mathrm{Im}D(t)\,, \qquad (5.6)$$

which directly relates mass shifts via unitarity so that one is always dealing with finite quantities. What is of interest is that the octet formulas emerge from the assumption of the Nambu-Goldstone realization of SU(3)×SU(3) without appeal to the symmetry-breaking transformation properties. So there is hope that one may compute mass shifts from the crossed-channel approach where the Cottingham formula fails. No explicit reference need be made to the quarks or counterterms as for example in (5.2). It is also clear that we avoid the Dashen sum rule since the Cottingham formula is no longer valid for the K^+-K^0 mass difference.

Although our approach of crossed-channel dispersion relations and the framework of our assumptions gives the octet mass formula, we have not yet indicated how one goes beyond this. In particular, for the meson sector can we understand relative sign of the π^+-π^0 and K^+-K^0 mass difference? If we can understand "sign reversal" in this sector then there is hope that we can understand it for the baryon $\Delta I = 1$ mass shifts as well. Besides the Goldstone-pair state the presence of electromagnetism requires us to consider the pair state plus a photon as illustrated in Fig. 4. This is what we will call the "nontadpole" piece in our work.

It is of interest to compare our t-channel approach with the Cottingham formula (the s-channel approach). For $\Delta I = 1$ mass shifts one can not

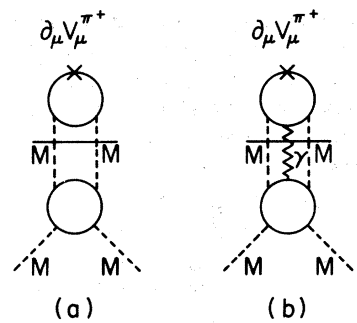

FIG. 4. (a) Tadpole graph. (b) Nontadpole graph.

carry out the comparison if, as we suggest, the Cottingham formula diverges for these quantities. However, for $\Delta I = 2$ mass shifts such a comparison could be carried out if everything is finite. An example is the π^+-π^0 mass difference. The term we call the tadpole from the pair states corresponds to highly inelastic states in the Cottingham formula as is illustrated in Fig. 5. Such an identification is valid only if the sum on the states n and m in Fig. 5, suitably integrated in the unsubtracted Cottingham dispersion relations, reproduces the appropriate scattering amplitudes. From our result (3.12) we have that such states are estimated to contribute

$$-\tfrac{1}{2}a(\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2) = -\tfrac{2}{3}(\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2) \qquad (5.7)$$

to the mass shift, which is not negligible.

A piece of the nontadpole contribution Fig. 4(b) is shown in Fig. 6. It is clear that this diagram represents part of the Born or elastic contribution to the Cottingham formula as in Fig. 6(a). However, it is not identical with this term, as is seen in the two diagrams of Fig. 6(b) which do not contribute to the elastic Born term in the Cottingham approach. It is clear that the states are counted in a very different fashion in this t-channel development and a simple identification with the elastic Born term in the Cottingham formula is not possible. It is also necessary in this t-channel method to ensure that the calculation maintains

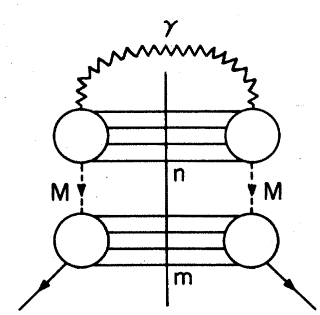

FIG. 5. Tadpole contribution to the Cottingham formula.

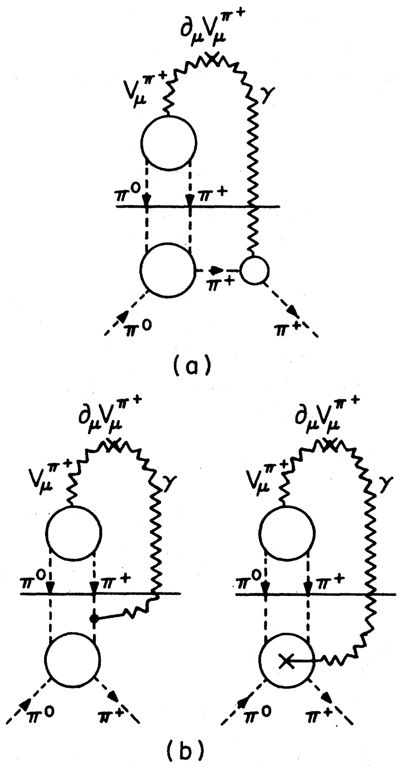

FIG. 6. (a) Nontadpole contribution seen as part of the elastic term. (b) Inelastic contributions to the nontadpole piece.

gauge invariance. First it should be remarked that the operator $\partial_\mu V_\mu^{\pi^+}(x)$ is not gauge-invariant although the equation for this operator that we have been considering (5.2) is gauge-invariant. Hence matrix elements of $\partial_\mu V_\mu^{\pi^+}(0)$ are in general not gauge-invariant. However, between single-particle states of the same isomultiplet at zero momentum transfer the matrix element is gauge-invariant and gives the physical mass difference of members of the isomultiplet. We discuss these problems associated with gauge invariance of the nontadpole graphs in Appendix B.

The general consequence of including the nontadpole terms to the calculation of the meson electromagnetic mass shifts is to alter the homogeneous equations (3.12) by the addition of driving terms $\delta_{\pi,K,\eta}$ proportional to $e^2/4\pi$

$$\begin{aligned} \mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2 &= -(a/2)(\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2) + \delta_\pi , \\ \mu_{K^+}{}^2 - \mu_{K^0}{}^2 &= (a/4)(\mu_{K^+}{}^2 - \mu_{K^0}{}^2) + (\sqrt{3}/2)a\Delta_{\pi^0\eta} + \delta_K , \\ \Delta_{\pi^0\eta} &= (\sqrt{3}/4)a(\mu_{K^+}{}^2 - \mu_{K^0}{}^2) + \delta_\eta . \end{aligned} \tag{5.8}$$

These equations imply

$$(1 + \tfrac{1}{2}a)[(\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2) - (\mu_{K^+}{}^2 - \mu_{K^0}{}^2) + \sqrt{3}\Delta_{\pi^0\eta}] = \delta_\pi - \delta_K + \sqrt{3}\delta_\eta , \tag{5.9}$$

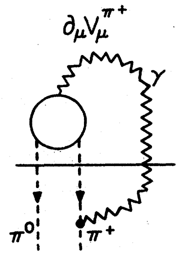

FIG. 7. Semidisconnected nontadpole piece.

and since in general if the divergence of the isospin current is given by (5.2), we have the Coleman-Glashow-type formulas in the SU(3)-symmetry limit

$$\begin{aligned} \mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2 &= \mu_{K^+}{}^2 - \mu_{K^0}{}^2 - \sqrt{3}\Delta_{\eta\pi^0} , \\ \delta_\pi &= \delta_K - \sqrt{3}\delta_\eta , \end{aligned} \tag{5.10}$$

so (5.9) is trivially satisfied. Using this relation and eliminating $\Delta_{\pi^0\eta}$ and a from (5.8), we find, assuming $\delta_{\pi,K,\eta} \neq 0$, $a \neq -2$,

$$\tfrac{1}{2}(\mu_{K^+}{}^2 - \mu_{K^0}{}^2) = (\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2)\left(\frac{(\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2) - \delta_\pi + \delta_K}{5(\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2) - 3\delta_\pi}\right) . \tag{5.11}$$

The point of view that we advocate is that the pion mass shift $\mu_{\pi^+}{}^2 - \mu_{\pi^0}{}^2$ is a given quantity since this is a $\Delta I = 2$ mass shift and can be calculated, with good experimental agreement, from the Cottingham formula, which we assume is finite in this application. No such statement can be made for the $\Delta I = 1$ mass shift $\mu_{K^+}{}^2 - \mu_{K^0}{}^2$ and to attempt to understand the sign and magnitude of this quantity on a theoretical basis we advocate our approach. This we see requires calculating $\delta_{\pi,K}$ from the dispersion relation and using a formula like (5.11).

The nontadpole contribution to the absorptive part can again be calculated exactly at threshold. We find that these pieces corresponding to the diagrams in Fig. 6 contribute at threshold $t \to 0$ to the absorptive part like $\mathrm{Im}D(t) = Ct^2 \ln t + Dt^2$. A semidisconnected piece as is shown in Fig. 7 does not contribute here because it contains the amplitude for a zero-zero transition of the physical photon which must vanish. From the standpoint of the threshold γ plane the nontadpole piece first contributes a pole and double pole at $\gamma = 2$.

In this paper we will not attempt to estimate the driving terms $\delta_{\pi,K}$ since this constitutes a whole study in its own right. We should remark however that besides the nontadpole contribution to the residue of the threshold pole at $\gamma = 2$ there is also a contribution from the nonthreshold behavior of the two-meson intermediate-state tadpole Fig. 4(a). These contributions should also be included if we retain the philosophy of keeping the next-to-leading threshold poles at $\gamma = 2$.

Within the framework of our original assumptions of keeping only the leading threshold singularity at $\gamma = 1$ we have the pure-octet formulas for the meson and baryon mass shifts. With the above remarks as to what is involved in looking at the next-to-leading terms, including the nontadpole piece, we conclude this study.

APPENDIX A

Here we will derive our representation for $F^a_{\alpha\beta}(\gamma)$. Starting with Eq. (2.7)

$$F^a_{\alpha\beta}(\gamma) = \frac{1}{\pi}\int_0^\infty \frac{dx}{x^{1+\gamma}} \operatorname{Im} D^a_{\alpha\beta}(x), \tag{A1}$$

which is a Mellin transform, we have that the singularities of $F^a_{\alpha\beta}(\gamma)$ in the right-half γ plane are determined by the threshold behavior of $\operatorname{Im} D^a_{\alpha\beta}(x)$ as $x \to 0$. The point $x = 0$ corresponds to the production threshold for $N = 2, 4, 6, \ldots$ ground-state mesons, and from unitarity one expects the behavior

$$\operatorname{Im} D^a_{\alpha\beta}(x) \underset{x\to 0}{\simeq} \sum_N C_N x^{r_N}, \tag{A2}$$

where C_2 and $r_2 = r$ are determined from current algebra and unitarity. This behavior (A2) for $\operatorname{Im} D^a_{\alpha\beta}(x)$ at $x \to 0$ implies from (A1) that $F^a_{\alpha\beta}(\gamma)$ will have poles at $\gamma = r_N$ along the right-half real axis with residues at these poles equal to C_N/π.

$F^a_{\alpha\beta}(\gamma)$ is a meromorphic function defined by analytic continuation by (A1). If $F^a_{\alpha\beta}(\gamma_0)$ exists with $\gamma_0 \leq 0$ and real, then (A1) implies $F^a_{\alpha\beta}(\gamma) \to 0$ for $\gamma \to \infty$ with $\operatorname{Re}\gamma > \gamma_0$. If we now consider a contour C in the half plane $\operatorname{Re}\gamma' > \gamma_0$ which encloses the point γ and the poles on the real axis, Cauchy's theorem implies

$$\frac{1}{2\pi i}\oint_C \frac{d\gamma' F^a_{\alpha\beta}(\gamma')}{\gamma' - \gamma} = F^a_{\alpha\beta}(\gamma) - \sum_N \frac{C_N/\pi}{r_N - \gamma}. \tag{A3}$$

Opening this contour integral to a line integral we have

$$F^a_{\alpha\beta}(\gamma) = \frac{C_2/\pi}{r - \gamma} + \sum_{N>2} \frac{C_N/\pi}{r_N - \gamma} + \frac{1}{2\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma' F^a_{\alpha\beta}(\gamma')}{\gamma' - \gamma}, \tag{A4}$$

which is Eq. (2.8) in the text with the background integral identified as

$$B_{\alpha\beta}(\gamma) = \frac{1}{2\pi i}\int_{\gamma_0 - i\infty}^{\gamma_0 + i\infty} \frac{d\gamma' F^a_{\alpha\beta}(\gamma')}{\gamma' - \gamma}. \tag{A5}$$

We can push this background integration line back into the left-half γ plane depending on the high-energy behavior of $\operatorname{Im} D^a_{\alpha\beta}(x)$ as $x \to \infty$. If, for example, $\operatorname{Im} D^a_{\alpha\beta}(x) \sim x^{-S}$ as $x \to \infty$, then the leading left-hand singularity of $F^a_{\alpha\beta}(\gamma)$ is at $\gamma = -S$, so the background integral could be pushed back to $\gamma_0 = -S$.

APPENDIX B

Here we will discuss the problems of gauge invariance in this approach of writing dispersion relations in the momentum transfer. The problem centers around the fact that $\partial_\mu V_\mu^{\pi^+}(x)$ is not a gauge-invariant operator. Under an infinitesimal gauge transformation $\partial_\mu V_\mu^{\pi^+} \to \partial_\mu V_\mu^{\pi^+} + e(\partial_\mu \delta\phi) V_\mu^{\pi^+}$. Hence, in general the matrix element of this operator transforms according to

$$\begin{aligned}\langle\alpha|\partial_\mu V_\mu^{\pi^+}(0)|\beta\rangle \to \langle\alpha|\partial_\mu V_\mu^{\pi^+}(0)|\beta\rangle &\\ - i(q^\alpha - q^\beta)_\mu e\langle\alpha|V_\mu^{\pi^+}(0)\delta\phi(0)|\beta\rangle &\\ - \langle\alpha|\partial_\mu V_\mu^{\pi^+} e\,\delta\phi|\beta\rangle, &\end{aligned}$$

where we have used

$$\partial_\mu(\delta\phi V_\mu^{\pi^+}) = \delta\phi\,\partial_\mu V_\mu^{\pi^+} + (\partial_\mu\delta\phi)V_\mu^{\pi^+}.$$

To the lowest relevant order of electromagnetism which we are considering here, we may drop the last term in this expression since it is of higher order in e. But we may only drop the second term if the momentum transfer $(q^\alpha - q^\beta)_\mu = 0$. For calculating mass shifts in an isomultiplet the momentum transfer vanishes, so to leading order in electromagnetism this matrix element is gauge-independent.

If we now calculate $D(0)$, which is the mass shift, from the dispersion relation (5.6),

$$D(0) = \frac{1}{\pi}\int_0^\infty \frac{dt}{t} \operatorname{Im} D(t), \tag{B1}$$

and consider the nontadpole contribution to the absorptive part, there enters into the unitarity condition the matrix element $\langle 0|i\partial_\mu V_\mu^{\pi^+}(0)|\pi^a(q_1)\pi^b(q_2)\gamma(k)\rangle$ corresponding to the two-meson–one-photon state. Using (5.2) we obtain for this matrix element

$$\langle 0|i\partial_\mu V_\mu^{\pi^+}(0)|\pi^a(q_1)\pi^b(q_2)\gamma(k)\rangle = ef_+^*((q_1+q_2)^2)\epsilon_\mu(k)(q_1-q_2)_\mu f^{\pi^+ab},$$

with $\epsilon_\mu(k)$ the polarization vector of the photon and $f_+((q_1+q_2)^2)$ the vector form factor. This matrix element is not gauge-invariant under $\epsilon_\mu \to \epsilon_\mu + \lambda k_\mu$. In computing the absorptive part we contract this matrix element with the scattering amplitude for $\gamma(k) + \pi^a(q_1) + \pi^b(q_2) \to \pi^c(p_1) + \pi^d(p_2)$ which we can compute exactly at threshold by low-energy theorems in a gauge-invariant fashion. In calculating the absorptive part there appears a sum on the photon polarizations $\sum \epsilon_\mu(k)\epsilon_\nu^*(k)$, which in the Feynman gauge is specified by $-g_{\mu\nu}$. We are required to calculate the absorptive part in this gauge if we use the dispersion relation (5.6) since other gauges will destroy the analyticity properties of $D(t)$ which we have assumed in writing (B1) by introducing additional singularities. The result of this calculation for $D(0)$ or the mass shift will be gauge-independent. Our conclusion is that we must use the Feynman gauge in conjunction with the dispersion relation (B1). A similar observation was made in Ref. 7, footnote 9.

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[1]M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

[2]S. Coleman, J. Math. Phys. 7, 787 (1966).

[3]J. Goldstone, Nuovo Cimento 19, 154 (1961).

[4]To leading order the meson masses vanish like $\mu^2 \sim \lambda$.

[5]Ling-Fong Li and Heinz Pagels, Phys. Rev. Letters 26, 1204 (1971).

[6]Ling-Fong Li and Heinz Pagels, Phys. Rev. Letters 27, 1089 (1971).

[7]S. D. Drell and H. Pagels, Phys. Rev. 140, B397 (1965).

[8]S. Coleman and S. Glashow, Phys. Rev. 134, B671 (1964).

[9]R. T. Dashen and S. C. Frautschi, Phys. Rev. 135, B1190 (1964).

[10]W. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963).

[11]T. P. Cheng and R. Dashen, Phys. Rev. Letters 26, 594 (1971).

[12]G. Höhler, H. P. Jakob, and R. Strauss, Phys. Letters 35B, 445 (1971).

[13]We conjecture that the four-meson state first gives rise to a pole at $\gamma = 3$.

[14]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[15]We would like to thank Professor B. W. Lee for a discussion on this point.

[16]We would like to thank R. Dashen and M. Weinstein for discussions relating to the K_{l3} decay theorem.

[17]R. Dashen and M. Weinstein, Phys. Rev. Letters 22, 1337 (1969).

[18]We assume that the amplitude $T_2(\nu, q^2)$ is also unsubtracted as is implied by Regge behavior.

[19]For our notation see, for example, J. D. Bjorken, Phys. Rev. 179, 1547 (1969).

[20]See T. D. Lee, Columbia Univ. report, 1971 (unpublished); H. Pagels, Phys. Rev. 185, 1990 (1969); Phys. Rev. D 3, 610 (1971); 4, 1932(E) (1971); R. Jackiw, R. Van Royen, and G. B. West, *ibid*. 2, 2473 (1970).

[21]E. D. Bloom *et al.*, MIT-SLAC Report No. SLAC-PUB-796, 1970 (unpublished), presented at the Fifteenth International Conference on High Energy Physics, Kiev, U.S.S.R., 1970.

[22]T. D. Lee and G. C. Wick, Nucl. Phys. B9, 209 (1969); B10, 1 (1969).

[23]M. Elitzur and H. Harari, Ann. Phys. (N.Y.) 56, 81 (1970).

[24]R. Dashen, Phys. Rev. 183, 1245 (1969). See the discussion in Sec. IV.

[25]Y. Nambu and G. Jona-Lasinio, Phys. Rev. 124, 246 (1961).

[26]K. Wilson, Phys. Rev. 179, 1499 (1960). The relevant discussion is Sec. VII B.