PHYSICAL REVIEW D VOLUME 5, NUMBER 6 15 MARCH 1972

σ Term in Pion-Nucleon Scattering*

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(Received 11 November 1971)

It is argued that the method of Cheng and Dashen gives a calculation of approximately *three times* the σ term, rather than the σ term itself. This conclusion is based on a correction derived from a unitarization of the t-channel $\pi\pi$ scattering amplitude, consistent with the constraints of current algebra. A detailed analysis of possible model dependence of the result is included.

I. INTRODUCTION

The study of the approximate symmetries of the strong interactions is one of the central questions of particle physics.[1] As a necessary part of these studies one must disentangle, from experiment, information concerning the mechanisms which break these symmetries. Although considerable effort has been expended in this direction, it is still far from a closed subject. For example, one still does not know conclusively whether chiral SU(2)×SU(2) is a better symmetry than SU(3), so that the manner in which chiral SU(3)×SU(3) symmetry is broken is still subject to uncertainties. The understanding of broken scale invariance is even further from our grasp.

One may obtain some information pertaining to these questions from the value of the so-called σ term in pion-nucleon scattering. The earliest estimates of this term gave support to the notion that chiral SU(2)×SU(2) symmetry was in fact a better symmetry than SU(3).[2] More recently the numerical value of this term has been the subject of some controversy,[3,4] throwing this conclusion into doubt. A reliable evaluation of the σ term takes on further importance, as it may be useful in providing an understanding of the mechanism by which scale invariance is broken.[5]

Unfortunately, the σ term is the value of the pion-nucleon scattering amplitude at an unphysical off-mass-shell point, so that ambiguities are introduced by any uncertainty in the extrapolation from off-shell to on-shell pions. Furthermore, the connection between the σ term and the mechanisms which break chiral SU(3)×SU(3) symmetry and scale invariance is by no means direct. At least three stages of argument can be identified.

(1) One must make assumptions or provide dynamical arguments which connect the scattering amplitude at the "σ point" to the on-shell amplitude.

(2) An evaluation of the on-shell amplitude must be made by means of dispersion relations,[6] or in terms of low-energy scattering data.[7]

(3) The "measured" value of the σ term must be compared with the value calculated from a postulated symmetry-breaking Hamiltonian.[8]

In order to deal with the first two issues, Cheng and Dashen (C-D) make a simple assumption concerning the off-shell behavior so as to evaluate the σ term from on-mass-shell dispersion relations, and obtain a surprisingly large result.

It is the purpose of this paper to present a detailed study of the first question, the extrapolation to the mass shell. Our conclusion is that C-D have in fact calculated approximately *three times* the σ term, rather than the σ term itself. This result follows from important corrections to the C-D result, whose origin is the unitarization of the t-channel $\pi\pi$ scattering amplitude consistent with the constraints of current algebra. Our argument leading to this conclusion is organized into two steps.

(1) A representation for πN scattering is derived from the Ward identities of chiral SU(2)×SU(2) symmetry which exhibits the one-particle *reducible* part of the σ field. This provides a framework for the discussion of the effects of a strong 0^+ t-channel exchange. This step is important irrespective of the existence of a low-lying 0^+ meson resonance, as the magnitude of the σ term itself sets the over-all scale for such exchanges.

(2) The one-particle reducible part of the σ field is obtained from our unitarization of $\pi\pi$ scattering, given in earlier work.[9] This additional contribution leads to the change in the interpretation of the numerical result of C-D. We end the discussion with a detailed analysis of this work, with an eye towards uncovering a possible model dependence which could weaken our conclusion. A number of supporting calculations, and a review of our unitarization of $\pi\pi$ scattering, are to be

found in the appendixes.

II. GENERAL CONSIDERATIONS

We construct a representation for πN scattering based on the Ward identities of chiral SU(2)×SU(2) symmetry, starting with the equal-time commutators

$$[A_0^a(x), A_\mu^b(y)]\,\delta(x^0-y^0) = i\epsilon_{abc}\,\delta^4(x-y)V_\mu^c(y) + \text{S.T.}, \tag{1a}$$

$$[A_0^a(x), \partial^\mu A_\mu^b(y)]\,\delta(x^0-y^0) = \delta_{ab}\,\sigma(x)\delta^4(x-y), \tag{1b}$$

with $V_\mu^a(x)$ $[A_\mu^a(x)]$ the vector [axial-vector] current with isospin a and S.T. the Schwinger terms. Equation (1b) defines a local scalar field $\sigma(x)$ by means of the indicated equal-time commutator. Define $\phi^a(x)$ and $\tilde{A}_\mu^a(x)$ by

$$\partial^\mu A_\mu^a(x) = m_\pi^{\,2} F_\pi \phi^a(x) \tag{2}$$

and

$$A_\mu^a(x) = \tilde{A}_\mu^a(x) - F_\pi \partial_\mu \phi^a(x), \tag{3}$$

where m_π (F_π) is the pion mass (decay constant). It is a straightforward matter to derive Ward identities for $\tilde{A}_\mu$ by replacing the axial-vector current in the usual Ward identities by Eq. (3). For pion-nucleon scattering, the result is[10]

$$\begin{aligned}\int d^4x\,d^4y\, e^{iq\cdot x} e^{-ik\cdot y}(\Box_x^{\,2}+m_\pi^{\,2})(\Box_y^{\,2}+m_\pi^{\,2})\langle p_2|T^*\{\phi^b(x)\phi^a(y)\}|p_1\rangle \\ = F_\pi^{-2} q^\mu k^\nu \int d^4x\,d^4y\, e^{iq\cdot x} e^{-ik\cdot y}\langle p_2|T^*\{\tilde{A}_\mu^b(x)\tilde{A}_\nu^a(y)\}|p_1\rangle \\ - \delta_{ab}F_\pi^{-2}\left(1-\frac{q^2}{m_\pi^{\,2}}-\frac{k^2}{m_\pi^{\,2}}\right)\int d^4x\, e^{i(q-k)\cdot x}\langle p_2|\sigma(x)|p_1\rangle + iF_\pi^{-2}\epsilon_{abc}k^\nu\int d^4x\, e^{i(q-k)\cdot x}\langle p_2|V_\nu^c(x)|p_1\rangle,\end{aligned} \tag{4}$$

where k (q) is the initial (final) pion four-momentum. It is evident that Eq. (4) is an off-shell continuation of the pion-nucleon scattering amplitude. This formulation of current algebra is useful since the pion poles are explicitly removed from the axial-vector current, and the contributions of the vector current and σ term are nonzero in the mass-shell limit.

Let us focus our attention on the isotopic-even amplitude. Define

$$T(p_2,q;p_1,k)^{ba} = -i\int d^4x\, e^{iq\cdot x}(q^2-m_\pi^{\,2})(k^2-m_\pi^{\,2}) \times\langle p_2|T^*\{\phi^b(x)\phi^a(0)\}|p_1\rangle, \tag{5}$$

and make the decomposition

$$T^{ba} = \delta_{ab}T^{(+)} + \tfrac{1}{2}[\tau_b,\tau_a]\,T^{(-)} \tag{6}$$

and

$$T^{(+)} = A^{(+)} + \gamma\cdot Q B^{(+)}, \tag{7}$$

where $Q=\frac{1}{2}(q+k)$. We will be particularly interested in

$$M^{(+)} = A^{(+)} + \nu B^{(+)}, \tag{8}$$

which is the amplitude studied by C-D. Useful variables for this problem, in addition to the usual Mandelstam variables, are

$$\begin{aligned}\nu &= (q+k)\cdot(p_2+p_1)/4m \\ &= (s-u)/4m\end{aligned} \tag{9}$$

and

$$\nu_B = -q\cdot k/2m. \tag{10}$$

Let us separate the nucleon pole contribution from the right-hand side of Eq. (4), which we indicate schematically as

$$T_{\mu\nu}(p_2,q;p_1,k)^{ba} + \text{“nucleon pole”} = -i\int d^4x\, e^{iq\cdot x}\langle p_2|T^*\{\tilde{A}_\mu^b(x)\tilde{A}_\nu^a(0)\}|p_1\rangle, \tag{11}$$

which also serves to define $T_{\mu\nu}$. The nucleon pole contribution to the even amplitude in (11) is

$$\begin{aligned}A_N(\nu,\nu_B,q^2,k^2)^{(+)} &= m g_A(q^2)g_A(k^2)/F_\pi^{\,2}, \\ B_N(\nu,\nu_B,q^2,k^2)^{(+)} &= \frac{m g_A(q^2)g_A(k^2)}{F_\pi^{\,2}}\left(\frac{\nu}{\nu_B^{\,2}-\nu^2}\right),\end{aligned} \tag{12}$$

where $g_A(q^2)$ is the axial-vector form factor of the nucleon, so that

$$\begin{aligned}M_N(\nu,\nu_B,q^2,k^2)^{(+)} &= \frac{m g_A(q^2)g_A(k^2)}{F_\pi^{\,2}}\left(\frac{\nu_B^{\,2}}{\nu_B^{\,2}-\nu^2}\right) \\ &\simeq \frac{g_{\pi N}^{\,2}}{m}\left(\frac{\nu_B^{\,2}}{\nu_B^{\,2}-\nu^2}\right)\end{aligned} \tag{13}$$

by virtue of the Goldberger-Treiman relation. Finally define

$$F_N(t) = i\langle p_2|\sigma(0)|p_1\rangle \tag{14}$$

so that on combining Eqs. (4), (5), (11), and (12) one arrives at the identity

$$T(p_2, q; p_1, k)^{(+)} = \text{“nucleon pole”} + F_\pi^{-2}\left(1 - \frac{q^2}{m_\pi^2} - \frac{k^2}{m_\pi^2}\right)F_N(t) + F_\pi^{-2} q^\mu k^\nu T_{\mu\nu}(p_2, q; p_1, k)^{(+)} \qquad (15)$$

The σ term of πN scattering, which is our primary interest, is

$$T(p, 0; p, 0)^{(+)} = M(0, 0; 0, 0)^{(+)} = F_\pi^{-2} F_N(0) \equiv C \,. \qquad (16)$$

The entire focus of our study, and that of others, is in obtaining a reliable estimate of C from mass-shell data.

Model of Cheng and Dashen

The model of C-D (Ref. 3) is easily formulated from Eq. (15). Assume that

$$F_\pi^{-2} F_N(t) \simeq C \qquad (17)$$

for sufficiently small values of t, and further assume that for sufficiently small ν and ν_B

$$M(\nu, \nu_B; q^2, k^2)^{(+)} = M_N(\nu, \nu_B; q^2, k^2)^{(+)} + \left(1 - \frac{q^2}{m_\pi^2} - \frac{k^2}{m_\pi^2}\right)C + \alpha\nu_B + \beta\nu^2, \qquad (18)$$

where α and β are constants and $M_N^{(+)}$ is given by Eq. (13). Equation (18) is based on the premise that $q^\mu k^\nu T_{\mu\nu}^{(+)}$ is a slowly varying function of ν and ν_B, for small ν and ν_B, as all important dynamical effects have already been exhibited. (It is this proposition which we challenge in our analysis of the one-particle reducible part of the σ field.) One observes that the C-D model has the properties

$$M(0, 0; 0, 0)^{(+)} = C, \qquad (19a)$$

$$M(0, 0; m_\pi^2, 0)^{(+)} = 0, \qquad (19b)$$

and

$$M(0, 0; m_\pi^2, m_\pi^2)^{(+)} = -C, \qquad (19c)$$

so that in this model the evaluation of the σ term is reduced to a calculation of the mass-shell amplitude at the unphysical point $\nu = 0$, $t = 2m_\pi^2$. [Note from Eq. (15) that the coefficient of the σ term varies rapidly in the region $0 \leq q^2$, $k^2 \leq m_\pi^2$.]

III. THE σ TERM

If the σ term C is large, then $q^\mu k^\nu T_{\mu\nu}^{(+)}$ has a strong contribution from σ exchange in the t channel, even if there is no low-lying 0^+ resonance, as $F_\pi^{-2}F_N(t) \simeq C$ sets the scale for these contributions In that case Eq. (18) may not be a reliable model for the off-shell extrapolation of the πN amplitude. This question can be dealt with by removing the σ one-particle reducible parts from $q^\mu k^\nu T_{\mu\nu}^{(+)}$, and then studying these effects separately. We pursue this line of analysis.

Guided by perturbation theory, one expects $T_{\mu\nu}^{(+)}$ to have the general structure

$$T_{\mu\nu}(p_2, q; p_1, k)^{(+)} = F_{\mu\nu}(q, k)\Delta_0(t)^{-1}F_N(t) + \tilde{T}_{\mu\nu}(p_2, q; p_1, k)^{(+)}, \qquad (20)$$

where $\tilde{T}_{\mu\nu}^{(+)}$ is one-particle *irreducible* with respect to the σ field. The $\tilde{A}\tilde{A}\sigma$ vertex, which appears in Eq. (20), is defined by

$$\delta^{ab} F_{\mu\nu}(q, k) = \int d^4x\, d^4y\, e^{iq\cdot x} e^{-ik\cdot y} \times \langle 0 | T^*\{\tilde{A}_\mu^b(x)\tilde{A}_\nu^a(y)\sigma(0)\} | 0\rangle \qquad (21)$$

and the propagator of the σ field by

$$\Delta_0(q^2) = \int d^4x\, e^{iq\cdot x}\langle 0 | T\{\sigma(x)\sigma(0)\} | 0\rangle \,. \qquad (22)$$

When Eq. (20) is inserted into (15) one encounters $q^\mu k^\nu F_{\mu\nu}(q, k)$, which itself can be calculated in terms of the Ward identity for the $\pi\pi\sigma$ vertex. This identity is[11]

$$-m_\pi^2 F_0(q, k) = F_\pi^{-2} q^\mu k^\nu F_{\mu\nu}(q, k) + F_\pi^{-2}\left(1 - \frac{q^2}{m_\pi^2} - \frac{k^2}{m_\pi^2}\right)\Delta_0(t) - m_\pi^2(m_\pi^2 - q^2)\Delta_\pi(q) - m_\pi^2(m_\pi^2 - k^2)\Delta_\pi(k) + m_\pi^4\Delta_\pi(0), \qquad (23)$$

where the $\pi\pi\sigma$ form factor is

$$\delta^{ab} F_0(q, k) = \int d^4x\, d^4y\, e^{iq\cdot x} e^{-ik\cdot y}(\Box_x^2 + m_\pi^2)(\Box_y^2 + m_\pi^2)\langle 0 | T^*\{\phi^b(x)\phi^a(y)\sigma(0)\} | 0\rangle \qquad (24)$$

and

$$\delta^{ab}\Delta_\pi(q) = \int d^4x\, e^{iq\cdot x}\langle 0 | T^*\{\phi^b(x)\phi^a(0)\} | 0\rangle \qquad (25)$$

is the pion propagator. Combining Eqs. (15), (20), and (23), one finds the representation

$$T(p_2, q; p_1, k)^{(+)} = \text{“nucleon pole”} - m_\pi^2[F_0(q,k) - (m_\pi^2 - q^2)\Delta_\pi(q) - (m_\pi^2 - k^2)\Delta_\pi(k) + m_\pi^2\Delta_\pi(0)]\,\Delta_0(t)^{-1}F_N(t) + F_\pi^{-2}q^\mu k^\nu \tilde{T}_{\mu\nu}(p_2, q; p_1, k)^{(+)}\,. \tag{26}$$

To evaluate $T^{(+)}$ at the σ point or verify the Adler condition, one must use the Ward identity (23).

On the pion mass shell, Eq. (26) becomes

$$T(\nu, \nu_B; m_\pi^2, m_\pi^2)^{(+)} = \text{“nucleon pole”} + m_\pi^2[1 - F_0(t)]\,\Delta_0(t)^{-1}F_N(t) + F_\pi^{-2}q^\mu k^\nu \tilde{T}^{(+)}_{\mu\nu}|_{q^2=k^2=m_\pi^2}\,, \tag{27}$$

where $F_0(t)$ is the $\pi\pi\sigma$ form factor with the pions on the mass shell. In arriving at Eq. (27) we have made the approximation

$$m_\pi^2\Delta_\pi(0) = 1\,. \tag{28}$$

This representation provides the framework for the analysis which we present as an alternative to that of C-D.

In Appendix A we study Eq. (27) in the linear σ model evaluated in the tree approximation, where we show that $q^\mu k^\nu \tilde{T}^{(+)}_{\mu\nu} \equiv 0$ in this approximation. This example from perturbation theory provides the interpretation of $[F_0(t) - 1]$ as the pseudovector coupling of $\pi\pi\sigma$, in accord with the PCAC (partial conservation of axial-vector current) estimate $F_0(0) = 1 + O(m_\pi^2/m_\sigma^2)$.

A. Ward-Identity Model

We now propose an alternative to the model of C-D based on Eqs. (26) and (27). We assume that

$$\begin{aligned} M(\nu, \nu_B; q^2, k^2)^{(+)} = {} & M_N(\nu, \nu_B; q^2, k^2)^{(+)} \\ & - m_\pi^2 F_\pi^2[F_0(q,k) - 1]\,\Delta_0(t)^{-1}C \\ & + \alpha\nu_B + \beta\nu^2 \end{aligned} \tag{29}$$

is valid for sufficiently small values of ν and ν_B, with α and β constants [although they are not the same constants that appear in Eq. (18)]. Equation (29) represents $q^\mu k^\nu \tilde{T}_{\mu\nu}$ by a slowly varying function of ν and ν_B after *both* the nucleon pole and σ one-particle reducible parts have been removed. (One might also remove the N^* contribution, but *a posteriori* this step can be shown to be unimportant for the issues raised here.[12]) On the pion mass shell, Eq. (29) becomes

$$\begin{aligned} M(\nu, \nu_B; m_\pi^2, m_\pi^2)^{(+)} = {} & \frac{g_{\pi N}^2}{m}\left(\frac{\nu_B^2}{\nu_B^2 - \nu^2}\right) \\ & - m_\pi^2 F_\pi^2[F_0(t) - 1]\,\Delta_0(t)^{-1}C \\ & + \alpha\nu_B + \beta\nu^2 \end{aligned} \tag{30}$$

which has the properties

$$\begin{aligned} & M(0,0;0,0)^{(+)} = C\,, \\ & M(0,0;m_\pi^2,0)^{(+)} = 0\,, \\ & M(0,0;m_\pi^2,m_\pi^2)^{(+)} = -m_\pi^2 F_\pi^2[F_0(2m_\pi^2) - 1] \times\Delta_0(2m_\pi^2)^{-1}C\,, \end{aligned} \tag{31}$$

which are demonstrated through use of Eqs. (13), (23), and (30).

This model is still incomplete, as one has not yet specified the behavior of $[F_0(t) - 1]\,\Delta_0(t)^{-1}$. Fortunately this quantity is available from earlier work,[9] where it is obtained as an intermediate step in the unitarization of $\pi\pi$ scattering satisfying current-algebra constraints. In Sec. III B we apply those results.

B. ππ Unitarity

Our unitarization of $\pi\pi$ scattering[9] implies

$$[F_0(t) - 1]\,\Delta_0(t)^{-1} = R(t) + O(\epsilon) \quad \text{for} \quad 4m_\pi^2 \leq t \leq t_0\,, \tag{32}$$

with $R(t)$ real to $O(\epsilon^2)$, also for $4m_\pi^2 \leq t \leq t_0$. The interval $4m_\pi^2 \leq t \leq t_0$ describes the region for which a basic hypothesis of our unitarization method is valid. For our purposes this interval need not be too large ($4m_\pi^2 \leq t \leq 6m_\pi^2$, say). Power counting and other heuristic arguments indicate that $\epsilon^2 \sim (m_\pi^4/m_\sigma^4)$, while a perturbation-theoretic study[13] gave

$$\epsilon^2 \sim (m_\pi^4/m_\sigma^4)\ln(m_\pi^2/m_\sigma^2).$$

Both estimates are valid for t sufficiently close to threshold. Not only is $R(t)$ real, but it must satisfy a self-consistency condition due to the crossing symmetry of $\pi\pi$ scattering, which gives enough information to determine $R(t)$ for small values of t. It is found that

$$R(t) = -m_\pi^{-2}F_\pi^{-2}\left[\left(1 - \frac{2t}{m_\pi^2}\right) + O\left(\frac{m_\pi^2}{m_\sigma^2}\right)\right]. \tag{33}$$

(The reader will find a sketch of the derivation of these results in Appendix B, but this should not be a substitute for a reading of the detailed arguments of Ref. 9.)

Unitarity has now provided enough information to complete our model, at least for on-shell πN scattering. The model, which combines Eq. (30) with Eqs. (32)–(33), is

$$M(\nu, \nu_B; m_\pi^2, m_\pi^2)^{(+)} = \frac{g_{\pi N}^2}{m}\left(\frac{\nu_B^2}{\nu_B^2 - \nu^2}\right) + \left[\left(1 - \frac{2t}{m_\pi^2}\right) + O\left(\frac{m_\pi^2}{m_\sigma^2}\right)\right] C + \alpha\nu_B + \beta\nu^2, \quad (34)$$

which leads to the important conclusion that

$$M(0, 0; m_\pi^2, m_\pi^2)^{(+)} = -[3 + O(m_\pi^2/m_\sigma^2)]\, C\,. \quad (35)$$

Hence a calculation of the on-shell πN scattering amplitude at $\nu=0$, $t=2$ does not calculate the σ term, but approximately *three times* the σ term. The isolation of t-channel exchanges, combined with $\pi\pi$ unitarity, has led to a large correction to the model of C-D. Let us now analyze our result to expose possible sources of model dependence which could weaken our conclusions.

IV. ANALYSIS OF RESULTS

This section is devoted to a detailed analysis of several issues, in order to provide a better understanding of our conclusion.

A. The Crucial Zero

There is a crude way of understanding the origin of the correction found in Eq. (35). To see this, let us accept (32) as valid, but not impose (33). Instead we replace Eq. (33) with the *ad hoc* assumption

$$[F_0(t) - 1)]\,\Delta_0(t)^{-1} = -m_\pi^{-2} F_\pi^{-2}\left(1 - \frac{t}{m_\pi^2 K}\right) \quad (36)$$

(with K a constant) so that Eq. (36) has a zero when $t = m_\pi^2 K$. With this replacement,

$$M(0, 0; m_\pi^2, m_\pi^2)^{(+)} = \left(1 - \frac{2}{K}\right) C \quad (37)$$

obtains, which shows that the location of the zero in Eq. (36) is intimately related to the magnitude of the correction to the C-D model. For example,

$$K = 1 \quad (38a)$$

in the σ model, tree approximation,[14] while

$$K = \tfrac{1}{2} \quad (38b)$$

for our theory. Strictly speaking, $K=1$ does not correspond to the C-D model, as they do not separate t-channel exchanges; however, if $K=1$, their conclusions are unaltered.

It is important to appreciate that this zero is determined dynamically, and *not* as a consequence of current algebra, PCAC, or kinematical requirements. For example, one can study $\pi\pi$ scattering in the *linear σ model*,[14,15] which provides a possible theoretical laboratory to illustrate this point. In the *tree approximation*

$$[F_0(t) - 1]\,\Delta_0(t)^{-1} = -F_\pi^{-2}(m_\pi^2 - t)\,, \quad (39)$$

while for the *one-loop approximation*[13,15]

$$[F_0(t) - 1]\,\Delta_0(t)^{-1} = -F_\pi^{-2}((m_\pi^2 - t) - \lambda(\lambda F_\pi^2)\{12\overline{B}_{\sigma\pi}(m_\pi^2) - 4\overline{B}_{\sigma\pi}(0) - 4(m_\pi^2 - t)\overline{B}'_{\sigma\pi}(m_\pi^2) - 12\overline{B}_{\sigma\sigma}(t) + 4\overline{B}_{\pi\pi}(t) + 8(\lambda F_\pi^2)[C_\sigma(t) + 3C_\pi(t)]\})\,. \quad (40)$$

In this model PCAC and current-algebra constraints are satisfied in each order of perturbation theory, while in principle the position of the zero depends on the meson-meson coupling constant λ. However, since our effect is due to $\pi\pi$ unitarity, it may require summing an infinite set of diagrams to check our claim in perturbation theory. [The functions $\overline{B}_{xy}(t)$ and $C_x(t)$ are integrals defined by Lee[15] in his study of the σ model. Note that the meson coupling λ is of opposite sign to that of Ref. 15.]

One can also study πN scattering in the linear σ model, as in Appendix A. In the *tree approximation*

$$M(\nu, \nu_B; q^2, k^2)^{(+)} = \frac{g^2}{m}\left(\frac{\nu_B^2}{\nu_B^2 - \nu^2}\right) + gF_\pi^{-1}[F_0(t) - 1]\,, \quad (41)$$

which means the remainder $\alpha\nu_B + \beta\nu^2 = 0$ in this order of approximation. Notice that in the tree approximation, the Adler condition requires

$$F_0(m_\pi^2) = 1\,. \quad (42)$$

However in higher orders of perturbation theory, the Adler condition does not provide a constraint strong enough to locate the zero accurate to $O(m_\pi^2/m_\sigma^2)$. But it is the *magnitude* of the term which is of $O(m_\pi^2/m_\sigma^2)$ that is crucial. The Adler condition implies (42) in the tree approximation because $F_0(q, k)$ has no off-shell dependence on q^2 or k^2, i.e., $F_0(q, k) = F_0(t)$ in lowest order, which is no longer true in higher order, so that one cannot determine the correction of $O(m_\pi^2/m_\sigma^2)$ from the Adler condition. It is possible that our unitarization procedure is equivalent to the summation of an infinite set of perturbation-theory diagrams, so that it is certainly possible that the zero of $[F_0(t) - 1]\,\Delta_0(t)^{-1}$ is no longer correctly given by the tree approximation.

If the Adler condition does not determine the position of this zero, what does? We suggest that it is the Weinberg zero of $\pi\pi$ scattering.[16] To illustrate this point, let us consider another result from our current-algebra unitarization program, where we have shown, as an intermediate result, that the S-wave $\pi\pi$ scattering amplitude can be written

$$T_0(s)=F_0(s)[F_0(s)-1]\,\Delta_0(s)^{-1}+O(\epsilon)$$
$$=F_0(s)[R(s)+O(\epsilon)]\,, \qquad (43)$$

for energies sufficiently close to threshold. [See Eqs. (B15)–(B18), or (3.5b) of Ref. 9.] Equation (43) states that $T_0(s)$ has the same phase as $F_0(s)$, as it must to satisfy unitarity. Since $F_0(0)=1+O(m_\pi^2/m_\sigma^2)$, by virtue of PCAC, the variation of $T_0(s)$ for small values of s is due almost entirely to $R(s)$. But if Eq. (43) is continued below threshold, it should exhibit the Weinberg zero, which $R(s)$ does, as can be verified by comparing Eq. (33) with the Weinberg S-wave $\pi\pi$ scattering amplitude.[16] It appears that t-channel unitarity for πN scattering has replaced the Adler zero that appears in lowest order by the Weinberg zero of $\pi\pi$ scattering. We think that this is the most satisfactory way of understanding the correction to the work of C-D as given by Eqs. (34) and (35).

B. Off-Shell Behavior

It should be apparent that if there is a large correction to the C-D model for on-mass-shell pions, there should be a corresponding correction to the off-shell behavior. This can be investigated by means of Eq. (29), which expresses the off-shell behavior of πN scattering in terms of $F_0(q,k)$, which has not as yet been specified except for its mass-shell limit. The off-shell behavior of $F_0(q,k)$ cannot be fixed from the Ward identity (23) and the mass-shell limit alone; additional information is required. By adopting a simple smoothness assumption for $F_{\mu\nu}(q,k)$, we show in Appendix C that

$$-m_\pi^2F_\pi^2[F_0(q,k)-1]\,\Delta_0(t)^{-1}=\left[\left(1-\frac{q^2}{m_\pi^2}-\frac{k^2}{m_\pi^2}\right)+m_\pi^{-4}[-t(q^2+k^2)+(q^4+k^4)+\tfrac{1}{4}\delta(t-q^2-k^2)(2-q^2-k^2)]\right], \qquad (44)$$

consistent with the Ward identity and mass-shell constraint. The parameter δ is a constant undetermined by these considerations. If Eq. (44) is combined with (29), one obtains the following model for off-shell πN scattering:

$$M(\nu,\nu_B;q^2,k^2)^{(+)}=M_N(\nu,\nu_B;q^2,k^2)^{(+)}+\left(1-\frac{q^2}{m_\pi^2}-\frac{k^2}{m_\pi^2}\right)C$$
$$+m_\pi^{-4}[-t(q^2+k^2)+(q^4+k^4)+\tfrac{1}{4}\delta(t-q^2-k^2)(2-q^2-k^2)]\,C+\alpha\nu_B+\beta\nu^2\,. \qquad (45)$$

The striking difference between (45) and (18) is in the presence of terms of $O(q^4/m_\pi^4)$ which are not present in the C-D model. These additional terms provide a distinguishing feature of our picture of πN scattering. We can think of no *a priori* reason to reject these terms, as Eq. (45) satisfies all the requirements of current algebra and PCAC. It is indeed true that $M^{(+)}$, as given by (45), varies rapidly for $0\le q^2$, $k^2\le m_\pi^2$, but the C-D amplitude also varies rapidly in this domain, so that this does not seem to be a sufficient reason to reject (45). We reiterate, the choice can only be made on the basis of dynamics, and not from current algebra or PCAC alone.

C. Model Dependence of Results

It is worthwhile to give some consideration to the extent to which our results are model dependent. Although it is true that our off-shell πN amplitude depends on a smoothness assumption, we would like to believe that our treatment of mass-shell πN scattering in Eqs. (32)–(35) is relatively free from these uncertainties. This is an important consideration, as our correction to C-D is derived from the mass-shell equations.

One can derive a representation for $\pi\pi$ scattering which is analogous to Eq. (30) for πN scattering,[9] with a "background term" of the form

$$q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}\tilde{T}^{\mu\nu\lambda\sigma}\,,$$

in analogy with the background term that appears in (27)–(30). The background term in $\pi\pi$ scattering also vanishes identically in the tree approximation to the linear σ model.[13] If it is assumed in general that the S- and P-wave projections of this background term of $\pi\pi$ scattering are predominately real for energies sufficiently close to threshold, it then follows as a theorem, based on unitarity, that $R(t)$ is real to $O(\epsilon^2)$, and is given explicitly by Eq. (32) to $O(\epsilon)\sim O(m_\pi^2/m_\sigma^2)$. The detailed behavior of $R(t)$, as given by Eq. (33), depends on the additional assumption that $R(t)$ is a linear function of t which can be analytically continued to small, negative values of t. If our basic assumptions are justified, then Eqs. (34) and (35) follow to the stated accuracy.[9]

These assumptions may lead to possible sources of model dependence, which could provide the loopholes to save the C-D analysis. Let us summarize some questions which deserve additional study, together with some indication as to our present understanding of these issues.

(1) We may have underestimated the error ϵ. However, the work of Ford and the present author[13] suggest that the estimate $\epsilon \sim m_\pi^2/m_0^2$ may not be unreasonable. Nevertheless, this is the most likely likely place to look for a source of model dependence of our result.

(2) $R(t)$ may not be a linear function of t even for small values of t. However, Höhler *et al.*[4] have verified that corrections to a linear extrapolation in t are negligible for low-energy πN scattering.

(3) The required analytic continuation of $R(t)$ to small, negative values of t, required by the self-consistency part of our unitarization program, may not be valid. However, this question has been studied in part by Jen,[17] who supports our conclusions.

Although these are all possibilities for model dependence of our result which must be admitted, one does not get a strong indication from preliminary studies of these questions that our basic results are grossly in error.

D. Experiment: Can Models Be Distinguished?

One might hope that one could distinguish our theory from the Cheng-Dashen model by sufficiently accurate low-energy πN scattering data. This hope can be quickly dispelled. All models of the type (36) can be characterized by

$$M(\nu, \nu_B; m_\pi^2, m_\pi^2)^{(+)} - \frac{g_{\pi N}^2}{m}\left(\frac{\nu_B^2}{\nu_B^2-\nu^2}\right) - \beta\nu^2 = \left(C - \frac{\alpha m_\pi^2}{2m}\right) + m_\pi^{-2}\left(\frac{\alpha m_\pi^2}{4m} - \frac{C}{K}\right)t\,. \tag{46}$$

The linear t dependence is not sufficient to determine the three parameters α, C, and K independently, so that experiment *alone* is unable to settle the question. Of course if one chooses a particular model to fix K, one can then determine the σ term. Furthermore, α and β can be calculated from specific models.[12,18] These models entail a calculation of $q^\mu k^\nu \tilde{T}_{\mu\nu}^{(+)}$ in Eq. (27), usually based on dominance by low-lying N^* states, together with a background coming from a high-energy continuum or Regge behavior. Indications are that these N^* states are not important, and may be neglected.

A consideration of the isotopic-even S-wave πN scattering length $a_{0+}^{(+)}$ is also informative. Omitting the negligible nucleon pole contribution, one obtains from Eq. (46)

$$a_{0+}^{(+)} = \frac{m}{8\pi(m+m_\pi)}\left[C - m_\pi^2\left(\frac{\alpha}{2m} - \beta\right)\right], \tag{47}$$

while the Cheng-Dashen model gives

$$a_{0+}^{(+)} = \frac{m}{8\pi(m+m_\pi)}\left[-C - m_\pi^2\left(\frac{\tilde{\alpha}}{2m} - \tilde{\beta}\right)\right]. \tag{48}$$

(We distinguish α and β from $\tilde{\alpha}$ and $\tilde{\beta}$ to emphasize that these constants are distinct.) Estimates from models all indicate that the contribution of low-lying N^* states to $m_\pi^2(\alpha/2m - \beta)$ is negligible.[12,18] If C is as large as claimed by C-D, then the small experimental value of $a_{0+}^{(+)}$ requires a cancellation of C by $(\alpha/2m - \beta)m_\pi^2$. [Note that C has opposite sign in Eqs. (47) and (48). Also note that the σ model, evaluated in the tree approximation, is described by Eq. (47) and not Eq. (48).] If one accepts the claim that $(\alpha/2m - \beta)m_\pi^2$ receives a small contribution from low-lying states, then this cancellation is effected by either a large subtraction constant (in the sense of dispersion relations), or by a surprisingly large Regge tail. Either of these modes of cancellation seems surprising to the author, although neither seems to be ruled out by first principles. If C is very small, then all treatments will be consistent.

Note Added in Proof

Since our conclusions are obviously controversial, it seems appropriate to emphasize the differences between our work and the accepted point of view. The manner in which the term $q^\mu k^\nu T_{\mu\nu}^{(+)}$, appearing in Eq. (15), is evaluated is crucial to these considerations.

(1) The *conventional wisdom* follows from a study of $T_{\mu\nu}^{(+)}$ expanded in invariant amplitudes,

$$T_{\mu\nu}^{(+)} = g_{\mu\nu}T_1 + q_\mu k_\nu T_2 + k_\mu q_\nu T_3 + \cdots,$$

so that

$$q^\mu k^\nu T_{\mu\nu}^{(+)} = q\cdot k T_1 + q^2 k^2 T_2 + (q\cdot k)^2 T_3 + \cdots.$$

Along the plane $\nu = q\cdot k = 0$, $q^\mu k^\nu T_{\mu\nu}^{(+)}$ is of $O(q^2k^2)$. One then conventionally argues that this remainder provides a correction estimated to be of $O(m_\pi^2/M^2)$ smaller than C, the σ term itself, with M some characteristic large mass taken to be roughly the nucleon mass. This estimate is based on the tree approximation or low orders of perturbation theory. This estimate also follows if one *formally* expands the scattering amplitude in powers of a chiral-symmetry-breaking parameter ϵ.[19] Note, however, that this formal expansion has been questioned by Li and Pagels.[20] Similar considerations may be applied to the three-point function studied in Appendix C.

(2) In contradistinction to the above, *our theory* treats $q^\mu k^\nu T_{\mu\nu}^{(+)}$ *dynamically* without appealing to the conventional intuitive arguments. An important step in our analysis is the separation of the one-particle reducible part of the σ field from $T_{\mu\nu}^{(+)}$, as described in Sec. III. There is no commitment to the existence of a 0^+ resonance here. This step merely reflects structure present in the scattering amplitude due to the σ field generated by the commutator Eq. (1b). As Eqs. (29)–(31) reveal, our final result follows from the t dependence of $R(t) \equiv [F_0(t)-1]\Delta_0(t)^{-1}$. If $R(t)$ is evaluated in the tree approximation, as in Appendix A, there is no departure from the standard conclusion.

Our evaluation of $R(t)$ follows from the unitarization of the $\pi\pi$ scattering amplitude satisfying current-algebra constraints. This treatment of $\pi\pi$ scattering does not involve an expansion in powers of the chiral-symmetry-breaking parameter ϵ, but is in fact valid to all orders in ϵ. It is significant that Li and Pagels[20] have argued that scattering amplitudes are not analytic in ϵ, due to the appearance of $\epsilon^2\ln\epsilon$ terms, etc., so that the arguments of Dashen and Weinstein need not be applicable to our model of $\pi\pi$ scattering.

Since our results do not agree with the conventional arguments, it follows *a posteriori* that the usual intuitive estimates of the remainder term are not valid in our theory, as shown explicitly in Sec. IV B and Appendix C. Thus the unitarization of the $\pi\pi$ scattering amplitude leads to a modification of the naive power-counting argument due to dynamical considerations. We conjecture that this departure from the usual power counting comes from the summation of terms of order $\epsilon^2(\ln\epsilon)^n$. Thus $q^\mu k^\nu T_{\mu\nu}^{(+)}$ is $O(\epsilon^2)$ in the tree approximation, $O(\epsilon^2\ln\epsilon)$ in the one-loop approximation, ..., $O(\epsilon^2\ln^n\epsilon)$ in the n-loop approximation, etc. Hence our conjecture is that the unitarization of the $\pi\pi$ scattering amplitude, involving a sum over all loops, leads to a remainder which is actually of $O(\epsilon)$ rather than the expected $O(\epsilon^2)$, with the summation of logarithms changing naive power counting. It would be interesting to demonstrate this explicitly.

(3) Our considerations do *not* spoil the usual successes of PCAC. Writing the S-wave $\pi\pi$ scattering amplitude as $T(t)=R(t)F(t)$, as in Eq. (43), demonstrates that our correction derives from the t dependence of the numerator, rather than from enhancements by the denominator function. This implies that our correction is only expected to be relevant in applications involving a 2π 0^+ intermediate state, which does not alter the Goldberger-Treiman relation, etc. In fact the calculation of the σ term is the only practical example we have found.

(4) We emphasize that our correction factor of 3 is not universal. More precisely, the correction is $[3+O(m_\pi^2/m_\sigma^2)]$, as can be seen in Eq. (34). If m_σ is small, our entire theory fails. If we adapt a more accurate calculation of $R(t)$ by Jen,[17] one finds the correction factor of 2.7.

(5) The validity of the approximation $m_\pi^2\Delta_\pi(0)=1$ has been questioned. In fact, this approximation is not essential to our method. If care is exercised to include $m_\pi^2\Delta_\pi(0)$, where appropriate, throughout our treatment of both $\pi\pi$ and πN scattering, one finds that $[F_0(t)-1]$ is replaced systematically by $[F_0(t)-m_\pi^2\Delta_\pi(0)]$, so that $R(t)=[F_0(t)-m_\pi^2\Delta_\pi(0)]\Delta_0(t)^{-1}$. Since Eqs. (29) and (30) are similarly modified, there are no changes in our final conclusion. Further, if one wishes to give *serious* consideration to the one-loop approximation to the σ model, then one should modify Eq. (40) accordingly. We do not pursue this last point further.

ACKNOWLEDGMENTS

This work was undertaken as a result of numerous conversations with Professor Sidney Coleman on this subject. I have also benefited from conversations with Professor G. Höhler, Professor B. Renner, Professor J. Wess, and Professor B. Zumino, and Dr. L. Maiani and Dr. J. Ellis. Thanks are also due Professor Roger Dashen for his comments on our work, and Professor Lowell Brown for a stimulating correspondence. In addition, Peter Ford has aided us with some numerical work.

I wish to express my appreciation to Professor Bruno Zumino and the theory group of CERN for their generous hospitality during the summer of 1971, where this work was completed.

APPENDIX A: πN SCATTERING IN THE σ MODEL

We restrict the discussion to the isotopin-even πN scattering amplitude evaluated in the tree approximation.[14] There are three diagrams which contribute, the direct and exchange nucleon poles, and the t-channel σ-exchange pole. In this model the Lagrangian does not contain a nucleon mass term, but the nucleon acquires a mass when the σ field is "translated" to remove its nonvanishing vacuum expectation value.

The isotopic-even πN scattering amplitude in the tree approximation is

$$T^{(+)}=A^{(+)}+\gamma\cdot QB^{(+)},$$
$$A^{(+)}=\frac{-2g\lambda^2F}{t-m_\sigma^{\,2}}, \tag{A1}$$
$$B^{(+)}=\frac{-g^2}{m}\left(\frac{\nu}{\nu^2-\nu_B^{\,2}}\right),$$

where

g = the meson-nucleon coupling constant ,

λ = the meson-meson coupling constant ,

$F=\langle\sigma\rangle$,

$m=gF$, the nucleon mass , (A2)

$m_\sigma^{\,2}=\mu^2+3\lambda^2F^2$,

$m_\pi^{\,2}=\mu^2+\lambda^2F^2$,

μ = the O(4)-invariant meson mass ,

$2\lambda^2F^2=m_\sigma^{\,2}-m_\pi^{\,2}$.

Therefore,

$$M^{(+)}=A^{(+)}+\nu B^{(+)}$$
$$=\frac{-2g\lambda^2F}{t-m_\sigma^{\,2}}+\frac{g^2}{m}\left(\frac{\nu^2}{\nu_B^{\,2}-\nu^2}\right). \tag{A3}$$

Equation (A3) may be rewritten using (A2) as follows:

$$M^{(+)}=gF^{-1}\left[\left(\frac{m_\sigma^{\,2}-m_\pi^{\,2}}{m_\sigma^{\,2}-t}\right)-1\right]+\frac{g^2}{m}\left(\frac{\nu_B^{\,2}}{\nu_B^{\,2}-\nu^2}\right)$$
$$=gF^{-1}[F_0(t)-1]+\frac{g^2}{m}\left(\frac{\nu_B^{\,2}}{\nu_B^{\,2}-\nu^2}\right), \tag{A4}$$

where $F_0(q,k)=F_0(t)$ in the tree approximation.

Equation (A4) shows that the representation (29) is satisfied with the background term, $\alpha\nu_B+\beta\nu^2=q^\mu k^\nu\tilde{T}^{(+)}_{\mu\nu}$, identically zero in the tree approximation. One can understand (A2), (A4), and (29), in the following way: Equation (A3) is the πN scattering amplitude for the σ model, described by *pseudoscalar* coupling for the πNN and $\pi\pi\sigma$ vertices. On the other hand (A4) is the same scattering amplitude given by Feynman rules appropriate to *pseudovector* coupling for the same vertices, which makes the Adler condition *prima facie* evident. Thus one can understand Eq. (29) as a representation for πN scattering appropriate to the pseudovector couplings of pions, which is what one would expect for an amplitude which satisfies current-algebra constraints explicitly. In particular, $[F_0(q,k)-1]$ is the pseudovector $\pi\pi\sigma$ vertex, as is consistent with the PCAC estimate $F_0(0)=1+O(m_\pi^{\,2}/m_\sigma^{\,2})$.

APPENDIX B: $\pi\pi$ SCATTERING AND UNITARITY

Let us review our method for unitarizing the $\pi\pi$ scattering amplitude.[9,13,17] Consider the amplitude for the scattering process $\pi_a(q_1)+\pi_b(q_2)\to\pi_c(q_3)+\pi_d(q_4)$. Proceeding by analogy with Eqs. (1)–(4) and Eqs. (20)–(27) of the text, one finds for the mass-shell $\pi\pi$ amplitude the representation

$$\begin{aligned}T(s,t,u)^{abcd}&=q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}\tilde{T}(q_1,q_2;q_3,q_4)^{\mu\nu\lambda\sigma}-\tfrac{1}{3}F_\pi^{\,-2}[5P_0+2P_2]^{ab,cd}\\&\quad+F_\pi^{\,-2}(-\tfrac{1}{3}+K_1)\left[[P_1]^{ab,cd}(u-t)+\binom{s}{b}\leftrightarrow\binom{t}{c}+\binom{s}{a}\leftrightarrow\binom{u}{c}\right]\\&\quad+\{[P_0]^{ab,cd}[F_0(s)-1]^2\Delta_0(s)^{-1}+\text{crossed terms}\}\\&\quad-2\{[P_1]^{ab,cd}(u-t)[F_1(s)-K_1]^2\Delta_1(s)^{-1}+\text{crossed terms}\},\end{aligned} \tag{B1}$$

where the $[P_I]^{ab,cd}$ are s-channel projection operators for isospin I; $F_1(s)$ is the electromagnetic form factor of the pions; $\Delta_1(s)$ is the coefficient of $g_{\mu\nu}$ in the two-point function of isovector currents, $C_V=\Delta_1(0)<0$; $K_1=1+\frac{1}{2}C_VF_\pi^{\,-2}$; and we have set $m_\pi=1$ in this appendix. Equation (B1), which is an exact consequence of Ward identities, is a special case of Eq. (2.1) of Ref. 9, restricted to the $(\frac{1}{2},\frac{1}{2})$ representation of chiral SU(2)×SU(2) to which $\partial\cdot A$ and σ belong. To simplify our notation, define

$$t_c(s,t)=q_{1\mu}q_{2\nu}q_{3\lambda}q_{4\sigma}\tilde{T}^{\mu\nu\lambda\sigma}, \tag{B2}$$

which is the background term. Once again, as in Appendix A, we are able to identify $[F_0(s)-1]$ as the $\pi\pi\sigma$ vertex appropriate to pseudovector coupling.

Consider the decomposition of $T(s,t,u)$ into partial waves. The S- and P-wave partial-wave amplitudes have the general form

$$T(s)=h(s)+F(s)^2\Delta(s)^{-1}, \tag{B3}$$

where

$$h(s)=R(s)-2K\Gamma(s)+K^2\Delta(s)^{-1}, \tag{B4}$$

with

$$\Gamma(s)\equiv F(s)\Delta(s)^{-1}$$

and

$$\begin{aligned} K&=1 \quad \text{for } S \text{ waves}\\ &=K_1 \quad \text{for } P \text{ waves}\,. \end{aligned}$$

[Do not confuse this K with the one that appears in Eq. (36) of the text.] We have defined

$$\begin{aligned} R_0(s)=(S\text{-wave projection})\Big\{&t_c(s,t)-\tfrac{1}{3}F_\pi^{-2}[5P_0+2P_2]^{ab,cd}+F_\pi^{-2}(-\tfrac{1}{3}+K_1)\Big[[P_1]^{ab,cd}(u-t)+\binom{s}{b}\leftrightarrow\binom{t}{c}+\binom{s}{a}\leftrightarrow\binom{u}{c}\Big]\\ &+\Big[[P_0]^{ac,bd}[F_0(t)-1]^2\Delta_0(t)^{-1}+\binom{t}{c}\leftrightarrow\binom{u}{b}\Big]-2\Big[[P_1]^{ac,bd}(u-s)[F_1(t)-K_1]^2\Delta_1(t)^{-1}+\binom{t}{c}\leftrightarrow\binom{u}{b}\Big]\Big\}, \end{aligned} \tag{B5}$$

and $R_1(s)$ is defined similarly for P waves. [The $R(s)$ that appears in the text is the function $R_0(s)$ defined here.] It is clear that both $R_0(s)$ and $R_1(s)$ have right- and left-hand cuts although the right-hand cuts are due entirely to $t_c(s,t)$.

The two-particle unitarity equations are

$$\mathrm{Im}\,T(s)=\rho(s)\,|T(s)|^2, \tag{B6}$$

$$\mathrm{Im}\,F(s)=\rho(s)T^*(s)F(s), \tag{B7}$$

and

$$\mathrm{Im}\,\Delta(s)=\rho(s)\,|F(s)|^2, \tag{B8}$$

with

$$\begin{aligned} \rho(s)&=-\frac{1}{32\pi}\left(\frac{s-4}{s}\right)^{1/2}\theta(s-4) \quad \text{for } S \text{ waves}\\ &=-\frac{1}{48\pi}\frac{(s-4)^{3/2}}{\sqrt{s}}\theta(s-4) \quad \text{for } P \text{ waves}\,. \end{aligned} \tag{B9}$$

It is straightforward to show that

$$\mathrm{Im}\,h(s)=\rho(s)\,|h(s)|^2, \tag{B10}$$

$$\mathrm{Im}\,\Gamma(s)=\rho(s)h^*(s)\Gamma(s), \tag{B11}$$

and

$$\mathrm{Im}\,t_c(s)=|h(s)+K\Gamma(s)|^2, \tag{B12}$$

where $t_c(s)$ is the appropriate S- or P-wave projection of $t_c(s,t)$ appearing in (B5).

One argues that $t_c(s)$ is small in absolute magnitude, slowly varying, and predominately real for energies sufficiently close to threshold, for $4\leq s\leq s_0$, say. This behavior gives support to the hypothesis that

$$\mathrm{Im}\,t_c(s)=0+O(\epsilon^2) \quad \text{for} \quad 4\leq s\leq s_0, \tag{B13}$$

which leads to a basic theorem, whose proof depends on (B3)–(B12).

Theorem. If $\mathrm{Im}\,t_c(s)=0+O(\epsilon^2)$ for $4\leq s\leq s_0$, then

$$h(s)=-\Gamma(s)+O(\epsilon), \tag{B14}$$

$$T(s)=F(s)[F(s)-K]\Delta(s)^{-1}+O(\epsilon), \tag{B15}$$

$$R(s)=T(s)F(s)^{-1}+O(\epsilon), \tag{B16}$$

and

$$R(s)=\text{real to } O(\epsilon^2) \tag{B17}$$

with Eqs. (B14)–(B17) valid for $4\leq s\leq s_0$. Combining Eq. (B15) with (B16), one has

$$R(s)=[F(s)-K]\Delta(s)^{-1}+O(\epsilon) \quad \text{for} \quad 4\leq s\leq s_0\,. \tag{B18}$$

Since $K=1$ for S waves, this reduces to Eq. (32) of the text.

It is to be observed that $R(s)$ is specified in two distinct ways, which implies the self-consistency requirement that Eq. (B5) be compatible with (B18). This self-consistency requirement, applied to both S and P waves, leads to two independent equations which couple $R_0(s)$ and $R_1(s)$ and provide a means for their determination. To carry out this program in detail requires a knowledge of $[F(t)-K]\Delta(t)^{-1}$ for $t<0$, which is again given by $R(t)$ if it is postulated that Eq. (B18) also holds (to a good approximation) for small, negative values of t.[9,17] For sufficiently small values of s, one may expand

$$R_0(s) = -(d_0+f_0 s), \tag{B19}$$

and similarly for $R_1(s)$. The self-consistency equations can also be expanded in a Taylor series about threshold, with a neglect of quadratic and higher-order terms in $(s-4)$. The coupled self-consistency equations can then be solved explicitly for $R_0(s)$ and $R_1(s)$. Adopting the highly plausible estimates

$$F_0(0) = 1 + O(m_\pi{}^2/m_\sigma{}^2) \text{ from PCAC},$$

$$\frac{dF_0(0)}{dt} = O(m_\pi{}^2/m_\sigma{}^2),$$

and

$$\frac{dF_1(0)}{dt} = O(m_\pi{}^2/m_\rho{}^2), \tag{B20}$$

one finds

$$R_0(s) = -F_\pi{}^{-2}\left[\left(1-\frac{2s}{m_\pi{}^2}\right)+O(\epsilon)\right], \tag{B21}$$

which is Eq. (33) of the text.

APPENDIX C: OFF-SHELL BEHAVIOR OF $\pi\pi\sigma$ VERTEX

In this appendix we correlate the t dependence of the on-shell $\pi\pi\sigma$ form factor $F_0(t)$ with the q^2 and k^2 dependence of the off-shell vertex $F_0(q,k)$. We begin with the Ward identity for the $\tilde{A}\tilde{A}\sigma$ vertex (23), simplified by the approximation $(m_\pi{}^2-q^2)\Delta_\pi(q)=1$, which is valid for small q^2. Then

$$-m_\pi{}^2F_\pi{}^2[F_0(q,k)-1] = q^\mu k^\nu F_{\mu\nu}(q,k) + \left(1-\frac{q^2}{m_\pi{}^2}-\frac{k^2}{m_\pi{}^2}\right)\Delta_0(t). \tag{C1}$$

On the pion mass shell this becomes

$$-m_\pi{}^2F_\pi{}^2[F_0(t)-1] = q^\mu k^\nu F_{\mu\nu}(q,k)\big|_{q^2=k^2=m_\pi{}^2} - \Delta_0(t). \tag{C2}$$

However, from our treatment of $\pi\pi$ unitarity, we know that

$$m_\pi{}^2F_\pi{}^2[F_0(t)-1] = -\left(1-\frac{2t}{m_\pi{}^2}\right)\Delta_0(t), \tag{C3}$$

which imposes a restriction on Eq. (C2). We now assume that $\Delta_0(t)^{-1}F_{\mu\nu}(q,k)$ is the most general quadratic function of momenta consistent with the Ward identity and Eq. (C3). It is then a straightforward exercise to show that

$$F_{\mu\nu}(q,k) = \{g_{\mu\nu}[A+B(q^2+k^2)+D(q\cdot k)] - 2m_\pi{}^{-4}q_\mu k_\nu - m_\pi{}^{-2}[2m_\pi{}^{-2}+(\tfrac{1}{2}A+B)](q_\mu q_\nu + k_\mu k_\nu) - Dk_\mu q_\nu\}\Delta_0(t) \tag{C4}$$

satisfies these criteria, with A, B, and D undetermined constants. When Eqs. (C1) and (C4) are combined, this gives

$$-m_\pi{}^2F_\pi{}^2[F_0(q,k)-1]\Delta_0(t)^{-1} = \left[\left(1-\frac{q^2}{m_\pi{}^2}-\frac{k^2}{m_\pi{}^2}\right)+m_\pi{}^{-4}[-t(q^2+k^2)+(q^4+k^4)+\tfrac{1}{4}\delta(t-q^2-k^2)(2-q^2-k^2)]\right], \tag{C5}$$

which is Eq. (44) of the text.

If we had assumed that $\Delta_0(t)^{-1}F_{\mu\nu}(q,k)=Ag_{\mu\nu}$, as suggested by the tree approximation, the mass-shell constraint (C3) would have forced $F_{\mu\nu}$ to have a kinematical singularity of the form $(t-m_\pi{}^2)/(t-2m_\pi{}^2)$, which we consider unacceptable. The avoidance of such kinematical singularities, combined with the t dependence given by Eq. (C3), dictates the presence of terms of $O(q^4/m_\pi{}^4)$ in $[F_0(q,k)-1]\Delta_0(t)^{-1}$.

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*Work supported in part by the U. S. Atomic Energy Commission under Contract No. AT(30-1) 3179.

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[1]See, e.g., M. Gell-Mann, in *Particle Physics*, edited by W. A. Simmons and S. F. Tuan (Western Periodicals, Los Angeles, 1970).

[2]F. von Hippel and J. Kim, Phys. Rev. D 1, 151 (1970).

[3]T. P. Cheng and R. Dashen, Phys. Rev. Letters 26, 594 (1971).

[4]G. Höhler, H. P. Jacob, and R. Strauss, Phys. Letters 35B, 445 (1971); see also M. Ericson and M. Rho, *ibid.* 36B, 93 (1971), and B. Renner, Hamburg report (unpublished).

[5]H. Fritzsch and M. Gell-Mann, *Tracts in Mathematics and the Natural Sciences*, edited by M. Dal Cin, G. J. Iverson, and A. Perlmutter (Gordon and Breach, New York, 1971), Vol. 2.

[6]This is the approach of Ref. 3 and Höhler *et al.*, Ref. 4.

[7]The on-shell amplitude, evaluated at $\nu=0$, $t=2$, can be expressed in terms of the S-wave scattering length and effective range, and P-wave scattering length; G. Altarelli, N. Cabibbo, and L. Maiani, Phys. Letters 35B, 415 (1971); S. Coleman and H. Schnitzer (unpublished). See also G. Höhler, Karlsruhe report (unpublished).

[8]S. Glashow and S. Weinberg, Phys. Rev. Letters 20, 224 (1968); M. Gell-Mann, R. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).
[9]H. J. Schnitzer, Phys. Rev. Letters 24, 1384 (1970); Phys. Rev. D 2, 1621 (1970).
[10]S. Weinberg, in *Lectures on Elementary Particles and Quantum Field Theory,* 1970 Brandeis University Summer Institute in Theoretical Physics, edited by S. Deser, M. Grisaru, and H. Pendleton (MIT Press, Cambridge, Mass., 1971), Vol. 1.
[11]I. S. Gerstein and H. J. Schnitzer, Phys. Rev. 170, 1638 (1968).
[12]H. J. Schnitzer, Phys. Rev. 158, 1471 (1967); L. S. Brown, W. J. Pardee, and R. D. Peccei, Phys. Rev. D 4, 2801 (1971).
[13]P. Ford and H. J. Schnitzer, Phys. Rev. D 4, 2402 (1971).
[14]M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).
[15]B. W. Lee, Nucl. Phys. B9, 649 (1969).
[16]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).
[17]C.-I. Jen, Phys. Rev. D 4, 1246 (1971). See also G. Ecker, Nucl. Phys. (to be published).
[18]K. Raman, Phys. Rev. 164, 1736 (1968); R. D. Peccei, *ibid.* 176, 2515 (1969); D. Wray, Ann. Phys. (N.Y.) 51, 162 (1969); G. Höhler and P. Stichel, Z. Physik 245, 387 (1971); R. Brandt and G. Preparata, New York University report (unpublished).
[19]R. Dashen and M. Weinstein, Phys. Rev. 188, 2330 (1969).
[20]L.-F. Li and H. Pagels, Phys. Rev. Letters 26, 1204 (1971).

PHYSICAL REVIEW D VOLUME 5, NUMBER 6 15 MARCH 1972

Pathologies of Noncanonical Dimensions

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(Received 28 June 1971)

Noninteger-power and logarithmic light-cone singularities for forward matrix elements of current commutators are examined in connection with the scaling properties of the Compton amplitude in the Bjorken limit and with the asymptotic properties in the Regge limit. Properties of compact support for the scaling functions are lost for noninteger-power singularities, and fixed (simple or multiple) Regge poles occur with complex residues. The residues of the fixed Regge poles are related to integrals over the scaling functions.

I. INTRODUCTION

In this note we shall examine some general properties of the forward virtual Compton amplitude for noncanonical light-cone singularities of the current commutator.[1,2] Scaling of deep-inelastic scattering is known to be connected to canonical behavior of the leading light-cone singularity of the current correlation function.[3,4] In examining noncanonical situations one is interested in finding out whether any pathological features emerge, which could be interpreted as arguments in favor of canonical singularities. We shall see that although, as expected, no definite contradictions arise, noncanonical cases show up rather peculiar features. A first distinguishing result is found when analyzing the support of the scaling functions: Whereas in the canonical case the support is between -1 and $+1$ in the scaling variable ω, for noninteger dimensions the support stretches from $+1$ down to $-\infty$. Another interesting feature obtains by examining the implications of the Regge limit. Such a limit is generally not derivable from the light cone alone, as it involves the whole interior of the light cone. Nevertheless one can conclude that, barring accidental compensations (see Sec. IV), fixed Regge poles (which are multiple Regge poles for logarithmic light-cone singularities) occur, of the type found in weak amplitudes.[5,6] Their residues are in general complex numbers[7] and can be expressed as dispersive integrals over the structure functions. The ratio of their real to imaginary part depends on the power of the singularity near the light cone. For canonical dimensions the imaginary part vanishes. Also, the residue of the fixed pole can be related to the physical