in the study of dual-resonance models. It is particularly useful for studying the parity content of possible self-bootstrapping configurations in the evasive case (i.e., when all trajectory functions are different). The price one pays for this simplicity in parity is that kinematic constraints between different amplitudes must be enforced "by hand." However, a number of conclusions can be drawn about any particular model even before these kinematic constraints are enforced.

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Local Current-Algebra Sum Rules in the Rest Frame*

Michael S. Chanowitz†‡
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
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We consider finite-moment sum rules in the rest frame. We show (a) that the usual derivations are incorrect for kinematical reasons, and (b) that the disconnected contributions are large. The largeness of the disconnected contributions is an advantage since they alone can be measured. We derive three pion sum rules, one of which implies the first spectral-function sum rule in the chiral limit; the derivation leads naturally to a proposal to test the spectral-function sum rule experimentally. We also derive five nucleon sum rules, from which we calculate the $I=1$ charge radius of the nucleon and the slopes at threshold for $\sigma(e^+e^-\to N\bar{N})_{I=1}$ and $\sigma(e^+e^-\to N\bar{\Delta})_{I=1}$.

I. INTRODUCTION

After nearly ten years of current algebra, we still know very little about local current commutators. The chiral charge algebra is (at least in Lagrangian models) not affected by a large class of symmetry-breaking interactions, but the local commutators may be quite dependent on the details of hadron interactions. As a result, it is difficult even to conjecture plausible forms for the local commutators. A second obstacle is that the infinite-momentum-frame method, which has been used successfully to study the charge algebra, can only be applied to a restricted class of local commutators. Commutators involving spatial current components are the most likely to depend on the detailed nature of hadron interactions, and they are also the least accessible to investigation by the infinite-momentum-frame method.[1]

Local commutators are more difficult to study than global commutators, but there is potentially much more to be learned from the local commutators. The equal-time commutators of axial charges have been used to calculate axial-vector form factors at zero momentum transfer.[2] From the local commutators we could potentially learn about the full momentum dependence of weak and electromagnetic form factors.

Here we study local commutators by means of rest-frame, finite-moment sum rules, which apply to all species of local current commutators. Such sum rules were first used to show that results previously obtained from SU(6) symmetry could also be obtained from current algebra.[3] More recently they have been used to attempt to distinguish between quark-model and field-algebra commutation relations.[4,5]

In addition to the particular results to be presented in this paper, there are two points which we wish to establish. The first is purely kinematical:

The usual method of deriving finite-moment sum rules from commutators of multipole operators is in general ambiguous and incorrect. The usual manipulations with multipole commutators are only formal, since they involve meaningless quantities such as $\delta(\vec{0})$. When a well-defined procedure is followed, the orbital angular momentum enters in ways not appreciated by the naive formal argument. Several well-defined sum rules may correspond to a single sum rule derived by the formal method, and the formal sum rule is not in general correct. A simple alternative method for deriving finite-moment sum rules is presented, which is used throughout this paper.

The second point has to do with disconnected contributions, in particular, the pair and semiconnected contributions (the terms are defined in Sec. III). Perhaps because they do not contribute to sum rules derived by the now familiar infinite-momentum-frame method, there has been a tendency to overlook their presence in rest-frame sum rules.[6] In the examples considered below, we show that the disconnected contributions are no less important than the familiar connected contributions. Furthermore, this is an advantage since it increases the usefulness of the sum rules. The connected contributions are measurable in principle but not in practice (in general one would have to observe the decay of a many-hadron state into a nucleon and a lepton-antilepton pair). Only the pair contributions, which can be measured in electron-positron colliding-beam experiments, are directly accessible to twentieth century experimental physics.[7] For this reason, it is fortunate that the pair diagrams are enhanced by kinematical factors in the sum rules involving spatial components (see Sec. V).

The plan of the paper is as follows.

In Sec. II we criticize the formal use of multipole operators to derive finite-moment sum rules and propose a simple alternative method.

In Sec. III we work through a purely theoretical exercise, a sum rule to which *only* the semiconnected terms contribute. We also introduce the techniques to be used in Secs. IV and V.

In Sec. IV we derive three pion sum rules from the three species of local SU(2) current commutators (time-time, time-space, and space-space). In the limit of exact $SU(2)\otimes SU(2)$ with $m_\pi = 0$, one becomes Weinberg's first spectral-function sum rule,[8] while the other two put constraints on anomalous contributions to the local commutators. The finite-moment sum rule from which the spectral-function sum rule emerges may be used to study corrections to the spectral-function sum rule due to $SU(2)\otimes SU(2)$ breaking. Furthermore, the derivation leads naturally to a proposal for experimentally testing the saturated spectral-function sum rule.[9]

In Sec. V we derive five nucleon sum rules from the three species of local SU(2) current commutators. Saturating the sum rules with the leading nucleon resonances, we calculate the nucleon isovector charge radius and the slopes at threshold for $\sigma(e^+e^- \to N\bar{N})_{I=1}$ and $\sigma(e^+e^- \to N\bar{\Delta})_{I=1}$. We predict that the cross section near threshold for production of isovector nucleon-antinucleon pairs is of the order of magnitude of a pointlike nucleon.[10]

II. AN UNEXPECTED PROPERTY OF MULTIPOLE COMMUTATORS

In this section we only discuss kinematics, but we nevertheless obtain a rather surprising result. We find that the usual derivations of finite-moment sum rules are ambiguous and incorrect. We then present a method which is simple and unambiguous, and which is used in the remainder of this paper.

We first review the usual argument, taking an example considered in Ref. 4. The SU(2) electric dipole operator is (at $t=0$)

$$E_a^i = \int d^3x\, x^i V_a^0(\vec{x}), \tag{2.1}$$

and the magnetic quadrupole operator is

$$M_a^{ij} = \int d^3x\, x^i [\vec{x}\times\vec{V}_a(\vec{x})]^j. \tag{2.2}$$

The angular momentum and parity of these operators is $J^P = 1^-$ and $J^P = 2^-$, respectively. Their equal-time commutator, deduced from the local current commutator

$$[V_a^0(x), \vec{V}_b(0)] = i\epsilon_{abc}\delta(\vec{x})\vec{V}_c(\vec{x}) + \cdots, \tag{2.3}$$

is

$$[E_a^i, M_b^{jk}] = i\epsilon_{abc}\int d^3x\, x^i x^j [\vec{x}\times\vec{V}_c(\vec{x})]^k + \cdots. \tag{2.4}$$

Extra terms in (2.4) would be due to a third-derivative Schwinger term in (2.3); only a third-derivative, operator Schwinger term could contribute to the sum rule that we will discuss.

The sum rule is derived by taking the matrix element of (2.4) between nucleons at rest and inserting a complete set of intermediate states:

$$2\left\langle N_+(\vec{0},+)\left|\int d^3x\, x^i x^j[\vec{x}\times\vec{V}_3(\vec{x})]^k\right|N_+(\vec{0},+)\right\rangle = \sum_n \langle N_+(\vec{0},+)|E_+^i|n\rangle\langle n|M_-^{jk}|N_+(\vec{0},+)\rangle - \{E_+^i \leftrightarrow M_-^{jk}\}. \tag{2.5}$$

$|N_t(\vec{p},s)\rangle$ is a nucleon state of momentum $\vec{p}$, isospin component $T_3 = t = \pm\frac{1}{2}$, and spin component

$S_3 = s = \pm\frac{1}{2}$, normalized by

$$\langle N_t(\vec{p}, s)|N_{t'}(\vec{p}', s')\rangle = (2\pi)^3 \delta_{tt'}\delta_{ss'}\delta(\vec{p}-\vec{p}'). \tag{2.6}$$

Since E and M are integrated quantities, they conserve spatial momentum, so the intermediate states $|n\rangle$ must be at rest. Then by the Wigner-Eckart theorem, the angular momentum and parity of the states $|n\rangle$ is restricted to be $J^P = \frac{3}{2}^-$. Thus, it seems plausible that the sum rule may be saturated by the "second" resonance, $N^*(1520)$.[4]

The derivation just presented may appear to be clear and correct. But we will show (a) that it is ambiguous, since it does not really imply a unique sum rule, and (b) that it is incorrect, since none of the sum rules related to (2.5) have the properties which (2.5) appears to have. In particular, the sum rules related to (2.5) have contributions from states of positive and negative parity and of angular momentum $\frac{1}{2}$ and $\frac{3}{2}$, so that saturation with just the $N^*(1520)$ is not plausible. (In subsequent sections we emphasize the importance of disconnected contributions, but here we are discussing only the connected contributions.)

The remainder of this section is in three parts. In part A we review the kinematics of matrix elements of a single multipole operator; we find that the formal arguments are correct. In part B we consider multipole commutators and find that the formal arguments are not correct in general. In part C we present a simple alternative method for deriving finite-moment sum rules.

We reassure the reader in advance that our remarks do not apply to the nonrelativistic sum rules of atomic and nuclear physics. This point is discussed briefly in part B.

A. Matrix Element of a Single Multiple Operator

Consider, for example, $\langle\sigma(\vec{0})|E^k|X(\vec{0})\rangle$, where $|\sigma(\vec{0})\rangle$ is a scalar meson at rest, $|X(\vec{0})\rangle$ is a state at rest whose angular momentum and parity are to be determined, and E^k is the electric dipole operator, (2.1). For σ, $J^P = 0^+$, and for E^k, $J^P = 1^-$, so by the Wigner-Eckart theorem, $|X\rangle$ must have $J^P = 1^-$.

The conclusion is correct, but it is necessary to be more careful in establishing it. With continuum normalization as in (2.6), $\langle\sigma(\vec{0})|E^k|X(\vec{0})\rangle$ is not a well-defined quantity, but is proportional to (a derivative of) $\delta(\vec{0})$. We obtain a well-defined quantity by considering $\int d^3p\,\langle\sigma(\vec{p})|E^k|X(\vec{0})\rangle$. Using the definition (2.1) and translation invariance, we have

$$\int \frac{d^3p}{(2\pi)^3}\langle\sigma(\vec{p})|E^k|X(\vec{0})\rangle = -i\frac{d}{dp^k}\langle\sigma(\vec{p})|V^0|X(\vec{0})\rangle\Big|_{\vec{p}=\vec{0}}. \tag{2.7}$$

The momentum dependence of $|\sigma(\vec{p})\rangle$ may be displayed by using the Lorentz boost operator,[11]

$$\begin{aligned}|\sigma(\vec{p})\rangle &= (m_\sigma/p_0)^{1/2} e^{-i\vec{K}\cdot\vec{\zeta}_\sigma}|\sigma(\vec{0})\rangle \\ &= [1 - i\vec{K}\cdot\vec{p}/m_\sigma + O(\vec{p}^{\,2})]|\sigma(\vec{0})\rangle,\end{aligned} \tag{2.8}$$

where $\vec{\zeta}_\sigma = (\vec{p}/|\vec{p}|)\sinh^{-1}(|\vec{p}|/m_\sigma)$. Then

$$\int \frac{d^3p}{(2\pi)^3}\langle\sigma(\vec{p})|E^k|X(\vec{0})\rangle = \frac{1}{m_\sigma}\langle\sigma(\vec{0})|K^k V^0|X(\vec{0})\rangle. \tag{2.9}$$

The amplitude in (2.9) is well defined. The operator K^kV^0 has $J^P = 1^-$, so by the Wigner-Eckart theorem (2.9) vanishes unless X has $J^P = 1^-$. This verifies the usual argument. From (2.7)–(2.9) we see explicitly how the orbital angular momentum enters the problem, a feature which is obscured by the usual, formal argument and which will be important in what follows.

B. Multipole Commutators

Consider the commutator (2.4) between protons:

$$\langle N_+(\vec{p}_1,+)|[M_+^{33}, E_-^3]|N_+(\vec{p}_2,+)\rangle = 2\int d^3x\,(x^3)^2\epsilon^{3ij}x^i\langle N_+(\vec{p}_1,+)|V_3^j(x)|N_+(\vec{p}_2,+)\rangle. \tag{2.10}$$

Equation (2.10) states the equality of two distributions; to obtain an equation in the ordinary sense, we may let $\vec{p}_2 = \vec{0}$ and integrate on $\vec{p}_1$. Inserting a complete set of states and using translation invariance, the result is

$$\begin{aligned}2\epsilon^{3ij}\frac{\partial^3}{\partial(p^3)^2\partial p^i}&\langle N_+(\vec{p},+)|V_3^j|N_+(\vec{0},+)\rangle\Big|_{\vec{p}=\vec{0}} \\ &= -\epsilon^{3ij}\sum_n\int d^3q\,d^3p\left[\left(\frac{\partial^2}{\partial p^3\partial p^i}\delta(\vec{p}-\vec{q})\right)\left(\frac{\partial}{\partial q^3}\delta(\vec{q})\right)\langle N_+(\vec{p},+)|V_+^j|n(\vec{q})\rangle\langle n(\vec{q})|V_-^0|N_+(\vec{0},+)\rangle\right. \\ &\qquad\left. - \left(\frac{\partial}{\partial p^3}\delta(\vec{p}-\vec{q})\right)\left(\frac{\partial^2}{\partial q^3\partial q^i}\delta(\vec{q})\right)\langle N_+(\vec{p},+)|V_-^0|n(\vec{q})\rangle\langle n(\vec{q})|V_+^j|N_+(\vec{0},+)\rangle\right].\end{aligned} \tag{2.11}$$

To evaluate (2.11) it is helpful (except when n is a single nucleon, which case must be considered separately) to use current conservation

$$\langle A|V_-^0|B\rangle = \frac{\vec{p}_A - \vec{p}_B}{E_A - E_B}\cdot\langle A|\vec{V}_-|B\rangle, \tag{2.12}$$

which is correct to lowest order in weak and electromagnetic interactions. Calculating the neutron contribution separately and using (2.12) for the other contributions, we perform the $\vec{p}$ and $\vec{q}$ integrations in (2.11), obtaining

$$2\epsilon^{3ij}\frac{\partial^3}{\partial(p^3)^2\partial p^i}\langle N_+(\vec{p},+)|V_3^j|N_+(\vec{0},+)\rangle\Big|_{\vec{p}=\vec{0}} = \epsilon^{3ij}\sum_{n\neq N}\frac{1}{m_n - m_N}\Bigg(\frac{\partial^2}{\partial p^3\partial p^i}\langle N_+(\vec{p},+)|V_+^j|n(\vec{0})\rangle\Big|_{\vec{p}=\vec{0}}\langle n(\vec{0})|V_-^3|N_+(\vec{0},+)\rangle$$
$$+\frac{\partial^2}{\partial q^3\partial q^i}\langle N_+(\vec{q},+)|V_-^3|n(\vec{q})\rangle\langle n(\vec{q})|V_+^j|N_+(\vec{0},+)\rangle\Big|_{\vec{q}=\vec{0}}\Bigg)$$
$$+\,(\text{neutron contribution}). \tag{2.13}$$

[There is indeed a neutron contribution; we will verify its presence below when we examine (2.13) in the free-field-theory case.]

Except for the neutron contribution, the first term of the commutator in (2.13) is what we expect from the formal manipulation of the multipole operators. The factor $\epsilon^{3ij}(\partial^2/\partial p^3\partial p^i)\langle N(\vec{p})|V_+^j|n(\vec{0})\rangle$ allows n to have $J^P=\frac{3}{2}^-, \frac{5}{2}^-$, and $\langle n(\vec{0})|V_-^3|N(\vec{0})\rangle$ allows $J^P=\frac{1}{2}^-, \frac{3}{2}^-$, so only states of $J^P=\frac{3}{2}^-$ can contribute. The big surprise is from the second term of the commutator, because $\partial^2/\partial q^3\partial q^i$ acts on *both* matrix elements. The contribution we expect from the usual argument is indeed present, i.e.,

$$\epsilon^{3ij}\langle N_+(\vec{0},+)|V_-^3|n(\vec{0})\rangle\frac{\partial^2}{\partial q^3\partial q^i}\langle n(\vec{q})|V_+^j|N_+(\vec{0},+)\rangle\Big|_{\vec{q}=\vec{0}}, \tag{2.14}$$

but other contributions are also present, for instance,

$$\epsilon^{3ij}\frac{\partial}{\partial q^3}\langle N_+(\vec{q},+)|V_-^3|n(\vec{q})\rangle\Big|_{\vec{q}=\vec{0}}\frac{\partial}{\partial q^i}\langle n(\vec{q})|V_+^j|N_+(\vec{0},+)\rangle\Big|_{\vec{q}=\vec{0}}. \tag{2.15}$$

Using the Lorentz boost operator (and the fact that $\langle N(\vec{0})|[V_-^3,K^3]|n(\vec{0})\rangle = i\langle N(\vec{0})|V_-^0|n(\vec{0})\rangle = 0$, which follows from $m_n\neq m_N$ and current conservation), (2.15) may be rewritten as

$$\frac{m_n - m_N}{m_n^2 m_N}\langle N_+(\vec{0},+)|V_-^3K^3|n(\vec{0})\rangle\langle n(\vec{0})|(\vec{K}\times\vec{V}_j)^3|N_+(\vec{0},+)\rangle. \tag{2.16}$$

The operator V^3K^3 has $J^P=0^+, 2^+$ while $\vec{K}\times\vec{V}$ has $J^P=1^+$, so (2.16) allows contributions of intermediate states with $J^P=\frac{1}{2}^+, \frac{3}{2}^+$, in contradiction of the formal argument. Other terms in (2.13) allow contributions from intermediate states of $J^P=\frac{1}{2}^-, \frac{5}{2}^+$ so that altogether contributions of $J^P=\frac{1}{2}^\pm, \frac{3}{2}^\pm, \frac{5}{2}^+$ are allowed (with isospin $I=\frac{3}{2}$).[12] Since the $\Delta(1236)$ and the neutron contribute, saturation with just the $N^*(1520)$ is implausible.

In a field theory of free nucleons, the sum rule (2.13) is a trivial identity, and the neutron contribution is necessary to satisfy that identity. With the normalization (2.6), the free-field isovector form factor is

$$\langle N_+(\vec{p},+)|V_3^\mu|N_+(\vec{q},+)\rangle = \tfrac{1}{2}(m^2/p_0q_0)^{1/2}\bar{u}(\vec{p},+)\gamma^\mu u(\vec{q},+),$$

and the left-hand side of (2.13) is found to be $-3i/4m^3$. On the right-hand side of (2.13) the only intermediate states are the nucleon and the nucleon pair (i.e., Z graph – see Sec. III). The neutron contribution is

$$\epsilon^{3ij}\sum_s\left[\frac{\partial}{\partial q^3}\frac{\partial^2}{\partial p^3\partial p^i}\langle N_+(\vec{p},+)|V_+^j|N_-(\vec{q},s)\rangle\Big|_{\vec{p}=\vec{q}}\langle N_-(\vec{q},s)|V_-^0|N_+(\vec{0},+)\rangle\right]_{\vec{q}=\vec{0}} = -\frac{3i}{4m^3}, \tag{2.17}$$

and the nucleon pair contribution (which appears in the cross term of the commutator) is[13]

$$\epsilon^{3ij}\sum_s\frac{\partial^2}{\partial q^3\partial q^i}\left\{\frac{\partial}{\partial p^3}\langle\Omega|V_-^0|N_+(\vec{0},+)\bar{N}_-(\vec{q}-\vec{p},s)\rangle\langle\bar{N}_-(\vec{q}-\vec{p},s)N_+(\vec{p},+)|V_+^j|\Omega\rangle\Big|_{\vec{p}=\vec{q}}\right\}_{\vec{q}=\vec{0}} = 0, \tag{2.18}$$

so that the sum rule is completely satisfied by the neutron contribution, contradicting the formal argument which claims that only $J^P=\frac{3}{2}^-$ contributions are allowed.

The above example taken from a relativistic free-field theory provides a clue to why the problem discussed here does not appear in nonrelativistic quantum mechanics. The nonzero contributions in the above example were due to derivatives acting on the relativistic factors, m/p_0. In a nonrelativistic theory of free nucleons, these factors are not present and there is no contribution from the neutron or anything else. This is also true of nonrelativistic theories of interacting particles, because a nonrelativistic state depends on its total momentum only through the trivial phase factor which specifies the motion of the center of mass. Commutators of multipole operators taken between momentum eigenstates yield interesting sum rules because the relativistic amplitudes have additional kinematical and dynamical dependence on the total momenta; the corresponding nonrelativistic sum rules embody nothing more than conservation of momentum. An interesting class of nonrelativistic sum rules (e.g., that of Thomas, Reiche, and Kuhn) is obtained by considering the expectation value between atomic or nuclear states of the commutator of a multipole operator with the time derivative of a multipole operator. We have checked that these sum rules are not affected by the problem raised here. The kinematical problem discussed in this section arises only in relativistic field theories.[14]

In (2.13), except for the neutron, all the unexpected contributions ($J^P=\frac{1}{2}^\pm, \frac{3}{2}^+, \frac{5}{2}^+$) appear in the cross term of the commutator and therefore have isospin $I=\frac{3}{2}$. This is because we proceeded from (2.10) by setting $\vec{p}_2=\vec{0}$ and integrating on $\vec{p}_1$. If instead we put $\vec{p}_1=\vec{0}$ and integrate on $\vec{p}_2$, we obtain a dynamically *different* sum rule, in which the unexpected contributions appear in the first term of the commutator and therefore have isospin values $I=\frac{1}{2}, \frac{3}{2}$. Thus, the naive sum rule (2.5) is ambiguous, in the sense that its precise form in the theory of distributions (2.10) implies more than one sum rule. This ambiguity suggests that there might be a prescription by which the naively expected sum rule (with only $J^P=\frac{3}{2}^-$ contributions) could be deduced from (2.10). We will discuss in part C why no such prescription exists.

The problem discussed here does not apply to all multipole commutators. If the currents are conserved and no higher moments appear in the commutator than the first moment of the time component or the zeroth moment of the space component, then the naive argument turns out to be valid. In particular, the Cabibbo-Radicati sum rule,[15] which is deduced from the commutator of SU(2) electric dipole operators, is not affected. But our remarks do apply to the sum rule obtained from SU(3) magnetic dipole operators.[3,5] The effect is perhaps not serious in this case, since the contributions which are correctly anticipated by the naive argument, $J^P=\frac{1}{2}^+, \frac{3}{2}^+$, are more important than those which are overlooked, $J^P=\frac{1}{2}^-, \frac{3}{2}^-$ (provided we assume resonance saturation).

We have also studied these sum rules using unit-normalized wave packets in place of the continuum-normalized momentum eigenstates treated here. (The width of the wave packets in momentum space must be small compared with the particle masses in order for any multipole selection rules to be present.) In this case, the sum rules are well defined, but they do not agree with the sum rules expected on the basis of the naive, formal argument. For instance, the unique sum rule obtained by taking (2.4) between wave packets is just a particular combination of the several sum rules which follow from the equality of distributions, (2.10). The contributions which are not anticipated by the formal argument are due to terms in which momentum derivatives act on the shape of the wave packets.

Finally, we remark that the inadequacy of the naive argument is related to its failure to appreciate the role of orbital angular momenta. The naive expectation for the quantum numbers of the states may apply to the moving states which appear at an early stage in the calculation but not to the rest-frame states which remain at the end of the calculation.[16]

C. A Simple Alternative Method

We present a simple, unambiguous method for deriving finite-moment sum rules. This method preserves the principal advantage of rest-frame, finite-moment sum rules, which is the restriction of allowed angular momentum and parity of the intermediate states.

Consider a local, equal-time ($t=0$) current commutator

$$[V_a^\mu(\vec{x}), V_b^\nu(\vec{0})]=\delta(\vec{x})C_{ab}^{\mu\nu}(\vec{x})+\cdots . \tag{2.19}$$

The three-dimensional Fourier transform is

$$\int d^3x\, e^{-i\vec{q}\cdot\vec{x}}[V_a^\mu(\vec{x}), V_b^\nu(\vec{0})]=C_{ab}^{\mu\nu}(\vec{0})+\cdots . \tag{2.20}$$

Take (2.20) between states of momentum $\vec{p}_1$ and $\vec{p}_2$, insert a complete set of states, and use translation invariance to evaluate the integral on the total three-momentum of the intermediate states. The resulting sum rule exhibits the most general three-momentum dependence:

$$\langle \vec{p}_1|C^{\mu\nu}_{ab}|\vec{p}_2\rangle+\cdots=\sum_n[\langle\vec{p}_1|V^\mu_a|n(\vec{p}_1+\vec{q})\rangle\langle n(\vec{p}_1+\vec{q})|V^\nu_b|\vec{p}_2\rangle-\langle\vec{p}_1|V^\nu_b|n(\vec{p}_2-\vec{q})\rangle\langle n(\vec{p}_2-\vec{q})|V^\mu_a|\vec{p}_2\rangle]. \tag{2.21}$$

Finite-moment, rest-frame sum rules are obtained from (2.21) by differentiating with respect to $\vec{p}_1$, $\vec{p}_2$, and $\vec{q}$ at the point $\vec{p}_1=\vec{p}_2=\vec{q}=\vec{0}$.

This simple method may be used to retrieve the naive multipole commutator sum rules when the latter are correct. For instance, the sum rule obtained from the commutator of two electric dipole operators (of conserved currents) can be derived (cf. Secs. IV and V) by choosing $\vec{p}\equiv\vec{p}_1=-\vec{q}=-\vec{p}_2$, $\mu=\nu=0$, and taking the second derivative of (2.21) with respect to p^i at $\vec{p}=\vec{0}$. [There are also other ways to obtain this sum rule from (2.21).]

Examining (2.21), we can see why it is in general not possible to derive sum rules which fulfill the expectations of the naive argument following (2.5). For instance, from the first term of (2.21) we can obtain the $J^P=\frac{3}{2}^-$ contributions expected from (2.5) by setting $\vec{q}+\vec{p}_1=\vec{0}$ and calculating

$$\epsilon^{klm}\frac{\partial}{\partial p_1^i}\frac{\partial^2}{\partial p_2^j\partial p_2^l}\{\langle N_+(\vec{p}_1,+)|V^0_+|n(\vec{0})\rangle\langle n(\vec{0})|V^m_-|N_+(\vec{p}_2,+)\rangle\}\Big|_{\vec{p}_1=\vec{p}_2=0}. \tag{2.22}$$

But when this prescription is applied to the cross term in (2.21),

$$-\epsilon^{klm}\frac{\partial}{\partial p_1^i}\frac{\partial^2}{\partial p_2^j\partial p_2^l}\{\langle N_+(\vec{p}_1,+)|V^m_-|n(\vec{p}_2-\vec{q})\rangle\langle n(\vec{p}_2-\vec{q})|V^0_+|N_+(\vec{p}_2,+)\rangle\}\Big|_{\vec{p}_1=\vec{p}_2=\vec{q}=\vec{0}}, \tag{2.23}$$

we obtain unwanted, positive-parity contributions. For instance, when $\partial^2/\partial p_2^j\partial p_2^l$ acts on the momentum dependence of the proton, $|N_+(\vec{p}_2,+)\rangle$, and for $i=j=k=3$, we find by using the Lorentz boost operators and the Wigner-Eckart theorem that the intermediate states have $J^P=\frac{3}{2}^+$.

Generally, we could derive the naively expected sum rules if we could find two variables $\vec{s}_1$ and $\vec{s}_2$, depending linearly on $\vec{p}_1$, $\vec{p}_2$, and $\vec{q}$, such that (i) $\langle\vec{p}_1|V^\mu_a|n(\vec{p}_1+\vec{q})\rangle$ and $\langle n(\vec{p}_2-\vec{q})|V^\mu_a|\vec{p}_2\rangle$ depend on $\vec{s}_1$ but not on $\vec{s}_2$, and such that (ii) $\langle n(\vec{p}_1+\vec{q})|V^\nu_b|\vec{p}_2\rangle$ and $\langle\vec{p}_1|V^\nu_b|n(\vec{p}_2-\vec{q})\rangle$ depend on $\vec{s}_2$ but not on $\vec{s}_1$. Then by taking appropriate derivatives with respect to $\vec{s}_1$ and $\vec{s}_2$, we could obtain only the desired contributions from both terms of the commutator. But it is clear that, except for special cases, no such variables $\vec{s}_1$ and $\vec{s}_2$ exist, since the two matrix elements in (i) depend on the same three independent variables as the matrix elements in (ii). Therefore, only in special cases, such as the ones mentioned in part B, can we construct the sum rules which we would expect on the basis of the naive formal argument.

III. AN EXAMPLE FROM THE ALGEBRA OF CHARGES

In this section we consider an example which illustrates the importance of disconnected contributions to rest-frame sum rules: *Only* the semiconnected terms contribute. We also introduce the kinematical techniques to be used in the subsequent sections, and, incidentally, we find theoretical support for Weinberg's remark[17] (in discussing K_{l4} decay) that Bose symmetry must be maintained in soft-pion limits.

The sum rule is obtained by taking the commutator of conserved axial charges between pions at rest and inserting a complete set of states,

$$2\langle\pi_+(\vec{0})|Q_3|\pi_+(\vec{0})\rangle=\sum_n[\langle\pi_+(\vec{0})|Q^5_+|n\rangle\langle n|Q^5_-|\pi_+(\vec{0})\rangle-\langle\pi_+(\vec{0})|Q^5_-|n\rangle\langle n|Q^5_+|\pi_+(\vec{0})\rangle]. \tag{3.1}$$

If the exact SU(2)⊗SU(2) symmetry were manifest in the mass spectrum, (3.1) would be trivially satisfied by the contribution of the scalar, isoscalar chiral partner of the pion. But we assume that SU(2)⊗SU(2) is spontaneously broken, so that $m_\pi=0$ and chiral multiplets do not appear in the mass spectrum. Then the pion has no chiral partner, and it is not evident how (3.1) is realized. We may wonder whether (3.1) is only a tautology in this case or whether it has physical content. This is the problem which we now proceed to solve.

To illustrate the derivation of finite-moment sum rules from local commutators, we will treat (3.1) as a "zeroth-moment" sum rule, obtained from the local equal-time ($t=0$) commutator

$$[A^0_+(\vec{x}),A^0_-(\vec{0})]=2V^0_3(\vec{x})\delta(\vec{x}). \tag{3.2}$$

We take the three-dimensional Fourier transform of (3.2) and bracket it by pions [normalized noncovariantly as in (2.6)] of equal and opposite momenta:

$$2\langle\pi_+(\vec{p})|V_3^0|\pi_+(-\vec{p})\rangle=\int d^3x\,e^{-i\vec{q}\cdot\vec{x}}\langle\pi_+(\vec{p})|[A_+^0(\vec{x}),A_-^0(\vec{0})]|\pi_+(-\vec{p})\rangle\,. \tag{3.3}$$

We insert a complete set of states, use translation invariance, and perform the integration on the total three-momentum of the intermediate states, with the result

$$2\langle\pi_+(\vec{p})|V_3^0|\pi_+(-\vec{p})\rangle=\sum_n[\langle\pi_+(\vec{p})|A_+^0|n(\vec{p}+\vec{q})\rangle\langle n(\vec{p}+\vec{q})|A_-^0|\pi_+(-\vec{p})\rangle-\langle\pi_+(\vec{p})|A_-^0|n(-\vec{p}-\vec{q})\rangle\langle n(-\vec{p}-\vec{q})|A_+^0|\pi_+(-\vec{p})\rangle]\,. \tag{3.4}$$

In (3.4) the argument of the currents is $(\vec{x},t)=(\vec{0},0)$ and n represents a complete set of quantum numbers except the total three-momentum.

Next we display the disconnected contributions.[18] When $|n\rangle$ contains a positive pion,

$$|n(\vec{p})\rangle=|n'(\vec{p}-\vec{k})\pi_+(\vec{k})\rangle\,, \tag{3.5}$$

then a matrix element $\langle\pi_+|A^\mu|n\rangle$ has a connected and a disconnected part,

$$\langle\pi_+(\vec{p}_1)|A^\mu|n(\vec{p}_2)\rangle=\langle\pi_+(\vec{p}_1)|A^\mu|n(\vec{p}_2)\rangle_c+(2\pi)^3\delta(\vec{p}_1-\vec{k})\langle\Omega|A^\mu|n'(\vec{p}_2-\vec{k})\rangle\,, \tag{3.6}$$

where the subscript c denotes the connected part. When we substitute (3.6) into the sum rule (3.4), we find four classes of contributions, which are illustrated in Fig. 1. The connected terms, Fig. 1(a), come from the product of connected parts with connected parts. The "pair" (or Z-graph) terms, Fig. 1(b), occur when the intermediate state contains two or more pions; they come from the product of disconnected parts in which different pions are disconnected in the two factors. The semiconnected terms, Fig. 1(c), come from the product of connected parts with disconnected parts. Finally, the fully disconnected terms, Fig. 1(d), come from the product of disconnected parts in which the same pion is disconnected in both factors.

We choose $\vec{q}=-\vec{p}$, so the intermediate states are at rest, and write the sum rule (3.4) with the structure of the disconnected contributions displayed:

$$\begin{aligned}2\langle\pi_+(\vec{p})|V_3^0|\pi_+(-\vec{p})\rangle=\sum_n[&\langle\pi_+(\vec{p})|A_+^0|n(\vec{0})\rangle\langle n(\vec{0})|A_-^0|\pi_+(-\vec{p})\rangle+\langle\Omega|A_+^0|\pi_+(-\vec{p})n(\vec{0})\rangle\,\langle n(\vec{0})\pi_+(\vec{p})|A_-^0|\Omega\rangle\\&+\langle\Omega|A_+^0|n(-\vec{p})\rangle\langle n(-\vec{p})\pi_+(\vec{p})|A_-^0|\pi_+(-\vec{p})\rangle+\langle\pi_+(\vec{p})|A_+^0|\pi_+(-\vec{p})n(\vec{p})\rangle\langle n(\vec{p})|A_-^0|\Omega\rangle\\&+(2\pi)^3\delta(2\vec{p})\langle\Omega|A_+^0|n(\vec{0})\rangle\langle n(\vec{0})|A_-^0|\Omega\rangle]-\{A_+^0\longleftrightarrow A_-^0\}\,.\end{aligned} \tag{3.7}$$

In (3.7) and all subsequent equations, the subscript c is suppressed and all matrix elements are understood to be connected.

In (3.7) the fully disconnected contributions cancel between the two terms of the commutator. In general, these terms are related to possible c-number contributions to the commutator. In Sec. IV we will encounter a sum rule in which the fully disconnected terms provide the familiar spectral representation for the c-number Schwinger term.

We can regard (3.7) as a double power series in $\vec{p}$ and m_π^2. Equating coefficients of like powers of $\vec{p}$, we obtain finite-moment sum rules, as discussed in Sec. II. Equating coefficients of like powers of m_π^2, we obtain sum rules reflecting the (presumably small) breaking of $SU(2)\otimes SU(2)$.[19] Here we evaluate (3.7) to zeroth order in $\vec{p}$ and m_π^2, which means we are studying the charge algebra in the limit of exact $SU(2)\otimes SU(2)$.

To zeroth order in $\vec{p}$, the connected and pair contributions must have angular momentum and parity $J^P=0^+$, while the semiconnected contributions have $J^P=0^-$. Using the parity operator and with some isospin rotations, (3.7) is

$$\begin{aligned}2+O(\vec{p})=\sum_n(&|\langle\pi_+(\vec{p})|A_+^0|n(\vec{0})\rangle|^2+|\langle\Omega|A_+^0|\pi_+(\vec{p})n(\vec{0})\rangle|^2-|\langle\pi_+(\vec{p})|A_-^0|n(\vec{0})\rangle|^2-|\langle\Omega|A_-^0|\pi_+(\vec{p})n(\vec{0})\rangle|^2\\&+2\,\mathrm{Re}\{\langle\Omega|A_+^0|n(\vec{p})\rangle[\langle n(\vec{p})\pi_+(-\vec{p})|A_-^0|\pi_+(\vec{p})\rangle+\langle n(\vec{p})\pi_+(-\vec{p})|A_+^0|\pi_+(\vec{p})\rangle]\})+O(\vec{p})\,,\end{aligned} \tag{3.8}$$

where Re denotes the "real part." (To avoid ambiguity later, we must include the $\vec{p}$ dependence, though finally we will let $\vec{p}=\vec{0}$.)

Using crossing symmetry for the pion with $m_\pi=0$, $\vec{p}=\vec{0}$, the connected contributions cancel with the pair contributions. The first term of (3.8) cancels with the fourth, and the second cancels with the third. Thus, the existence of scalar mesons is irrelevant, and only the semiconnected terms remain.

The axial-vector current is conserved, so $\langle\Omega|A_+^0|n(\vec{p})\rangle=\langle\Omega|A_+^0|n(\vec{0})\rangle+O(\vec{p})$ can only contribute to zeroth order in $\vec{p}$ if $|n(\vec{0})\rangle$ is a state of zero energy. Therefore, $|n\rangle$ can only be a single-pion state. It may seem that $|n\rangle$ might be a many-pion state, but the sum on n would then include an integration on the relative pion momenta. The constraint that $|n(\vec{0})\rangle$ have zero energy means that only one point (the threshold) in the region of integration can contribute. Unless there is a singularity at this point, the contribution to the sum rule is zero. There is in fact a pion-pole singularity at the point, but its residue is proportional to the π-π scattering amplitude, which vanishes to zeroth order in m_π.[20]

Now the sum rule is reduced to

$$1+O(\vec{p})=\mathrm{Re}\{\langle\Omega|A_+^0|\pi_+(\vec{p})\rangle[\langle\pi_-(\vec{p})\pi_+(-\vec{p})|A_-^0|\pi_+(\vec{p})\rangle+\langle\pi_+(\vec{p})\pi_+(-\vec{p})|A_+^0|\pi_+(\vec{p})\rangle]\}+O(\vec{p},m_\pi). \tag{3.9}$$

The first factor is

$$\langle\Omega|A_+^0|\pi_+(\vec{p})\rangle=i\sqrt{2}\,(F_\pi/\sqrt{2m_\pi})m_\pi+O(\vec{p}). \tag{3.10}$$

To calculate the other terms, we use Adler's "massive" partial conservation of axial-vector current (PCAC) prescription, $\partial_\mu A_i^\mu=m_\pi^2F_\pi\phi_i^\pi$, and calculate to zeroth order in m_π. This procedure is equivalent to Nambu's procedure, in which $\partial_\mu A^\mu=0$ and $m_\pi=0$ (see Chap. 2 of Ref. 1 for a discussion of this point).

Using PCAC and the Lehmann-Symanzik-Zimmermann (LSZ) reduction, we have

$$\langle\pi_\pm(p)\pi_+(q)|A_\pm^0|\pi_+(k)\rangle=\mp\frac{i}{\sqrt{2}\,F_\pi m_\pi^2}\left(\frac{-i}{\sqrt{2}\,F_\pi m_\pi^2}\right)^2\frac{1}{(8k^0p^0q^0)^{1/2}}$$
$$\times\int dx\,dy\,dz\,e^{i(px+qy-kz)}K_xK_yK_z\langle T(D_\mp(x)D_-(y)D_+(z)A_\pm^0(0))\rangle_\Omega, \tag{3.11}$$

where K is the Klein-Gordon operator and $D_i\equiv\partial_\mu A_i^\mu$. We can reduce (3.11) to a known three-point function by the usual Ward-identity procedure of moving a derivative through the time-ordering, etc. However, the result would be ambiguous since it would depend on which of the three derivatives we chose. The clue to the solution is Bose symmetry: We must average the three possible Ward identities. This prescription is well defined and it is also correct [if the reader is skeptical, he may glance ahead to the conclusion, Eq. (3.14)]. The Bose-symmetric Ward identity is

$$\begin{aligned}T(\partial_\mu A_\mp^\mu(x)\partial_\nu A_-^\nu(y)\partial_\tau A_+^\tau(z)A_\pm^0(0))=\tfrac{1}{3}\{&\partial_\mu T(A_\mp^\mu(x)\partial_\nu A_-^\nu(y)\partial_\tau A_+^\tau(z)A_\pm^0(0))+\partial_\nu T(\partial_\mu A_\mp^\mu(x)A_-^\nu(y)\partial_\tau A_+^\tau(z)A_\pm^0(0))\\&+\partial_\tau T(\partial_\mu A_\mp^\mu(x)\partial_\nu A_-^\nu(y)A_+^\tau(z)A_\pm^0(0))-\delta(x_0)T([A_\mp^0(x),A_\pm^0(0)]\partial_\nu A_-^\nu(y)\partial_\tau A_+^\tau(z))\\&-\delta(y_0)T([A_-^0(y),A_\pm^0(0)]\partial_\mu A_\mp^\mu(x)\partial_\tau A_+^\tau(z))-\delta(z_0)T([A_+^0(z),A_\pm^0(0)]\partial_\mu A_\mp^\mu(x)\partial_\nu A_-^\nu(y))\}\\&+O(m_\pi^2).\end{aligned} \tag{3.12}$$

In (3.12) the "σ commutators" (commutators of currents with divergences) are omitted since they do not contribute to zeroth order in m_π.

When we substitute (3.12) into (3.11) and integrate by parts, the contributions from the first three terms of (3.12) vanish as $p,k,q\to 0$ (since no external-line insertions[1] are possible). In the remaining three terms we evaluate the commutators and reverse the LSZ procedure, with the result

$$\begin{aligned}\langle\pi_-(\vec{p})\pi_+(-\vec{p})|A_-^0|\pi_+(\vec{p})\rangle&=\frac{-i}{3F_\pi p_0^{1/2}}[\langle\pi_+(-\vec{p})|V_3^0|\pi_+(\vec{p})\rangle+\langle\pi_-(\vec{p})\pi_+(-\vec{p})|V_3^0|\Omega\rangle+O(\vec{p},m_\pi)]\\&=\frac{-i}{3F_\pi m_\pi^{1/2}}+O(\vec{p},m_\pi),\end{aligned} \tag{3.13a}$$

$$\begin{aligned}\langle\pi_+(\vec{p})\pi_+(-\vec{p})|A_+^0|\pi_+(\vec{p})\rangle&=\frac{-i}{3F_\pi p_0^{1/2}}[\langle\pi_+(\vec{p})|V_3^0|\pi_+(\vec{p})\rangle+\langle\pi_+(-\vec{p})|V_3^0|\pi_+(\vec{p})\rangle+O(\vec{p},m_\pi)]\\&=\frac{-2i}{3F_\pi m_\pi^{1/2}}+O(\vec{p},m_\pi).\end{aligned} \tag{3.13b}$$

Finally, we substitute (3.10) and (3.13) into (3.9), with the conclusion that

$$1=1. \tag{3.14}$$

The fact that only the semiconnected terms were important is of physical interest, since it dramatically illustrates the dangers of neglecting disconnected contributions to rest-frame sum rules. Incidentally, we have seen in a purely tautological context the importance of maintaining Bose symmetry while going to the chiral limit.

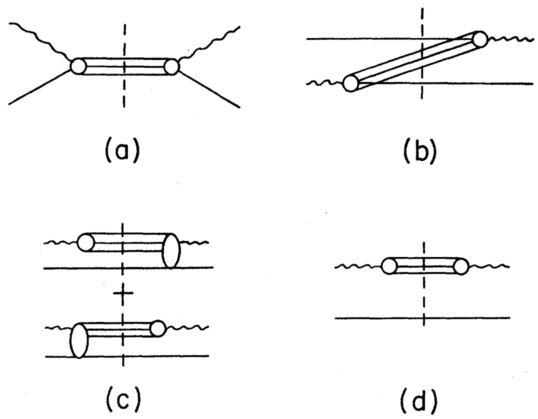

FIG. 1. Contributions to the sum rules: (a) connected, (b) pairs, (c) semiconnected, and (d) fully disconnected. (The wavy lines denote the currents.)

IV. "PHYSICAL" SUM RULES AND THE FIRST SPECTRAL-FUNCTION SUM RULE

In this section we will derive three sum rules from local commutators of SU(2) vector currents. In the limit of exact SU(2)$\otimes$SU(2) with $m_\pi=0$, one of these sum rules becomes the first spectral-function sum rule of Weinberg.[8] The derivation is interesting, not simply for the novelty of obtaining the spectral-function sum rule as a finite-moment sum rule, but for two more cogent reasons. First, the "physical" sum rule, which becomes the spectral-function sum rule in the chiral limit, may be examined in order to study the corrections to the spectral-function sum rule due to the breaking of SU(2)$\otimes$SU(2). Second, the way in which the spectral-function sum rule emerges from the physical sum rule naturally suggests an experimental test of the (saturated) spectral-function sum rule. (The same test has also been proposed by Pais and Treiman.[9])

We now proceed to derive the three physical sum rules from the local equal-time commutation relations of the SU(2) vector currents. For the sake of definiteness, we use the space-space commutators of the quark model[1] and the algebra of fields.[21] We assume that the Schwinger term is a c number (we will see that this assumption can be weakened somewhat). Then the commutation relations are

$$[V^0_+(\vec x,0),\, V^0_-(\vec 0,0)] = 2V^0_3(\vec x,0)\delta(\vec x)\,, \tag{4.1}$$

$$[V^0_+(\vec x,0),\, V^3_-(\vec 0,0)] = 2V^3_3(\vec x,0)\delta(\vec x) + iC_V\frac{\partial}{\partial x_3}\delta(\vec x)\,, \tag{4.2}$$

$$[V^3_+(\vec x,0),\, V^3_-(\vec 0,0)] = \begin{cases}2V^0_3(\vec x,0)\delta(\vec x)\,, & \text{quark} \\ 0\,, & \text{field algebra}.\end{cases} \tag{4.3}$$

Higher-derivative Schwinger terms in (4.2) (which are in fact present in the quark model[22]) would have no effect on the finite-moment sum rules we will derive here.

As in Eqs. (3.3) and (3.4), we take three-dimensional Fourier transforms, take the expectation value between pions of equal and opposite momenta, and insert complete sets of intermediate states. As in Sec. III, we choose $\vec p=-\vec q$ so that the intermediate states are at rest. The result is three sum rules, analogous to Eq. (3.7), which we will not bother to write down.

We neglect weak and electromagnetic interactions, so that the SU(2) vector currents are conserved. Then choosing $\vec p\equiv p\hat z$ along the third axis, we may explicitly display the first-order momentum dependence of matrix elements of time components, e.g.,

$$\langle\pi_+(\vec p)|V^0_+|n(\vec 0)\rangle = \frac{p}{p^0_\pi-p^0_n}\langle\pi_+(\vec p)|V^3_+|n(\vec 0)\rangle\,, \tag{4.4}$$

where p^0_n is the energy of the state $|n\rangle$ (the single-pion intermediate state requires separate treatment). Finally, we choose finite-moment sum rules by taking derivatives with respect to $\vec p$ at $\vec p=0$. For the sum rule obtained from (4.1), we evaluate $d^2/dp^2|_{\vec p=0}$; this is equivalent to taking the commutator of dipole operators. For the sum rule obtained from (4.2), we take $d/dp|_{\vec p=0}$, which is equivalent to considering the commutator of the dipole operator with the "spatial" charge. For the sum rule obtained from (4.3), we put $\vec p=\vec 0$ equivalent to considering the commutator of two "spatial" charges. (By "spatial" charge, we mean the space integral of a spatial component of the current.) Then the three "physical" sum rules, which follow from (4.1), (4.2), and (4.3), respectively, are

$$\begin{aligned}\tfrac{2}{3}\langle r_\pi^2\rangle = \sum_{n\neq\pi}\bigg(&\frac{1}{(m_n-m_\pi)^2}|\langle\pi_+(\vec 0)|V^3_+|n(\vec 0)\rangle|^2 + \frac{1}{(m_n+m_\pi)^2}|\langle\Omega|V^3_+|\pi_+(\vec 0)n(\vec 0)\rangle|^2 \\ &+\frac{1}{m_n^2}\langle\Omega|V^3_+|n(\vec 0)\rangle\langle n(\vec 0)\pi_+(\vec 0)|V^3_-|\pi_+(\vec 0)\rangle + \frac{1}{m_n^2}\langle\pi_+(\vec 0)|V^3_+|\pi_+(\vec 0)n(\vec 0)\rangle\langle n(\vec 0)|V^3_-|\Omega\rangle\bigg) - \{V^3_+ \leftrightarrow V^3_-\}\,,\end{aligned} \tag{4.5}$$

$$0 = \frac{1}{m_\pi} + \sum_{n\neq\pi}\bigg(\frac{1}{m_n-m_\pi}|\langle\pi_+(\vec 0)|V^3_+|n(\vec 0)\rangle|^2 + \frac{1}{m_n+m_\pi}|\langle\Omega|V^3_+|\pi_+(\vec 0)n(\vec 0)\rangle|^2$$

$$+\frac{1}{m_n}\langle\Omega|V_+^3|n(\vec{0})\rangle\langle n(\vec{0})\pi_+(\vec{0})|V_-^3|\pi_+(\vec{0})\rangle+\frac{1}{m_n}\langle\pi_+(\vec{0})|V_+^3|\pi_+(\vec{0})n(\vec{0})\rangle\langle n(\vec{0})|V_-^3|\Omega\rangle\Big)+\{V_+^3\longleftrightarrow V_-^3\}, \tag{4.6}$$

$$\left.\begin{matrix}\text{quark} & 2\\ \text{field algebra} & 0\end{matrix}\right\}=\sum_n\big[|\langle\pi_+(\vec{0})|V_+^3|n(\vec{0})\rangle|^2+|\langle\Omega|V_+^3|\pi_+(\vec{0})n(\vec{0})\rangle|^2$$
$$+\langle\Omega|V_+^3|n(\vec{0})\rangle\langle n(\vec{0})\pi_+(\vec{0})|V_-^3|\pi_+(\vec{0})\rangle+\langle\pi_+(\vec{0})|V_+^3|\pi_+(\vec{0})n(\vec{0})\rangle\langle n(\vec{0})|V_-^3|\Omega\rangle\big]-\{V_+^3\longleftrightarrow V_-^3\}. \tag{4.7}$$

In (4.6), r_π is the charge radius of the pion, and m_n is the energy of the state $|n(\vec{0})\rangle$.

In (4.5) and (4.7) the fully disconnected contributions from the two terms of the commutator canceled with one another. But in (4.6) they added, contributing

$$2(2\pi)^3\delta(2\vec{p})\sum_n\frac{1}{m_n}|\langle\Omega|V_+^3|n(\vec{0})\rangle|^2 \tag{4.8}$$

to the right-hand side. The c-number Schwinger term contributed a similar term to the left-hand side,

$$\langle\pi_+(\vec{p})|C_V|\pi_+(-\vec{p})\rangle=C_V(2\pi)^3\delta(2\vec{p}). \tag{4.9}$$

Since the sum rule has no other terms containing singular δ functions, (4.8) and (4.9) must cancel with one another. Indeed,

$$C_V=2\sum_n\frac{1}{m_n}|\langle\Omega|V_+^3|n(\vec{0})\rangle|^2 \tag{4.10}$$

is just a clumsy way to write the usual spectral representation for the c-number Schwinger term.[22]

We obtained (4.6) from the assumption that the Schwinger term is a c number. This assumption can be relaxed slightly without changing (4.6). In general, instead of (4.2), we could have

$$[V_a^0(\vec{x},0),V_b^3(\vec{0},0)]=if_{abc}V_c^3(\vec{x},0)\delta(\vec{x})+i\frac{\partial}{\partial x_3}[S_{ab}(\vec{x})\delta(\vec{x})]+\cdots, \tag{4.11}$$

where S_{ab} might contain an operator part, Q_{ab}, i.e.,

$$S_{ab}(\vec{x})=\delta_{ab}C_V+Q_{ab}(\vec{x}). \tag{4.12}$$

Then the operator Schwinger term would contribute $\langle\pi_+(\vec{0})|Q_{+-}|\pi_+(\vec{0})\rangle$ to the left-hand side of (4.6). However, the right-hand side of (4.6) is symmetric under the interchange of the isospin indices, $+\longleftrightarrow-$. Hence, only an isospin-symmetric operator Schwinger term, i.e., $Q_{ab}=Q_{ba}$, could contribute to the sum rule. Thus, an $I=0$ or $I=2$ operator Schwinger term would contribute to (4.6), but an $I=1$ term would not.

The isospin and G parity (I^G) of the intermediate states is restricted to $I^G=(0,1,2)^-$ for the connected and pair contributions and $I^G=1^+$ for the semiconnected contributions. The principal advantage of resorting to a finite-moment sum rule is that the angular momentum and parity (J^P) of the intermediate states is also sharply restricted. From the Wigner-Eckart theorem, we see that the connected and pair contributions have $J^P=1^+$, while the semiconnected contributions have $J^P=1^-$.

The importance of the disconnected contributions is most evident by inspection of (4.6). The connected and pair terms make a positive definite contribution to (4.6), so the semiconnected terms must supply an equally large negative contribution. If instead of pions, we had derived a fermion sum rule (as in Sec. V), then the pair terms would make a negative definite contribution to (4.6); in this case, the connected terms would be canceled by the sum of the pair and semiconnected terms. *In either case we see that the disconnected terms are at least as important as the connected ones, so that it is not a reasonable approximation to consider only the connected contributions.*

We refer to Eqs. (4.5)–(4.7) as "physical" sum rules because they are consequences only of the commutation relations (4.1)–(4.3) and may therefore be valid in the real world of massive pions and broken SU(2)⊗SU(2). But, although the sum rules are "physical," their usefulness is limited by the experimental inaccessibility of the matrix elements on the right-hand sides. For instance, to measure a connected contribution $\langle\pi_+(\vec{0})|V_3^3|n_+(\vec{0})\rangle$, it would be necessary to observe the decay of the state $|n\rangle$ into a pion and a lepton-antilepton pair. This is possible in principle but not likely to be realized in practice – especially when $|n\rangle$ is a state of several hadrons. The only exception is the pair contributions, which are accessible to electron-positron annihilation experiments. We will see that the experimental accessibility of the pair contributions offers a possible means of testing Weinberg's first spectral-function sum rule.

Before we study the sum rules in the chiral limit, we should mention a feature of (4.5)–(4.7) which seems puzzling upon first examination. Consider the semiconnected contribution when $|n\rangle$ is a ρ meson. The matrix element $\langle\rho(\vec{0})\pi(\vec{0})|V^3|\pi(\vec{0})\rangle$ then contains a ρ-meson pole with a residue given by the amplitude for $\rho\pi$ elastic scattering at threshold. In the zero-width limit, $\Gamma_\rho=0$, this pole makes a divergent contribution to the sum rules. The divergence cannot be trivially canceled by the cross term of the commutator, because different isospin channels are involved. Furthermore, the cross term appears with a negative sign in (4.5) and (4.7), but with a positive sign in (4.6). A similar difficulty also appears in the sum rule (3.7) of Sec. III.

This problem also arises in the Adler-Weisberger sum rule, and was resolved by Weisberger in his original letter.[2] The point is that there are also singularities from the connected contributions which, using unitarity, are seen to cancel the singularities from the semiconnected contributions. In Appendix A we present a demonstration of the cancellation which is more detailed and differs in some respects from Weisberger's presentation. The cancellation means that to zeroth order in Γ_ρ/m_ρ, we may neglect the pole contribution to the ρ-meson semiconnected term.

We now proceed to consider the "physical" sum rules (4.5)–(4.7) in the limit of exact $SU(2)\otimes SU(2)$ with $m_\pi=0$. We will evaluate the matrix elements on the right-hand sides according to the PCAC method, which is exact in the limit under consideration [see the remark following Eq. (3.10)]. Then for the connected and pair contributions we have

$$\langle\pi_+(\vec{0})|V_+^3|n(\vec{0})\rangle=\frac{-i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|n(\vec{0})\rangle+\lim_{p\to0}\frac{-p_\mu}{2F_\pi m_\pi^{1/2}}\int dx\,e^{ipx}\langle\Omega|T(A_-^\mu(x)V_+^3(0))|n(\vec{0})\rangle, \tag{4.13a}$$

$$\langle\pi_+(\vec{0})|V_-^3|n(\vec{0})\rangle=\lim_{p\to0}\frac{-p_\mu}{2F_\pi m_\pi^{1/2}}\int dx\,e^{ipx}\langle\Omega|T(A_-^\mu(x)V_-^3(0))|n(\vec{0})\rangle\ , \tag{4.13b}$$

$$\langle\Omega|V_-^3|\pi_+(\vec{0})n(\vec{0})\rangle=\frac{i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|n(\vec{0})\rangle+\lim_{p\to0}\frac{p_\mu}{2F_\pi m_\pi^{1/2}}\int dx\,e^{-ipx}\langle\Omega|T(A_+^\mu(x)V_-^3(0))|n(\vec{0})\rangle, \tag{4.13c}$$

$$\langle\Omega|V_+^3|\pi_+(\vec{0})n(\vec{0})\rangle=\lim_{p\to0}\frac{p_\mu}{2F_\pi m_\pi^{1/2}}\int dx\,e^{-ipx}\langle\Omega|T(A_+^\mu(x)V_+^3(0))|n(\vec{0})\rangle. \tag{4.13d}$$

(We are forced to keep track of the factors $m_\pi^{-1/2}$ because of our choice of noncovariant normalization; eventually we will put $m_\pi=0$.) The terms proportional to p_μ can only contribute "external-line insertion" pole terms in the limit $p=0$ [see Fig. 2(a)]. These insertions are possible if $|n\rangle$ is a many-particle state containing at least some particles which are not G-parity eigenstates and which have spin and isospin greater than zero.

The evaluation of the semiconnected terms requires some care because two pions appear in the matrix elements. As in Sec. III, the Ward identity must maintain the Bose symmetry of the pions. We have

$$\langle n(\vec{0})\pi_+(\vec{p}=\vec{0})|V_\pm^3|\pi_+(\vec{q}=\vec{0})\rangle=\left(\frac{-i}{\sqrt{2}F_\pi m_\pi^2}\right)^2\frac{(m_\pi^2-p^2)(m_\pi^2-q^2)}{2m_\pi}\int dx\,e^{ipx-iqy}\langle n(\vec{0})|T(\partial_\mu A_-^\mu(x)\partial_\nu A_+^\nu(y)V^3(0))|\Omega\rangle \tag{4.14}$$

and the Bose-symmetric Ward identity is

$$\begin{aligned}T(\partial_\mu A_-^\mu(x)\partial_\nu A_+^\nu(y)V_\pm^3(0))&\\=\partial_\mu\partial_\nu T(A_-^\mu(x)A_+^\nu(y)V_\pm^3(0))&+\tfrac12\delta(x_0)\delta(y_0)[A_+^0(y),[A_-^0(x),V_\pm^3(0)]]+\tfrac12\delta(x_0)\delta(y_0)[A_-^0(x),[A_+^0(y),V_\pm^3(0)]]\\&-\partial_\mu T(\delta(y_0)[A_+^0(y),V_\pm^3(0)]A_-^\mu(x))-\partial_\nu T(\delta(x_0)[A_-^0(x),V_\pm^3(0)]A_+^\nu(y))\\&-\tfrac12\partial_\mu T(\delta(x_0-y_0)[A_+^0(y),A_-^\mu(x)]V_\pm^3(0))-\tfrac12\partial_\nu T(\delta(x_0-y_0)[A_-^0(x),A_+^\nu(y)]V_\pm^3(0))\\&-\tfrac12 T(\delta(x_0-y_0)[A_-^0(x),\partial_\nu A_+^\nu(y)]V_\pm^3(0))-\tfrac12 T(\delta(x_0-y_0)[A_+^0(y),\partial_\mu A_-^\mu(x)]V_\pm^3(0)).\end{aligned} \tag{4.15}$$

The last line contains the so-called σ commutators which vanish in the chiral limit. Substituting the remaining terms in (4.15) into (4.14), integrating by parts the terms containing derivatives, and taking the soft-pion limit, we find

$$\begin{aligned}\langle n(\vec{0})\pi_+(\vec{0})|V_\pm^3|\pi_+(\vec{0})\rangle=&-\frac{1}{4F_\pi^2m_\pi}\langle n(\vec{0})|V_\pm^3|\Omega\rangle-\lim_{p,q\to0}\frac{p_\mu q_\nu}{4F_\pi^2m_\pi}\int dx\,dy\,e^{ipx-iqy}\langle n(\vec{0})|T(A_-^\mu(x)A_+^\nu(y)V_\pm^3(0))|\Omega\rangle\\&+\lim_{p\to0}\frac{ip_\mu}{2F_\pi^2m_\pi}\int dx\,e^{\pm ipx}\langle n(\vec{0})|T(A_\pm^\mu(x)A_3^3(0))|\Omega\rangle+\lim_{p\to0}\frac{ip_\mu}{4F_\pi^2m_\pi}\int dx\,\langle n(\vec{0})|T(V_3^\mu(x)V_\pm^3(0))|\Omega\rangle.\end{aligned} \tag{4.16}$$

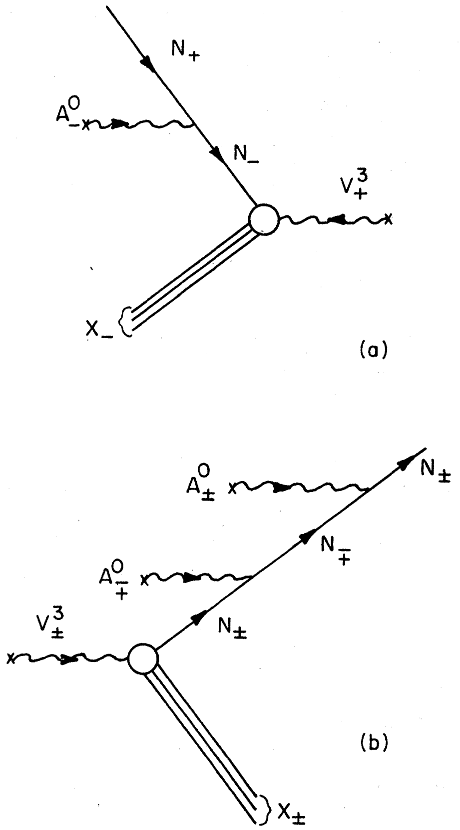

FIG. 2. External-line-insertion pole terms, corresponding to Eqs. (B2a) and (B3).

The integral in the fourth term in (4.16) has no p dependence, so the fourth term vanishes at $p=0$. The second and third terms contribute external-line insertions, as shown in Fig. 2.

In deriving (4.13) and (4.16) we have not used any local current-current commutators. We have used only charge-charge or charge-current commutators. So (4.13) and (4.16) are not affected by model-dependent features of the local commutators, such as the nature of the Schwinger terms.

Next we substitute (4.13) and (4.16) into the sum rules, (4.5)–(4.7). In all three sum rules, we find that the external-line insertions from (4.16) cancel with the ones from (4.13). The proof of cancellation is presented in Appendix B. The proof is straightforward except for one amusing point. The double insertions [Fig. 2(b)], which arise from the second term in (4.16) turn out to be undefined; that is, we can obtain any value depending on the path in (p, q) space along which we approach the origin, $(p, q)=(0,0)$. But if we average the result for any path with the result for the path obtained by interchanging p and q, we find that the result is well defined. Furthermore, the result obtained has just the right value to cancel the single insertion terms calculated from (4.13) which are free of any such ambiguity. So we see again the importance of maintaining Bose symmetry in calculations with several soft pions.

Now we substitute (4.13) and (4.16) into (4.5)–(4.7), neglecting the insertion terms, which all cancel. We multiply the three equations by m_π (to eliminate the kinematic pole, m_π^{-1}) and then put $m_\pi=0$. From (4.5) and (4.7) we learn only that $0=0$, i.e., the local commutators (4.1) and (4.3) are consistent with spontaneously broken SU(2) $\otimes$SU(2). This information is not completely trivial. For instance, the commutator (4.1) *could* contain a term proportional to the second derivative of a δ function, $\Delta(Q(\vec{x})\delta(\vec{x}))$. On the left-hand side of (4.5), this term would contribute

$$\langle \pi_+(\vec{0})|Q|\pi_+(\vec{0})\rangle = (1/2m_\pi)F(0), \qquad (4.17)$$

where F is a scalar form factor. Unless F vanished in the chiral limit with $m_\pi=0$, we would have a contradiction. So we see that our result puts restrictions on such anomalous contributions: These anomalous contributions must be of less than the leading order in m_π, so that they may be neglected in the chiral limit. [Of course, we have only proved this statement for anomalous terms in (4.1) and (4.3) which would contribute to (4.5) and (4.7).] In general, very little is known about whether such anomalous terms are present in the hadronic current commutators of the real world. Perhaps the only information comes from the success of the Cabibbo-Radicati sum rule,[15] which tells us that if a term like $\Delta(Q(\vec{x})\delta(\vec{x}))$ is present in (4.1), then the expectation value of Q between nucleon states is small.

Substituting (4.13) and (4.16) into (4.6), we find a nontrivial sum rule because in (4.6) the cross term of the commutator contributes with a positive sign. We have

$$\sum_n \frac{1}{m_n}|\langle\Omega|V^3_+|n(\vec{0})\rangle|^2 = F_\pi^{\,2} + \sum_n \frac{1}{m_n}|\langle\Omega|A^3_+|n(\vec{0})\rangle|^2 . \qquad (4.18)$$

This is precisely Weinberg's first spectral-function sum rule.[8] Referring to the "clumsy" form of the spectral representation, (4.10), we see that (4.18) may be rewritten as

$$C_V = 2F_\pi^{\,2} + C_A . \qquad (4.19)$$

The states contributing to the vector spectral function must have quantum numbers $I^G, J^P = 1^+, 1^-$; those contributing to the axial-vector spectral function have $I^G, J^P = 1^-, 1^+$. If we truncate the sums with the ρ and A_1 meson contributions, we find the familiar saturated form of the sum rule,

$$g_\rho^2/m_\rho^2 = F_\pi^2 + g_A^2/m_A^2\,, \qquad (4.20)$$

where g_ρ is defined by

$$\langle\Omega|V_a^\mu|\rho_b(p,\lambda)\rangle = \delta_{ab}\frac{\epsilon^\mu(p,\lambda)}{(2p_0)^{1/2}}g_\rho \qquad (4.21)$$

and g_A is defined analogously.

The assumptions which have gone into our derivation of (4.19) are similar to the assumptions used by Weinberg in his original derivation.[8] They differ in only one respect. Weinberg's derivation would fail if there were $I=1$ operator Schwinger terms in the A-A or V-A local commutators. The derivation we have presented would fail in the presence of an $I=0$ or $I=2$ operator Schwinger term in the V-V commutator, (4.2).

The principal attraction of the derivation presented here is that it establishes the connection between the spectral-function sum rule, (4.18), which holds in a theoretical world of massless pions, and the "physical" sum rule, (4.6), which may be valid in the real world. If we can study (4.6), either in models or experimentally, we can learn about the corrections to (4.18) due to the breaking of SU(2)⊗SU(2). The pair terms in (4.6) can be measured in e^+e^- annihilation experiments, and they contribute 50% of the axial-vector spectral representation in the sum rule (4.18). In particular, the cross section near threshold for $e^+e^-\to\pi A_1$ determines the pair contribution $|\langle\Omega|V_-^3|\pi_+(\vec{0})A_1(\vec{0})\rangle|^2$, which in the chiral limit (with our noncovariant normalization) is $g_A^2/2F_\pi^2 m_\pi m_A$ [see (4.13d)]. Since F_π, m_π, and m_A are all known, we have an experimental means to estimate g_A and to test (4.20). *Furthermore, our derivation of (4.18) shows that the error introduced into this estimate of g_A by the use of PCAC is the same as the error committed in the derivation of the spectral-function sum rule.*

More generally, Pais and Treiman[9] have observed that it may be possible to retrieve the SU(2) vector and axial-vector spectral functions below the nucleon-antinucleon threshold from the cross sections for electron-positron annihilation into one or two soft pions and anything else.[23] For details the reader should consult their paper.

A difficulty of rest-frame sum rules is that we have little experience which tells us whether they can be adequately saturated by low-mass contributions. This problem is only likely to be resolved by comparison with experiment, which is surely a disadvantage, since we are then testing the commutator and the saturation scheme together. (This is really a difference of degree and not of kind from other sum rules; e.g., consider the implications for the Adler-Weisberger sum rule if at the National Accelerator Laboratory the same value of $\sigma_{\pi^+N}-\sigma_{\pi^-N}$ is seen as was seen at Serpukhov.) However, on the basis of the material presented in this section, we may at least note that the saturation properties of the finite-moment sum rules may be related to those of the more familiar spectral-function sum rules. Assuming that the terms due to SU(2)⊗SU(2) breaking are at least as convergent as the SU(2)⊗SU(2) symmetric terms (which is reasonable in the context of arguments about asymptotic symmetries), we can conclude that the $[V^0, V^3]$ sum rule converges like the first spectral-function sum rule, while the $[V^0, V^0]$ ($[V^3, V^3]$) rule converges more rapidly (slowly) by a factor linearly proportional to the mass of the intermediate states. The saturated form of the spectral-function sum rule has been incorporated into literally hundreds of theoretical models, but its validity is not known. We may hope that it will eventually be determined, perhaps by the method outlined in the two preceding paragraphs.

To conclude this section, we emphasize again the importance of the disconnected contributions. For instance, in the spectral-function sum rule, we found that the pair diagrams accounted for half of the axial-vector spectral integral and that the semiconnected diagrams accounted for the entire vector spectral integral. We also saw that the importance of the pair contributions is a definite advantage since it means that the (saturated) spectral-function sum rule can be tested experimentally.

V. NUCLEON SUM RULES AND $e^+e^-\to$BARYONS

We derive five nucleon sum rules which are analogous to the "physical" pion sum rules of Sec. IV. One is the nucleon analog of (4.6), from which we deduced the first spectral-function sum rule in the chiral limit. Again we find that the pair and semiconnected contributions are just as important as the connected ones. This fact is responsible for whatever usefulness the sum rules may have since only the pair contributions are currently accessible to experimental observation. Neglecting the semiconnected terms, we saturate the pair and connected terms with the leading resonances and calculate the isovector nucleon radius and near-threshold values of $\sigma(e^+e^-\to N\bar{N})_{I=1}$ and $\sigma(e^+e^-\to N\bar{\Delta}(1236))_{I=1}$.

The assumption that the semiconnected terms can be neglected is very uncertain. As in (4.5)–(4.7), the leading semiconnected term is the ρ-meson contribution which is proportional to

$$g_\rho\langle\rho(\vec{0})N_+(\vec{0})|V^3|N_+(\vec{0})\rangle. \qquad (5.1)$$

Equation (5.1) contains a ρ-pole contribution, whose residue is the ρN scattering amplitude at threshold. As we remarked in Sec. IV, this contribution is infinite in the zero-width limit, $\Gamma_\rho=0$, but the infinity is canceled via unitarity by double pole contributions to the connected terms (see Appendix A or Ref. 2). Therefore, if we assume ρ-pole dominance of (5.1) and neglect terms of order Γ_ρ/m_ρ, then there is no ρ-meson semiconnected contribution to the sum rule. This is the argument we presented in Ref. 10.

However, we can see from Sec. IV that the argument is not completely leak-proof. In the soft-pion limit we found a contribution to $\langle\pi\rho|V|\pi\rangle$ which was proportional to g_ρ/F_π^2 and did not have the ρ-pole structure. It is an open question whether there are significant analogous contributions to (5.1). Such contributions are theoretically inaccessible since there is no nucleon analog of the soft-pion procedure, and they are experimentally inaccessible since they do not contribute to ρN scattering. So all we can say is that the contributions from (5.1) which are proportional to the ρN scattering amplitude are suppressed in the sum rule by at least a factor Γ_ρ/m_ρ.

In the pion sum rules, it would have been impossible to neglect the semiconnected contributions because in (4.6) the connected and pair contributions are positive definite and can only be canceled by the semiconnected contributions. But in the analogous nucleon sum rule the pair terms appear with a negative sign (because of Fermi statistics) so that it is at least possible that the semiconnected terms can be neglected. Furthermore, we will see below that in sum rules involving spatial components of the currents, the pair diagrams are significantly enhanced by kinematical factors.

We simply state the results because the method of derivation is identical to that of Sec. IV. In Sec. IV there were three sum rules, but here there are five because of the additional degrees of freedom due to the nucleon spin. Semiconnected terms are not recorded because they will be neglected in our saturation scheme. From the five equal-time ($t=0$) commutators

$$[V_+^0(\vec{x}),\ V_-^0(\vec{0})]=2\delta(\vec{x})V_3^0(\vec{x}),\tag{5.2}$$

$$[V_+^0(\vec{x}),\ V_-^3(\vec{0})]=2\delta(\vec{x})V_3^3(\vec{x})+iC_V\frac{\partial}{\partial x_3}\delta(\vec{x}),\tag{5.3}$$

$$[V_+^0(\vec{x}),\ V_-^-(\vec{0})]=2\delta(\vec{x})V_3^-(\vec{x})+iC_V\frac{\partial}{\partial x_-}\delta(\vec{x}),\tag{5.4}$$

$$[V_+^3(\vec{x}),\ V_-^3(\vec{0})]=\begin{cases}2\delta(\vec{x})V_3^0(\vec{x}), & \text{quark}\\ 0, & \text{field algebra}\end{cases}\tag{5.5}$$

$$[V_+^+(\vec{x}),\ V_-^-(\vec{0})]=\begin{cases}4\delta(\vec{x})[V_3^0(\vec{x})+\frac{2}{3}A_0^3(\vec{x})+\frac{1}{3}A_8^3(\vec{x})], & \text{quark}\\ 0, & \text{field algebra}\end{cases}\tag{5.6}$$

we obtain the corresponding sum rules

$$\frac{1}{8m^2}+\frac{\langle r_V^2\rangle}{3}-\frac{|G^V(4m^2)|^2}{2m^2}=\sum_{N^*}(2R_{11}+2R_{13}-R_{31}-R_{33})+\{R\to\overline{R}\},\tag{5.2$'$}$$

$$0=\frac{1}{8m}-\frac{|G^V(4m^2)|^2}{2m}+\sum_{N^*}(m^*-m)(R_{11}+R_{13}+R_{31}+R_{33})-\{R\to\overline{R},\ m\to -m\},\tag{5.3$'$}$$

$$\frac{G_M^V(0)}{2m}-2\,\frac{|G^V(4m^2)|^2}{m}=\sum_{N^*}(m^*-m)(-4R_{11}+2R_{13}+2R_{31}-R_{33})-\{R\to\overline{R},\ m\to -m\},\tag{5.4$'$}$$

$$\left.\begin{matrix}\text{quark} & \frac{1}{2}\\ \text{field algebra} & 0\end{matrix}\right\}=2|G^V(4m^2)|^2+\sum_{N^*}(m^*-m)^2(2R_{11}+2R_{13}-R_{31}-R_{33})+\{R\to\overline{R},\ m\to -m\},\tag{5.5$'$}$$

$$\left.\begin{matrix}\text{quark} & \frac{1}{2}+\langle N_+(\vec{0},+)|(\frac{2}{3})^{1/2}A_0^3+(\frac{1}{3})^{1/2}A_8^3|N_+(\vec{0},+)\rangle\\ \text{field algebra} & 0\end{matrix}\right\}=4|G^V(4m^2)|^2+\sum_{N^*}(m^*-m)^2(4R_{11}+R_{13}+R_{31}-2R_{33})$$

$$+\{R\to\overline{R},\ m\to -m\}.\tag{5.6$'$}$$

$\langle r_V\rangle$ is the isovector nucleon radius, $G^V(4m^2) \equiv G_M^V(4m^2)=G_E^V(4m^2)$ is the Sachs isovector form factor [normalized by $G_E^V(0)=\frac{1}{2}$] at $t=4m^2$, A_i^μ is a U(3) nonet of axial-vector currents, and the "inelastic mean square radii" are defined by

$$R_{2I,2J}(N^*)=\frac{1}{(m^*-m)^2}\,|\langle N_+(\vec{0})|\,V_3^3|N^*_{I,J}(\vec{0})\rangle|^2\,, \tag{5.7}$$

$$\overline{R}_{2I,2J}(N^*)=\frac{1}{(m^*+m)^2}|\langle\Omega|\,V_3^3|N_+(\vec{0})N^*_{I,J}(\vec{0})\rangle|^2\,.$$

m^* is the mass of the state N^*. In (5.3′)–(5.6′), $(m^*\pm m)$ is factored merely for brevity and not to imply a mass degeneracy of the intermediate states.

In (5.3′) the connected contribution, $(m^*-m)\times(R_{11}+R_{13}+R_{31}+R_{33})$, is positive definite so it must be compensated by an equally large contribution from the pair and semiconnected contributions. Therefore, it is not plausible to assume in such sum rules that both semiconnected and pair contributions can be neglected. This conclusion could be modified only if there were an operator Schwinger term which made a large positive contribution to the left-hand side of (5.3′).

In (5.3′)–(5.6′) the connected terms R are multiplied by factors (m^*-m) or $(m^*-m)^2$, while the pair terms $\overline{R}$ are multiplied by (m^*+m) or $(m^*+m)^2$. With typical values of m^* we see that the pair terms are enhanced relative to the connected terms by at least an order of magnitude in (5.5′) and (5.6′) and by factors of 3 to 5 in (5.3′) and (5.4′). Thus, the pair terms may contribute little to the nucleon radius in (5.2′) and still be very important in determining the over-all system of equations.

From (5.7) we see that the connected contributions have quantum numbers $I=\frac{1}{2}, \frac{3}{2}$ and $J^P=\frac{1}{2}^-, \frac{3}{2}^-$, while the pair terms have $I=\frac{1}{2}, \frac{3}{2}$ and $J^P=\frac{1}{2}^+, \frac{3}{2}^+$. Therefore, in addition to the nucleon and nucleon-pair contributions, we attempt to saturate the sum rules with the $N^*(1520)$ and the $\Delta(1236)$. $\langle r_V\rangle$ and $G_M^V(0)$ are known experimentally.[24] The three unknowns – $|G^V(4m^2)|^2$, $R(1520)$, and $\overline{R}(1236)$ – are determined from Eqs. (5.3′), (5.4′), and (5.5′). [(5.6′) is of little value since nothing is known about the neutral axial-vector currents.] The results for the quark model (field algebra) are $|G^V(4m^2)|^2=0.43$ (0.29), $R(1520)=0.87m^{-2}$ $(0.98m^{-2})$, and $\overline{R}(1236)=0.19m^{-2}$ $(0.25m^{-2})$. Substituting these results into (5.2′), we calculate that $\langle r_V\rangle=0.46$ F (0.48 F), to be compared with the experimental value,[24] $\langle r_V\rangle=0.62\pm0.01$ F. Agreement is as good as one might hope, considering the severity of the saturation assumptions. The quark-model and field-algebra results are not distinguishable.

From the values of $G^V(4m^2)$ and $\overline{R}(1236)$ we can estimate the values near threshold of $\sigma(e^+e^-\to N\overline{N})_{I=1}$ and $\sigma(e^+e^-\to N\overline{\Delta})_{I=1}$. Where p is the magnitude of the spatial momentum in the center-of-mass frame, we have

$$\frac{d\sigma}{dp}(e^+e^-\xrightarrow[I=1]{}N\overline{N})\bigg|_{p=0}=\frac{\pi\alpha^2}{2m^3}|G^V(4m^2)|^2=\begin{cases}1.6\times10^{-32}m^{-1}\,\mathrm{cm}^2, & \text{quark}\\ 1.1\times10^{-32}m^{-1}\,\mathrm{cm}^2, & \text{field algebra}\end{cases} \tag{5.8}$$

and

$$\frac{d\sigma}{dp}(e^+e^-\xrightarrow[I=1]{}N\overline{\Delta})\bigg|_{p=0}=\frac{16\pi\alpha^2 mm_\Delta}{(m+m_\Delta)^3}\overline{R}(1236)=\begin{cases}2.4\times10^{-32}m^{-1}\,\mathrm{cm}^2, & \text{quark}\\ 3.2\times10^{-32}m^{-1}\,\mathrm{cm}^2, & \text{field algebra}.\end{cases} \tag{5.9}$$

Since the cross sections vanish at $p=0$, (5.8) and (5.9) may be used to estimate the magnitude of the cross sections for small p. For instance, at $p=\frac{1}{10}m$, 100 MeV/c in the center-of-mass frame, (5.8) implies a cross section of the order of magnitude of 10^{-33} cm^2.

Considering the crudeness of the calculations, (5.8) and (5.9) should probably be regarded just as indications of the order of magnitude. As such, they are surprisingly large. The prediction (5.8) is of the order of magnitude of a pointlike nucleon. The predictions $|G^V(4m^2)|^2=0.43$ (0.29) may be compared with $|G^V(4m^2)|^2=0.25$ for a pointlike nucleon. The ρ-dominance model for the Pauli form factors gives $|G^V(4m^2)|^2=0.22$, while the dipole fit to the spacelike Pauli form factors gives a value which is smaller by an order of magnitude, $|G^V(4m^2)|^2=0.022$. [The pole and dipole calculations must be made for the Pauli form factors in order to satisfy $G_M^V(4m^2)=G_E^V(4m^2)$.]

The only experimental information about the nucleon form factors in the timelike region is an upper bound for the proton form factor, $|G_M^p(5.8m^2)|<0.2$.[25] This could be compatible with Eq. (5.8). For instance, the ρ-dominance model $|G_M^V(5.8m^2)_\rho|=0.36$ provides a smooth extrapolation from the order of magnitude of the experimental bound at $5.8m^2$ to the order of magnitude of the prediction at $4m^2$. Since the pole alone can extrapolate so well between the two values, it is possible that a smooth extrapolation occurs with the additional help of the cut singularities.

Being large, the prediction (5.8) has the virtue of being relatively easy to disprove. With the anticipated luminosity of the Frascati colliding beams, we would predict roughly one $(N\bar{N})_{I=1}$ event per hour at a nucleon center-of-mass momentum of 100 GeV/c. The prediction seems large, but we are encouraged by preliminary results[26] which suggest that the pion form factor may be half-pointlike near $s=4m^2$.

Note added in proof. Near the end of Sec. IV we state that the error in estimating g_A from $e^+e^- \to \pi A_1$ near threshold is the same as the error in deriving the spectral-function sum rule. This is not quite correct. The error in the estimate of g_A is $O(m_\pi)$, but in the sum rule the $O(m_\pi)$ corrections cancel and the leading correction is $O(m_\pi^2)$.

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APPENDIX A: CANCELLATION OF SINGULARITIES

The sum rules in Secs. III, IV, and V have infinite contributions from the semiconnected terms, which are canceled by contributions from the connected terms. This was first established by Weisberger.[2] The demonstration given here differs from Weisberger's in some respects and is presented in greater detail.

Consider, for example, the pion sum rule (4.7). The ρ-meson semiconnected contribution to the first term of the commutator is

$$2\,\mathrm{Re}[\langle\Omega|\,V_+^3\,|\rho_-(\vec{0},3)\rangle\langle\rho_-(\vec{0},3)\pi_+(\vec{0})|\,V_-^3\,|\pi_+(\vec{0})\rangle], \tag{A1}$$

where "Re" denotes the real part. The current V_-^3 carries four-momentum $(m_\rho,\vec{0})$, so that the ρ-pole contribution is infinite in the zero-width limit, $\Gamma_\rho=0$. The divergence is canceled by double-pole contributions from the connected terms

$$\sum_n |\langle\pi_+(\vec{0})|\,V_+^3\,|n(\vec{0})\rangle|^2\,. \tag{A2}$$

Weisberger established the cancellation by working directly with the infinite quantities, like (A1) and (A2), using the $i\epsilon$ prescription to define the pole denominators. Here we follow a somewhat less delicate course: We derive a kinematically different sum rule which has only finite contributions and consider it in the singular limit.

In the analog of (3.4) for the present case, we choose $\vec{p}\neq\vec{0}$ and $\vec{p}+\vec{q}=\vec{x}\neq\vec{0}$. Then (A1) is replaced by

$$\langle\Omega|\,V_+^3\,|\rho_-(\vec{0},3)\rangle[\langle\rho_-(\vec{x}-\vec{p},3)\pi_+(\vec{p})|\,V_-^3\,|\pi_+(-\vec{p})\rangle+\langle\rho_-(-\vec{x}-\vec{p},3)\pi_+(\vec{p})|\,V_-^3\,|\pi_+(-\vec{p})\rangle], \tag{A3}$$

where we have used parity and have taken the (harmless) limit $\vec{x},\vec{p}\to\vec{0}$ in the term $\langle\Omega|V|\rho\rangle$. Similarly, (A2) is replaced by

$$\sum_n \langle\pi_+(\vec{p})|\,V_+^3\,|n(\vec{x})\rangle\langle n(\vec{x})|\,V_-^3\,|\pi_+(-\vec{p})\rangle\,. \tag{A4}$$

(A3) is finite because the current V_-^3 is off the ρ mass shell if $\vec{x}\cdot\vec{p}\neq 0$. We will study the singularities of (A3) and (A4) as $\vec{x}$ and $\vec{p}$ approach zero.

The intermediate states in (A3) and (A4) are chosen to be "out" states and the $\rho_-\pi_+$ scattering amplitude is defined by

$$\langle\rho_-(\vec{k}_2,\lambda_2)\pi_+(\vec{p}_2)\,\mathrm{out}|\rho_-(\vec{k}_1,\lambda_1)\pi_+(\vec{p}_1)\,\mathrm{in}\rangle=(2\pi)^4\delta(p_1+k_1-p_2-k_2)\mathfrak{M}(\vec{p}_1,\vec{k}_1,\lambda_1;\,\vec{p}_2,\vec{k}_2,\lambda_2). \tag{A5}$$

Applying the LSZ reduction to the incoming ρ_- and inverting the spin-1, mass-shell projection operator, we obtain with the correct phase the ρ-pole contribution to semiconnected matrix elements,

$$\langle\rho_-(\pm\vec{x}-\vec{p},3)\pi_+(\vec{p})|\,V_-^3\,|\pi_+(-\vec{p})\rangle_{\text{pole}}=\mp i\frac{g_\rho(2m_\rho)^{1/2}}{4\vec{p}\cdot\vec{x}}\mathfrak{M}, \tag{A6}$$

where we have taken the (harmless) limit $\vec{x},\vec{p}\to 0$ in the factor $\mathfrak{M}\equiv\mathfrak{M}(\vec{0},\vec{0},3;\,\vec{0},\vec{0},3)$. We substitute (A6) into (A3) and find that the singular part of (A3) is

$$ig_\rho^{\,2}\,\mathrm{Re}\,\mathfrak{M}/\vec{p}\cdot\vec{x}\,. \tag{A7}$$

Next we consider the connected terms (A4). As in (A5), there are ρ-pole contributions

$$\langle\pi_+(\pm\vec{p})|\,V_+^3\,|n(\vec{x})\rangle = \frac{+i\sqrt{2}\,g_\rho(2m_\rho)^{1/2}\mathfrak{M}_n^*}{(p_n^0-p^0)^2-(\vec{x}\mp\vec{p})^2-m_\rho^{\,2}+i\epsilon}\,, \tag{A8}$$

where p_n^0 is the energy of $|n(\vec{x})\rangle$ and we have taken $\vec{p},\vec{x}\to 0$ in the factor $(2m_\rho)^{1/2}\mathfrak{M}_n \equiv (2m_\rho)^{1/2}\mathfrak{M}(\vec{0},\vec{0},3;\,n(\vec{0}))$ defined by

$$\langle n(\vec{p}_n)\mathrm{out}|\pi_+(\vec{p})\rho_-(\vec{k},\lambda)\mathrm{in}\rangle = (2\pi)^4\delta(p+k-p_n)\mathfrak{M}(\vec{p},\vec{k},\lambda;\,n(\vec{p}_n))\,. \tag{A9}$$

Defining the four-momenta

$$k_\pm = (k_\pm^0,\vec{k}_\pm) = ([m_\rho^{\,2}+(\vec{x}\mp\vec{p})^2]^{1/2},\,\pm\vec{p}-\vec{x})\,, \tag{A10}$$

the ρ-pole contributions to the connected terms (A4) are

$$\sum_n\int_{-\infty}^{\infty}dq_0\,\delta(p_0+q_0-p_n^0)\frac{4g_\rho^{\,2}|\mathfrak{M}_n|^2m_\rho}{[q_0^{\,2}-(k_+^0)^2+i\epsilon][q_0^{\,2}-(k_-^0)^2-i\epsilon]} \tag{A11}$$

or

$$\sum_n\int_{-\infty}^{\infty}dq_0\,\delta(p^0+q^0-p_n^0)\frac{4g_\rho^{\,2}|\mathfrak{M}_n|^2m_\rho}{(q^0-k_+^0+i\epsilon)(q^0+k_+^0-i\epsilon)(q^0-k_-^0-i\epsilon)(q^0+k_-^0+i\epsilon)}\,. \tag{A12}$$

We close the contour in the upper half-plane and use Cauchy's theorem to evaluate the integral, with the result

$$\frac{g_\rho^{\,2}}{\vec{p}\cdot\vec{x}}(2\pi i)\frac{1}{2}\sum_n\delta(p^0+k^0-p_n^0)|\mathfrak{M}_n|^2 \tag{A13}$$

(in the δ function we let $\vec{x},\vec{p}\to\vec{0}$ so that $k_\pm^0\to k^0=m_\rho$).

With the amplitudes defined by (A5) and (A9), the forward unitarity relation is

$$\mathrm{Re}\,\mathfrak{M}(\vec{p},\vec{k},\lambda;\,\vec{p},\vec{k},\lambda) = -\frac{1}{2}\sum_n\int\frac{d^3p_n}{(2\pi)^3}(2\pi)^4\delta(p+k-p_n)|\mathfrak{M}(\vec{p},\vec{k},\lambda;\,n)|^2. \tag{A14}$$

(The integration on $\vec{p}_n$ is indicated explicitly because in the notation of this appendix $\sum_n$ does not include the integration over the total three-momentum.) Using (A14) we see that the singular part of the connected contributions, (A13), is

$$-ig_\rho^{\,2}\,\mathrm{Re}\,\mathfrak{M}/\vec{p}\cdot\vec{x} \tag{A15}$$

which cancels the semiconnected singularity, (A7).

APPENDIX B: CANCELLATION OF EXTERNAL-LINE INSERTIONS

In this appendix we show that the external-line pole terms (Fig. 2) in (4.13) and (4.16) do not contribute to the sum rules (4.5)–(4.7). We calculate in the limit of exact $SU(2)\otimes SU(2)$ with $m_\pi=0$ (i.e., to leading order in m_π). The pole terms occur if $|n\rangle$ contains at least some particles which have spin and isospin greater than zero and which are not eigenstates of G parity. Such states always have projections both in the channel of the connected and pair contributions ($I^G, J^P=1^-,1^+$) and in the channel of semiconnected contributions ($I^G, J^P=1^+,1^-$). We find that the projections of the pole terms from the two channels cancel with one another in the sum rules. To illustrate the cancellation, we will consider in detail the case when $|n\rangle$ contains a nucleon and other particles. We will then briefly indicate the generalization to particles of arbitrary spin and isospin.

We consider the contribution of the state

$$|n(\vec{0})\rangle = |N_\alpha(\vec{k},s)X_\beta(-\vec{k})\rangle\,,$$

where $|N_\alpha(\vec{k},s)\rangle$ is a nucleon of spin s and third component of isospin $I_3=\alpha$, and $|X_\beta\rangle$ represents the remaining particles, with $I_3=\beta$. To spare ourselves some tedious bookkeeping, we stipulate that $|X\rangle$ has isospin $I=\frac{1}{2}$ (if $|X\rangle$ has a larger isospin, we must keep track of more terms, but nothing essential changes).

In the chiral limit, $m_n\pm m_\pi\to m_n$, and the contribution of the state $|n\rangle$ to the sum rules (4.5)–(4.7) is proportional to

$$\sum_s(|\langle\pi_+(\vec{0})|\,V_+^3\,|N_{1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle|^2+|\langle\pi_+(\vec{0})|\,V_+^3\,|N_{-1/2}(\vec{k},s)X_{1/2}(-\vec{k})\rangle|^2$$

$$-\eta|\langle\Omega|V_-^3|\pi_+(\vec{0})N_{1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle|^2-\eta|\langle\Omega|V_-^3|\pi_+(\vec{0})N_{-1/2}(\vec{k},s)X_{1/2}(-\vec{k})\rangle|^2$$
$$+2\,\mathrm{Re}\{\langle\Omega|V_+^3|N_{-1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle[\langle\pi_+(\vec{0})N_{-1/2}(\vec{k},s)X_{-1/2}(-\vec{k})|V_-^3|\pi_+(\vec{0})\rangle$$
$$+\eta\langle\pi_+(\vec{0})N_{1/2}(\vec{k},s)X_{1/2}(-\vec{k})|V_+^3|\pi_+(\vec{0})\rangle]\}),$$

where we define $\eta=+1$ for (4.5) and (4.7) and $\eta=-1$ for (4.6). According to (4.13) and (4.16), in the chiral limit

$$\langle\pi_+(\vec{0})|V_+^3|N_{\pm1/2}(\vec{k},s)X_{\mp1/2}(-\vec{k})\rangle=\frac{-i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|N_{\pm1/2}(\vec{k},s)X_{\mp1/2}(-\vec{k})\rangle$$
$$+\lim_{p\to0}\frac{-p^0}{2F_\pi m_\pi^{1/2}}\int dx\,e^{ipx}\langle\Omega|T(A_-^0(x)V_+^3(0))|N_{\pm1/2}(\vec{k},s)X_{\mp1/2}(-\vec{k})\rangle, \tag{B2a}$$

$$\langle\Omega|V_-^3|\pi_+(\vec{0})N_{\pm1/2}(\vec{k},s)X_{\mp1/2}(-\vec{k})\rangle=\frac{i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|N_{\pm1/2}(\vec{k},s)X_{\mp1/2}(-\vec{k})\rangle$$
$$+\lim_{p\to0}\frac{p^0}{2F_\pi m_\pi^{1/2}}\int dx\,e^{-ipx}\langle\Omega|T(A_+^0(x)V_-^3(0))|N_{\pm1/2}(\vec{k},s)X_{\mp1/2}(-\vec{k})\rangle, \tag{B2b}$$

$$\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})\pi_+(\vec{0})|V_\pm^3|\pi_+(\vec{0})\rangle=\frac{-1}{4F_\pi^2m_\pi}\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})|V_\pm^3|\Omega\rangle$$
$$+\lim_{p\to0}\frac{ip^0}{2F_\pi^2m_\pi}\int dx\,e^{\pm ipx}\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})|T(A_\pm^0(x)A_3^3(0))|\Omega\rangle$$
$$+\lim_{p,q\to0}\frac{-p^0q^0}{4F_\pi^2m_\pi}\int dx\,dy\,e^{ipx-iqy}\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})|T(A_-^0(x)A_+^0(y)V_\pm^3(0))|\Omega\rangle. \tag{B3}$$

(We have used the fact that $\vec{p}=\vec{q}=\vec{0}$.)

External-line insertions for Eq. (B2) are illustrated in Fig. 2(a). For instance, the external-line insertion in Eq. (B2a) is

$$\lim_{p\to0}\frac{-ip^0}{2F_\pi m_\pi^{1/2}}\bar{\chi}_{-1/2}(k_X)\Gamma_+^3(-k_X-k_N+p)\frac{\not{k}_N-\not{p}+m_N}{(k_N-p)^2-m_N^2+i\epsilon}\Gamma_{5,-}^0(-p)u_{1/2}(k_N,s)\left(\frac{m_Xm_N}{k_X^0k_N^0}\right)^{1/2}, \tag{B4}$$

where k_X and k_N are the four-momenta of $|X\rangle$ and the nucleon, χ is a generalized spinor for the state $|X\rangle$, and the vertex functions are defined by

$$\langle\Omega|V_\gamma^\mu|N_\alpha(k_N,s)X_\beta(k_X)\rangle=\bar{\chi}_\beta(k_X)\Gamma_\gamma^\mu(-k_N-k_X)u_\alpha(k_N,s)(m_Xm_N/k_X^0k_N^0)^{1/2},$$
$$\langle N_\alpha(p_2,s_2)|A_\gamma^\mu|N_\beta(p_1,s_1)\rangle=\bar{u}_\alpha(p_2,s_2)\Gamma_{5,\gamma}^\mu(p_2-p_1)u_\beta(p_1,s_1)m_N/(p_1^0p_2^0)^{1/2}. \tag{B5}$$

The zero in the numerator of (B4) is canceled by the pole, leaving a finite result:

$$\lim_{p\to0}p^0\frac{\not{k}_N-\not{p}+m_N}{(k_N-p)^2-m_N^2}=-\frac{\not{k}_N+m_N}{2k_N^0}=-\frac{m_N}{k_N^0}\sum_s u(k_N,s)\bar{u}(k_N,s). \tag{B6}$$

Using (B5) and (B6), we rewrite (B4) as

$$\frac{i}{2F_\pi m_\pi^{1/2}}\langle N_{-1/2}(k_N,s)|A_-^0|N_{1/2}(k_N,s)\rangle\langle\Omega|V_+^3|N_{-1/2}(k_N,s)X_{-1/2}(k_X)\rangle. \tag{B7}$$

Similarly, we calculate the pole contribution to (B2b), and we record the final expressions for (B2):

$$\langle\pi_+(\vec{0})|V_+^3|N_{1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle=\frac{i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|N_{1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle$$
$$+\frac{i}{2F_\pi m_\pi^{1/2}}\langle N_{-1/2}(\vec{k},s)|A_-^0|N_{1/2}(\vec{k},s)\rangle\langle\Omega|V_+^3|N_{-1/2}(\vec{k},s)X_{-1/2}(\vec{k},s)\rangle+\cdots,$$
$$\langle\pi_+(\vec{0})|V_+^3|N_{-1/2}(\vec{k},s)X_{1/2}(-\vec{k})\rangle=\frac{-i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|N_{-1/2}(\vec{k},s)X_{1/2}(-\vec{k})\rangle+\cdots, \tag{B8a}$$

$$\langle\Omega|\,V_-^3|\pi_+(\vec{0})N_{-1/2}(\vec{k},s)X_{1/2}(-\vec{k})\rangle=\frac{i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|N_{-1/2}(\vec{k},s)X_{1/2}(-\vec{k})\rangle$$
$$+\frac{i}{2F_\pi m_\pi^{1/2}}\langle N_{1/2}(\vec{k},s)|A_+^0|N_{-1/2}(\vec{k},s)\rangle\langle\Omega|\,V_-^3|N_{1/2}(\vec{k},s)X_{1/2}(\vec{k},s)\rangle+\cdots,$$
$$\langle\Omega|\,V_-^3|\pi_+(\vec{0})N_{1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle=\frac{i}{F_\pi m_\pi^{1/2}}\langle\Omega|A_3^3|N_{1/2}(\vec{k},s)X_{-1/2}(-\vec{k})\rangle+\cdots. \qquad \text{(B8b)}$$

In (B8) we have recorded the insertions onto external nucleon legs; we have of course omitted insertions onto particles contained in the state $|X\rangle$.

Next we consider the insertion terms in the semiconnected contribution, (B3). The term in (B3) containing a single insertion is evaluated just as we have evaluated (B2), but the double-insertion term [Fig. 2(b)] requires more care. The contribution of this term to (B3) is

$$\lim_{p,q\to 0}\frac{p^0q^0}{4F_\pi^2m_\pi}\bar{u}_{-1/2}(k_N,s)\Gamma^0_{5,-}(-p)\frac{\not{k}_N+\not{p}+m_N}{(k_N+p)^2-m_N^2}\Gamma^0_{5,+}(q)\frac{\not{k}_N+\not{p}-\not{q}+m_N}{(k_N+p-q)^2-m_N^2}\Gamma^3_-(k_X+k_N+p-q)\chi_{-1/2}(k_X)\left(\frac{m_Xm_N}{k_X^0k_N^0}\right)^{1/2}, \qquad \text{(B9)}$$

and we must evaluate the limit

$$\lim_{p,q\to 0}\left(p^0q^0\frac{\not{k}_N+\not{p}+m_N}{(k_N+p)^2-m_N^2}\Gamma^0_{5,+}(q)\frac{\not{k}_N+\not{p}-\not{q}+m_N}{(k_N+p-q)^2-m_N^2}\right). \qquad \text{(B10)}$$

This limit does not exist; the value of (B10) depends on the path in (p,q) space along which we approach the origin. In particular, if $x=dp^0/dq^0|_{p^0=q^0=0}$ is the slope at the origin, then for (B10) we find the x-dependent expression

$$\frac{1}{x-1}\frac{\not{k}_N+m_N}{2k_N^0}\Gamma^0_{5,+}(0)\frac{\not{k}_N+m_N}{2k_N^0}. \qquad \text{(B11)}$$

In evaluating matrix elements with several soft pions, Eqs. (3.11) and (4.14), we emphasized the need to calculate Bose-symmetric Ward identities, (3.12) and (4.15). The same remark applies in this instance; an arbitrary approach to the origin in (p,q) space clearly fails to maintain the Bose symmetry of the two pions. But if we choose an arbitrary path and take its average with the result obtained by interchanging p and q (i.e., the path obtained by reflecting the original path about the axis $p=q$), then the result is Bose-symmetric. This gives the (B12) prescription

$$\frac{1}{x-1}\to\frac{1}{2}\left(\frac{1}{x-1}+\frac{1}{(1/x)-1}\right)=-\frac{1}{2} \qquad \text{(B12)}$$

so that the expression in (B10) is equal to

$$-\frac{1}{2}\frac{\not{k}_N+m_N}{2k_N^0}\Gamma^0_{5,+}(0)\frac{\not{k}_N+m_N}{2k_N^0}, \qquad \text{(B13)}$$

and the double-insertion term, (B9), is

$$-\frac{1}{8F_\pi^2m_\pi}|\langle N_{-1/2}(\vec{k},s)|A_-^0|N_{1/2}(\vec{k},s)\rangle|^2\langle N_{-1/2}(\vec{k},s)X_{-1/2}(-\vec{k})|\,V_-^3|\Omega\rangle. \qquad \text{(B14)}$$

With the insertions onto the external nucleon leg calculated, we rewrite (B3) as

$$\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})\pi_+(\vec{0})|\,V_\pm^3|\pi_+(\vec{0})\rangle=\frac{-1}{4F_\pi^2m_\pi}\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})|\,V_\pm^3|\Omega\rangle$$
$$\mp\frac{1}{2F_\pi^2m_\pi}\langle N_{\pm1/2}(\vec{k},s)|A_\pm^0|N_{\mp1/2}(\vec{k},s)\rangle\langle N_{\mp1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})|A_3^3|\Omega\rangle$$
$$-\frac{1}{8F_\pi^2m_\pi}|\langle N_{\pm1/2}(\vec{k},s)|A_\pm^0|N_{\mp1/2}(\vec{k},s)\rangle|^2\langle N_{\pm1/2}(\vec{k},s)X_{\pm1/2}(-\vec{k})|\,V_\pm^3|\Omega\rangle. \qquad \text{(B15)}$$

It is now trivial to substitute (B8) and (B15) into (B1) and to see that the contributions due to insertions onto the external nucleon line cancel among themselves, for both $\eta=1$ and $\eta=-1$. The cancellation gives strong support for our use of Bose symmetry in (B12), since the (B12) prescription resolved the ambiguity of the double-insertion terms in just the right way for them to cancel with the unambiguous single-insertion terms. If the cancellation had not taken place, then in Sec. IV we would have contradicted the first spectral-function sum rule.

To generalize this example, we must consider insertions onto particles of arbitrary spin and isospin. The generalization to arbitrary isospin increases bookkeeping problems, but it is straightforward to see that the cancellation occurs as in our example. To generalize to arbitrary spin, we use Weinberg's treatment of perturbation theory for arbitrary spin.[27] In particular, we need only observe that the definitions (B5) are easily generalized and that the property of the spin-$\frac{1}{2}$ propagator used in (B6) is true for any spin, i.e., the propagator for arbitrary spin is just the usual scalar propagator multiplied by the appropriate spin projection operator.

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‡Address after 1 September 1971: Stanford Linear Accelerator Center, P.O. Box 4349, Stanford, Calif. 94305.

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Using the Wigner-Eckart theorem, it is easy to verify that these contributions do not cancel those from (2.15).

[13]We can anticipate the vanishing of the nucleon pair contribution in (2.13) by the following rule: Pair contributions in one term of the commutator correspond to connected contributions in the other term. In (2.13), the first term of the commutator has (except for the neutron) only expected connected contributions, $J^P = \frac{3}{2}^-$, so the cross term of the commutator has only expected pair contributions, $N\bar{N}^*$, where N^* has $J^P = \frac{3}{2}^+$.

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