$$T^{\omega}_{J(22)}=\frac{q_\omega{}^{2J}}{\pi}\int_{(m_\omega+1)^2}^{\infty}\frac{dt'}{t'-t}q_\omega'{}^{2J+1}\sqrt{t'}\,|U(t',t)|^2\,. \tag{A7}$$

Needless to say, although Figs. 4(a), 5(a), and 5(c) could be evaluated quite readily by the LTBS approximation, they could also be evaluated by the technique of the preceding paragraph. This then gives contributions

$$T^{2(\pi)}_{J}=\frac{\nu_\pi{}^{J}}{\pi}\int_4^{\infty}\frac{dt'}{t'-t}\frac{|V_J^{(\pi)}(q_\pi'^2,q_\pi'^2,t)|^2}{q_\pi'^{2J-1}\sqrt{t'}}\,, \tag{A8}$$

$$T^{\pi}_{J(12)}=\frac{(q_\pi q_\omega)^J}{\pi}\times\int_4^{\infty}\frac{dt'}{t'-t}\frac{V_J^{(\pi)}(q_\pi'^2,q_\pi'^2,t)V_J^{\omega}(q_\pi'^2,q_\omega'^2,t)}{q_\pi'{}^{J-1}q_\omega'{}^{J}\sqrt{t'}}\,, \tag{A9}$$

and

$$T^{\pi}_{J(22)}=\frac{q_\omega{}^{2J}}{\pi}\int_4^{\infty}\frac{dt'}{t'-t}q_\pi'\frac{|V_J^{\omega}(q_\pi'^2;q_\omega'^2,t)|^2}{q_\omega'{}^{2J}\sqrt{t'}} \tag{A10}$$

instead of Eqs. (22), (A1), and (A4). The latter equations are probably more accurate.

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Operator Formulation of a Dual Multiparticle Theory with Nonlinear Trajectories*

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An operator formalism for a dual-resonance theory with nonlinear trajectories is presented and an explicit, factorized operator representation is obtained for the N-point function. Operator Ward identities are also given.

In a previous paper,[1] we considered a meromorphic dual N-point Born term B_N with poles at energies $=s_l{}^{1/2}$ given by

$$1+(1-q)(a+bs_l)=q^{-l}\,. \tag{1}$$

a and b are constants and q is a parameter between zero and unity whose value determines the degree of nonlinearity of the trajectories.[2] In the limit $q\to 1$, the trajectories become linear and $B_N\to V_N$, the Veneziano N-point amplitude.[3]

In this note we present an operator representation of B_N from which it follows that B_N possesses the necessary factorization properties to be a satisfactory multiparticle Born term. From this representation the degeneracy of the lth level is determined to be $\leq 6^l$. An N-particle amplitude with two particles in excited levels can then be written as the matrix element of an operator Born term B_N^{op} between states representing particles in these excited levels. The excited states are generated by a *single* set of six *noncommuting* creation operators $a_\alpha^\dagger$, $\alpha=0, 1, 2, 3, 4, 5$ acting on a vacuum state. These operators and their adjoints a^α satisfy the following commutation relations[4]:

$$a^\alpha a_\beta^\dagger = q a_\beta^\dagger a^\alpha + (1-q)\delta_\beta^\alpha . \tag{2}$$

The 6^l independent states with energy $= s_l^{1/2}$ given by Eq. (1) are written as

$$|\alpha_1 \cdots \alpha_l\rangle \equiv a_{\alpha_l}^\dagger a_{\alpha_{l-1}}^\dagger \cdots a_{\alpha_1}^\dagger |0\rangle , \tag{3}$$

where the state $|0\rangle$ satisfies

$$a^\alpha |0\rangle = 0 , \quad \langle 0|0\rangle = 1 . \tag{4}$$

6^l independent states are generated by (3) because any set of indices $\alpha_1 \cdots \alpha_l$ produces an independent operator acting on $|0\rangle$. Thus, although B_N includes V_N as a special limiting case, B_N can be constructed from six operators, whereas if one starts directly from V_N, as was done by Fubini, Gordon, and Veneziano,[5] one is naturally led to introduce an infinite set of ordinary commuting operators satisfying the usual harmonic-oscillator commutation relations. In the general $q \neq 1$ theory, the single six-dimensional "generalized oscillator" plays a role equivalent to that of the infinite set of four-dimensional oscillators in the special $q=1$ Veneziano limit. This fact suggests that there is an underlying subtle organization to the usual infinite collection of Veneziano oscillator modes. Thus the simple operator structure which arises from the commutation relations (2) should be of interest also from the point of view of the Veneziano limit. We also note that in the proof given below, no restriction is put on the momenta p_i of the external lines in the N-point function B_N and hence factorization is demonstrated for a general off-mass-shell amplitude. In fact, there is essentially no difference between factorization on or off the mass shell in distinct contrast with the usual factorization procedures[6] in the generalized Veneziano model.

We begin by letting p_i, $i=1, \ldots, N$ denote the external momenta in the N-point amplitude. A typical graph for B_N is depicted in Fig. 1. The totality of planar tree graphs with the ordering of the momenta p_i given in Fig. 1 has poles in the $\frac{1}{2}N(N-3)$ variables s_{ij} defined by

$$s_{ij} \equiv (p_i + p_{i+1} + \cdots + p_j)^2 ,$$

where $1 \leq i \leq N-2$, $i+1 \leq j \leq N-1$ except for $i=1$, $j=N-1$. The positions of the poles are given by Eq. (1). B_N is then defined by the following multiple power series in s_{ij}:

$$B_N = \sum_{n_{ij}=0}^{\infty} \left(\prod_{j=2}^{N-2} \frac{(\sigma_{1j})^{n_{1j}}}{f_{n_{1j}}} \right) \left(\prod_{i=2}^{N-2} \prod_{j=i+1}^{N-1} \frac{(\sigma_{ij})^{n_{ij}}}{f_{n_{ij}}} \right) \left(\prod_{ijkl} q^{n_{ij}n_{kl}} \right) \tag{5}$$

where

$$f_n = (1-q)(1-q^2) \cdots (1-q^n) , \quad n>0 , \quad f_0 = 1 ,$$

and

$$\sigma_{ij} = 1 + (1-q)(a + b s_{ij}) , \tag{6}$$

and the product $\prod_{ijkl} q^{n_{ij}n_{kl}}$ runs over all indices $ijkl$ in the range $(1 \leq i < k \leq j < l \leq N-1)$. These inequalities mean that "duality factors," $q^{n_{ij}n_{kl}}$, are included for indices corresponding to overlapping variables σ_{ij} and σ_{kl}. The power series Eq. (5) involves $\frac{1}{2}N(N-3)$ σ_{ij} variables and summations. It converges when all the variables satisfy $|\sigma_{ij}| < 1$. The poles in the variable s_{ij} become explicit when we sum over the index n_{ij} as can be seen from the formula[1]

$$\sum_{n=0}^{\infty} \frac{\sigma^n}{f_n} = \prod_{j=0}^{\infty} \frac{1}{1-\sigma q^j} . \tag{7}$$

The duality factors $q^{n_{ij}n_{kl}}$ prevent simultaneous poles in B_N in the overlapping variables.[1]

To factorize B_N we first express σ_{ij} for $2 \leq i < j \leq N-1$ in terms of the $N-1$ independent momenta

$$\overline{Q}_i \equiv \sum_{k=1}^{i-1} p_k \tag{8}$$

of the multiperipheral configuration of Fig. 1. We define six-component quantities $Q_{i\alpha}$ and $Q_i'^{\alpha}$ as follows:

$$\left.\begin{aligned} Q_{i\mu} &= [2(1-q)b]^{1/2} \overline{Q}_{i\mu} \\ Q_i'^{\mu} &= -[2(1-q)b]^{1/2} \overline{Q}_i^{\mu} \end{aligned}\right\} , \quad \mu = 0, 1, 2, 3$$

$$Q_i'^4 = Q_{i5} = 1 ,$$

$$Q_i'^5 = Q_{i4} = \tfrac{1}{2}[1 + (1-q)(a + 2b\overline{Q}_i^{\,2})] . \tag{9}$$

We can then write

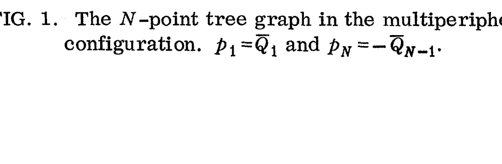

FIG. 1. The N-point tree graph in the multiperipheral configuration. $p_1 = \overline{Q}_1$ and $p_N = -\overline{Q}_{N-1}$.

$$\sigma_{ij} = 1 + (1-q)[a + b(\overline{Q}_j - \overline{Q}_{i-1})^2] = Q_{i-1} \cdot Q'_j , \qquad (10)$$

where we have used the notation

$$Q_{i-1} \cdot Q'_j = \sum_{\alpha=0}^{5} Q_{i-1\,\alpha} Q_j'^{\alpha} . \qquad (11)$$

Equation (10) expresses the variable σ_{ij} as a "dot product" of two "six-vectors" Q_{i-1} and Q'_j, where Q_{i-1} depends only upon external momenta to the left of p_i, and Q'_j (by momentum conservation) depends only upon momenta to the right of p_j. [In the remainder of this article all dot products are six-dimensional as in Eq. (11).]

We now use Eq. (2) to define a set of six operators a^α and their adjoints $a^\dagger_\alpha$. No commutation relation between a^α and a^β is postulated. We also define an inverse Hermitian Hamiltonian operator H^{-1} by the following commutation relations with the operators a^α and $a^\dagger_\alpha$:

$$a^\alpha H^{-1} = qH^{-1}a^\alpha , \quad H^{-1}a^\dagger_\alpha = qa^\dagger_\alpha H^{-1} , \qquad (12)$$

and the condition

$$H^{-1}|0\rangle = |0\rangle , \qquad (13)$$

where $|0\rangle$ is defined by Eq. (4). From Eqs. (3), (12), and (13) it follows that

$$H^{-1}|\alpha_1 \cdots \alpha_l\rangle = q^l |\alpha_1 \cdots \alpha_l\rangle . \qquad (14)$$

We now define an operator N-point function B_N^{op} by the following rules:

(i) We associate with each vertex in Fig. 1 the vertex operator

$$V(Q_{i-1}, Q_i) \equiv \sum_{n=0;\, m=0}^{\infty} \frac{(Q'_i \cdot a^\dagger)^n}{f_n} q^{nm} \frac{(Q_{i-1} \cdot a)^m}{f_m} . \qquad (15)$$

(ii) We associate with the ith internal line the propagator

$$P(Q_i) \equiv \sum_{n_{1i}=0}^{\infty} \frac{(\sigma_{1i}H^{-1})^{n_{1i}}}{f_{n_{1i}}} . \qquad (16)$$

The operator N-point function B_N^{op} is then defined as follows:

$$B_N^{\text{op}}(Q_1 \cdots Q_{N-1}) \equiv V(Q_1 Q_2) P(Q_2) V(Q_2 Q_3) \times P(Q_3) \cdots P(Q_{N-2}) V(Q_{N-2} Q_{N-1}) . \qquad (17)$$

The fundamental result is the following factorized operator representation for B_N of Eq. (5):

$$B_N = \langle 0 | B_N^{\text{op}}(Q_1 \cdots Q_{N-1}) | 0 \rangle . \qquad (18)$$

In order to prove Eq. (18) we commute all the $a^\dagger_\alpha$ to the left and all the a^α to the right of the H^{-1} operators in the middle of Eq. (17), using the commutation relations (2) and (12). No commutation relation between a^α and a^β is needed in order to carry out this operation, just as no such commutation relationship is needed in the corresponding work of Fubini, Gordon, and Veneziano and of Nambu.[5] Since B_N^{op} involves products of the form $(Q_i \cdot a)^m (Q'_j \cdot a^\dagger)^n$ the above reduction of B_N^{op} can be carried out by repeated application of the formula

$$\frac{(Q_i \cdot a)^m}{f_m} \frac{(Q'_j \cdot a^\dagger)^n}{f_n} = \sum_{l=0}^{\text{Min}(m,n)} \frac{(Q_i \cdot Q'_j)^l}{f_l} \frac{(Q'_j \cdot a^\dagger)^{n-l}}{f_{n-l}} q^{(n-l)(m-l)} \frac{(Q_i \cdot a)^{m-l}}{f_{m-l}} . \qquad (19)$$

Equation (19) is a direct consequence of Eq. (2), as is easily verified by induction. The only other products which appear in B_N^{op} are of the form $(\sigma_{1i}H^{-1})^n (Q'_j \cdot a^\dagger)^m$ and $(Q_j \cdot a)^m (\sigma_{1i}H^{-1})^n$. From Eq. (12) these products can be rewritten in reverse order multiplied by an additional factor q^{nm}. For example, repeated application of Eq. (12) yields

$$(Q \cdot a)^m H^{-n} = q^{nm} H^{-n} (Q \cdot a)^m . \qquad (20)$$

The above rearrangement of the operators in B_N^{op} introduces additional q^{nm} factors via Eqs. (19) and (20), and $\frac{1}{2}(N-2)(N-3)$ additional sums because Eq. (19) must be applied $1+2+\cdots+N-3 = \frac{1}{2}(N-2)(N-3)$ times. Each application of Eq. (19) introduces a subenergy variable $Q_{i-1} \cdot Q'_j = \sigma_{ij}$, $2 \le i < j \le N-1$. The $N-3$ momentum-transfer variables σ_{1i}, $2 \le i \le N-2$, already appear in the propagators $P(Q_i)$. Thus all of the $\frac{1}{2}N(N-3)$ σ_{ij} variables of Eq. (5) appear explicitly in B_N^{op}. Our reordering of the operators in B_N^{op} leaves us with a multiple power series in σ_{ij}, $Q_i \cdot a$, and $Q'_j \cdot a^\dagger$ which takes on the explicit form

$$B_N^{\text{op}} = \sum \frac{(Q'_2 \cdot a^\dagger)^{n_2}}{f_{n_2}} \frac{(Q'_3 \cdot a^\dagger)^{n_3}}{f_{n_3}} \cdots \frac{(Q'_{N-1} \cdot a^\dagger)^{n_{N-1}}}{f_{n_{N-1}}} \prod_{k \ge l \ge 2}^{N-1} q^{n_k m_l} \left(\prod_{j=2}^{N-2} \frac{(\sigma_{1j} H^{-1} q^{m_2 + \cdots + m_j + n_{j+1} + \cdots + n_{N-1}})^{n_{1j}}}{f_{n_{1j}}} \right)$$
$$\times \left(\prod_{i=2}^{N-2} \prod_{j=i+1}^{N-1} \frac{(\sigma_{ij} q^{n_i + \cdots + n_{j-1} + m_{i+1} + \cdots + m_j})^{n_{ij}}}{f_{n_{ij}}} \right) \left(\prod_{ijkl} q^{n_{ij} n_{kl}} \right) \frac{(Q_1 \cdot a)^{m_2}}{f_{m_2}} \frac{(Q_2 \cdot a)^{m_3}}{f_{m_3}} \cdots \frac{(Q_{N-2} \cdot a)^{m_{N-1}}}{f_{m_{N-1}}} , \qquad (21)$$

where $\prod_{ijkl} q^{n_{ij} n_{kl}}$ is the factor appearing in Eq. (5). In Eq. (21) all the indices n_{ij}, n_i, and m_i are summed over integers from zero to infinity. The indices n_i and m_i arise from the indices which originally appeared in $V(Q_{i-1}, Q)$. The explicit form (21) of B_N^{op} can be straightforwardly verified by using induction on N. If we then evaluate $\langle 0|B_N^{\text{op}}|0\rangle$ using Eq. (4) as well as the adjoint equation $\langle 0|\alpha_\alpha^\dagger=0$, we find that the only terms which contribute in Eq. (21) have $n_i=m_j=0$ and we are left with the desired result (18).

An N-point amplitude containing one or two particles with high spin will involve matrix elements of B_N^{op} between the states $|\alpha_1\cdots\alpha_l\rangle$, eigenstates of H^{-1} with eigenvalues q^l, Eq. (14). Using the commutation relations (2), we obtain the following values for the scalar products of these states and their adjoint states:

$$\langle\alpha_1\cdots\alpha_l|\beta_1\cdots\beta_j\rangle=\delta_{jl}N^{\alpha_1\cdots\alpha_l}_{\beta_1\cdots\beta_l}, \tag{22}$$

where

$$\langle\alpha_1\cdots\alpha_l|\equiv\langle 0|a^{\alpha_1}a^{\alpha_2}\cdots a^{\alpha_l} \tag{23}$$

and

$$N^{\alpha_1\cdots\alpha_l}_{\beta_1\cdots\beta_l}=\sum_P q^{T(P)}\delta^{\alpha_1}_{\beta_{P(1)}}\delta^{\alpha_2}_{\beta_{P(2)}}\cdots\delta^{\alpha_l}_{\beta_{P(l)}}. \tag{24}$$

Here the $\sum_P$ goes over all $l!$ permutations of the integers 1 to l and $T(P)$ is the number of transpositions associated with the permutation P. The 6^l-by-6^l matrix N is in general a nonsingular matrix since its determinant is a polynomial in q with a nonvanishing constant term.[7] The existence of the matrix N^{-1} allows us to define $|\beta_1\widetilde{\cdots}\beta_l\rangle$ by the equation

$$|\beta_1\widetilde{\cdots}\beta_l\rangle=N^{-1\,\alpha_1\cdots\alpha_l}_{\beta_1\cdots\beta_l}|\alpha_1\cdots\alpha_l\rangle. \tag{25}$$

From Eqs. (25) and (22) we have the orthonormality relation

$$\langle\alpha_1\cdots\alpha_j|\beta_1\widetilde{\cdots}\beta_l\rangle=\delta^{\alpha_1}_{\beta_1}\cdots\delta^{\alpha_l}_{\beta_l}\delta_{jl}. \tag{26}$$

The unit operator 1 can then be written

$$1=\sum_{l=0}^{\infty}|\alpha_1\widetilde{\cdots}\alpha_l\rangle\langle\alpha_1\cdots\alpha_l|, \tag{27}$$

where summation over α_i is understood. We insert this completeness relation (27) on both sides of $P(Q_i)$ in our factorized form (17) for B_N^{op} and use the fact that $\langle\alpha_1\cdots\alpha_l|$ is an eigenstate of H^{-1}. This leads to the result

$$B_n=\sum_{l=0}^{\infty}\langle 0|B_{i+1}^{\text{op}}|\alpha_1\widetilde{\cdots}\alpha_l\rangle\times\left(\sum_{n_{1i}=0}^{\infty}\frac{(\sigma_{1i}q^l)^{n_{1i}}}{f_{n_{1i}}}\right)\langle\alpha_1\cdots\alpha_l|B_{N-i+1}^{\text{op}}|0\rangle, \tag{28}$$

where B_{i+1}^{op} is the operator representation of the $(i+1)$-point function which depends on momenta $Q_1,\ldots,Q_i$, and B_{N-i+1}^{op} depends on the momenta $Q_i,\ldots,Q_{N-1}$. Using Eq. (8) and Eq. (25) we can rewrite Eq. (28) in the form

$$B_N=\sum_{l=0}^{\infty}\langle 0|B_{i+1}^{\text{op}}|\alpha_1\widetilde{\cdots}\alpha_l\rangle\times\frac{N^{\alpha_1\cdots\alpha_l}_{\beta_1\cdots\beta_l}}{\prod_{j=0}^{\infty}(1-\sigma_{1i}q^{l+j})}\langle\beta_1\widetilde{\cdots}\beta_l|B_{N-i+1}^{\text{op}}|0\rangle. \tag{29}$$

Finally if we insert the expression (27) for the unit operator into our amplitude $\langle 0|B_N^{\text{op}}|\alpha_1\widetilde{\cdots}\alpha_l\rangle$, we conclude that $\langle\beta_1\widetilde{\cdots}\beta_j|B_N^{\text{op}}|\alpha_1\widetilde{\cdots}\alpha_l\rangle$ is the N-point function when the first and the Nth particles of the multiperipheral configuration (Fig. 1) are in excited states $\{\beta\}$ and $\{\alpha\}$. This function can easily be evaluated explicitly from the formula (21) for B_N^{op}.

The factorized expression for B_N is valid for all values of the four-momenta $p_1\cdots p_N$. The denominator $\prod_{j=0}^{\infty}(1-\sigma_{1i}q^{l+j})$ vanishes when

$$\sigma_{1i}=q^{-(l+j)},\quad j=0,1,2,\ldots \tag{30}$$

which correspond to points in the mass spectrum of Eq. (1). The lth term in the sum contributes to all states with "energies" $\sigma_{1i}\geq q^{-l}$. Thus the residue of the pole in B_N at $\sigma_{1i}=q^{-L}$ will receive contributions from all terms in the sum (29) for which $l\leq L$. Since there are 6^l states $|\alpha_1\widetilde{\cdots}\alpha_l\rangle$ for each l, that means that the degeneracy at the Lth pole is at most $6^L+6^{L-1}+\cdots 6+1=\frac{1}{5}(6^{L+1}-1)$, independent of the number of external lines. The terms in Eq. (29) which contribute to the maximum angular momentum (L) at the pole $\sigma_{1i}=q^{-L}$ are terms with the maximum number of spatial tensor indices; for example, the term with $l=L$ and $\alpha_1=\alpha_2=\cdots=\alpha_l=3$. For the set of indices $\alpha_i=3$, the only nonvanishing component of N has all the $\beta_j=3$ and the coefficient of this component of N has the value $\sum_P q^{T(P)}>0$. Thus, we see that this highest-spin state, which is obviously nondegenerate, is a state of positive norm. However, states with negative norm have been found among the degenerate daughter states with spin $l=L-1$.

We now show that the apparent degeneracy of the Lth level can be reduced because the amplitudes $\langle 0|B_N^{\text{op}}|\alpha_1\cdots\alpha_l\rangle$ are not all linearly independent when the $(N-1)$th particle in Fig. 1 is on its mass shell with $(\text{mass})^2=p_{N-1}{}^2=s_0$, determined from Eq. (1). Equations (10) and (1) with $i=j=N-1$ then yield

$$Q_{N-2}\cdot Q'_{N-1}=1. \tag{31}$$

With the restriction (31) the vertex $V(Q_{N-2},Q_{N-1})$ satisfies the "operator Ward identity"

$$V(Q_{N-2},Q_{N-1})=V(Q_{N-2},Q_{N-1})(Q'_{N-1}\cdot a^\dagger). \tag{32}$$

Equation (32) is easily derived from the definition

TABLE I. Summary of "Feynman rules" of operator formalisms for dual theories.

	Nonlinear $q \neq 1$	Veneziano $q = 1$
OJJ' vertex	$\sum_{n,m=0}^{\infty} \frac{(p\cdot a^\dagger)^n}{f_n} q^{nm} \frac{(k\cdot a)^m}{f_m}$	$\prod_{n=1}^{\infty} \exp\left(\frac{(p+k)^\mu c_\mu^{\dagger(n)}}{\sqrt{n}}\right) \exp\left(-\frac{(p+k)^\mu c_\mu^{(n)}}{\sqrt{n}}\right)$
Propagator	$\sum_{n=0}^{\infty} \frac{(q^{-\alpha}H^{-1})^n}{f_n}$	$\int_0^1 dx\, x^{-\alpha+H_V-1}(1-x)^{\alpha(0)-1}$ where $H_V = \sum_{n=1}^{\infty} n c_\mu^{\dagger(n)} c^{\mu(n)}$
External lines	$\lvert\alpha_1 \cdots \alpha_l\rangle$ orthonormal states defined in the text	$\prod_{n,\mu} \lvert I_{n,\mu}\rangle$ harmonic-oscillator occupation-number states

(15) of V and the commutation relations (19). If we insert (32) into our expression (17) for $B_N^{\rm op}$, we obtain

$$\langle 0|\, B_N^{\rm op} |\, \alpha_1 \cdots \alpha_{l-1}\rangle = (Q'_{N-1})^{\alpha_l} \langle 0|\, B_N^{\rm op} |\, \alpha_1 \cdots \alpha_{l-1}\alpha_l\rangle\,. \tag{33}$$

Equation (33) is valid for arbitrary values of the spin indices $\alpha_1 \cdots \alpha_{l-1}$. Hence it gives us 6^{l-1} conditions on the amplitude with $N-1$ scalar particles and one higher-spin particle, provided only that the spin-zero particle which is adjacent to the spinning particle is on the mass shell. These linear dependence relations, which reduce the degeneracy of the lth level to 6^l, can be used as in the case of the Veneziano model to eliminate some of the ghosts among daughter poles. However, because of the much greater degeneracy of the lth level for $q \neq 1$, many more ghosts are undoubtedly present than in the $q=1$ limit. Understanding what should or could be done about them is a much more serious problem in the $q<1$ theory than in the Veneziano limit.

In conclusion we present a brief table (Table I) which illustrates the correspondence between the states and operators of the nonlinear and the Veneziano theories. In the table the $c_\mu^{(n)}$, $n = 1, 2, \ldots, \infty$ refer to the infinite set of four-dimensional harmonic-oscillator operators[5] of the Veneziano model. Because of the obvious economy of the single set of six-dimensional operators, it is hoped that they may also be of use in the $q=1$ Veneziano theory. As it stands, the $q<1$ operator formalism is not directly applicable in the limit $q \to 1$. Of course, the limit can be taken after matrix elements have been evaluated.

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