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One-Loop Renormalization of the Nonlinear σ Model

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We propose a method for regularizing and renormalizing the nonlinear σ model based on the limit of the linear σ model when the σ mass tends to infinity. The linear σ model being renormalized using Ward identities, this procedure ensures automatically the soft-pion theorems. It also retains the properties of transformations of the Green's functions under the chiral group. In the one-loop approximation, we show that the pion propagator has a finite limit when the σ mass tends to infinity, while the four-pion Green's function diverges only logarithmically with the σ mass.

I. INTRODUCTION

The problem of the renormalization of perturbation series in the theory of broken symmetries is an important problem in strong-interaction physics. One of the oldest cases where this situation occurs is SU(2) symmetry broken by electromagnetic interactions. This problem is not yet solved in the sense that we do not know how to express a certain number of physical quantities in a finite way in terms of the symmetrical quantities. For instance, we do not know how to compute electromagnetic differences of masses.

In the case of the chiral symmetry SU(2)×SU(2), recent progress has been made. First Lee[1] showed how to renormalize the linear σ model whose Lagrangian reads

$$\mathcal{L}=\tfrac{1}{2}[(\partial_\mu\sigma)^2+(\partial_\mu\vec{\pi})^2]-\tfrac{1}{2}\mu_0{}^2(\sigma^2+\vec{\pi}^2)-\tfrac{1}{4}\lambda_0(\sigma^2+\pi^2)^2+c_0\sigma. \tag{1.1}$$

Symanzik[2] then solved in a more systematic way the problem of the renormalization of theories in which the symmetry is broken in the Lagrangian by terms linear in the fields, making an extensive use of the Ward identities. He also generalized his method for breaking terms quadratic or cubic in the fields. In Symanzik's method, it makes sense to speak of broken symmetries only as long as the breaking terms in the Lagrangian are less singular than the symmetric terms, from the point of view of power counting in the perturbation series.

In all the preceding cases, we have to deal with Lagrangians which are renormalizable in the ordinary sense, and the fields are linear representations of the symmetry group. However, in the case of SU(2)×SU(2) chiral symmetry, a very popular idea is that the pion transforms nonlinearly under the action of the chiral group. Moreover, certain attempts to use the linear σ model to compute πN phase shifts have not given very satisfying results. (References on calculation with the Padé method can be found in Ref. 3).

Therefore it is interesting to study nonlinear realizations of the chiral symmetry. A possible model is the nonlinear σ model[4] which also fulfills the requirements of current algebra, but in which the σ field is eliminated through the condition

$$\sigma^2(x)+\vec{\pi}^2(x)=F_0{}^2. \tag{1.2}$$

The Lagrangian with pions only becomes

$$\mathcal{L}=\frac{1}{2}\left[(\partial_\mu\vec{\pi})^2+\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{F_0{}^2-\vec{\pi}^2}\right]+C_0[F_0{}^2-\vec{\pi}^2(x)]^{1/2}. \tag{1.3}$$

In this case we are faced with a completely different problem. Because the Lagrangian is not renormalizable in the usual sense, and because the symmetry acts nonlinearly on the pion field, the problem of defining a finite perturbation series, even in the limit of the exact symmetry ($c=0$) without destroying the symmetry properties is still open. Even by the "superpropagator" methods[5,6] it does not seem easy to solve this problem.[7]

If one expands the perturbation series in power of a parameter associated with the number of loops (here $1/F^2$), considering functions of $F^2-\vec{\pi}^2(x)$ as equivalent to their Taylor series expansions, then even the renormalization of the one-loop graphs in the symmetric case is not obvious, even if one is careful in the definition of the Feynman rules.[8]

If one takes the point of view of the Feynman integral, which generates the time-ordered Green's functions, it is important to integrate with an invariant measure.[8]

A convenient way to write the functional generating the Green's functions is to introduce an auxilliary field $\sigma(x)$ and to make explicit the constraint (1.2):

$$G=\int \delta(\vec{\pi}^2+\sigma^2-F_0{}^2)d\vec{\pi}d\sigma \exp\left\{i\int[\tfrac{1}{2}(\partial_\mu\vec{\pi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2+c_0\sigma]d^4x\right\}; \tag{1.4}$$

$\delta(\vec{\pi}^2+\sigma^2-F_0{}^2)$ can be formally rewritten introducing a new field $\alpha(x)$:

$$G=\int d\vec{\pi}d\sigma d\alpha \exp\left\{i\int[\tfrac{1}{2}(\partial_\mu\vec{\pi})^2+\tfrac{1}{2}(\partial_\mu\sigma)^2+c_0\sigma+\alpha(\vec{\pi}^2+\sigma^2-F_0{}^2)]d^4x\right\}. \tag{1.5}$$

Now from this expression it is easy to recover the linear σ model. If we add a term $\alpha^2(x)/\lambda_0$ in the Lagrangian we obtain

$$G(\lambda_0)=\int d\vec{\pi}d\sigma d\alpha \exp\left\{i\int d^4x\left[\mathcal{L}(\vec{\pi},\sigma)+\alpha(\vec{\pi}^2+\sigma^2-F_0{}^2)+\frac{\alpha^2}{\lambda_0}\right]\right\}, \tag{1.6}$$

$$\mathcal{L}(\vec{\pi},\sigma)=\tfrac{1}{2}[(\partial_\mu\vec{\pi})^2+(\partial_\mu\sigma)^2]+c_0\sigma-\tfrac{1}{2}\mu_0{}^2(\vec{\pi}^2+\sigma^2). \tag{1.7}$$

If λ_0 goes to infinity we recover $G=G(\infty)$. If we integrate over the field $\alpha(x)$ we see that $G(\lambda)$ generates the Green's functions of the linear σ model,

$$G(\lambda_0)=\int d\sigma d\vec{\pi} \exp\left\{i\int d^4x[\mathcal{L}(\vec{\pi},\sigma)-\tfrac{1}{4}\lambda_0(\vec{\pi}^2+\sigma^2-F^2)^2]\right\}. \tag{1.8}$$

The fact that, if λ_0 tends towards infinity, the generating functional of the linear σ model has for a limit the generating functional of the nonlinear σ model, reflects the well-known fact[9] that the tree graphs of the linear σ model tend towards the tree graphs of the nonlinear σ model.

But it is possible now to renormalize completely the linear σ model[10] and give a finite sense to the perturbation series. Because the tree graphs of the linear σ model have the tree graphs of the nonlinear model for their limit, we can therefore consider the renormalized perturbation series of the nonlinear model. We have then to study the behavior of the series, in the linear model, when the coupling constant λ_0 goes to infinity. Divergent terms will appear. Because the model is symmetric for any value of λ_0, the divergent terms will be also symmetric, and we can compensate them by adding counterterms to the Lagrangian. Following those lines we have a method of regularizing and renormalizing at each finite order the nonlinear σ model.

We have made an explicit calculation for the pion propagator and the four-point function in the one-loop approximation. *A priori* one could think that terms will appear rising as powers of λ_0 or, equivalently, as powers of $M_\sigma{}^2$ (M_σ is the σ mass and is related by a Ward identity to λ). Actually the pion propagator has a finite limit and the pion-pion amplitude rises as $\ln M_\sigma{}^2$. Only one counterterm is needed at this order and only one renormalization constant occurs. Similar results will be published later for the σ model with pions and nucleons. We do not know if this result generalizes for an arbitrary number of loops, but in any case the method seems a promising method of renormalizing the nonlinear σ model. Also the method can be used for more general models.

We shall now discuss the infinite-mass limit in the one-loop approximation.

II. THE TREE APPROXIMATION

We first expand the nonlinear Lagrangian (1.3) in powers of the pion field and get

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\vec{\pi})^2-\frac{1}{2}\left(\frac{c_0}{F_0}\right)\vec{\pi}^2-\frac{1}{8}\left(\frac{c_0}{F_0}\right)\frac{\vec{\pi}^4}{F_0{}^2}+\frac{1}{2}\frac{(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{F_0{}^2}-\frac{1}{16}\left(\frac{c_0}{F_0}\right)\frac{\vec{\pi}^6}{F_0{}^4}+\frac{1}{2}\frac{\vec{\pi}^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{F_0{}^2}+O(\vec{\pi}^8). \tag{2.1}$$

We have expanded $\mathcal{L}$ nonlinear only up to the sixth degree in the pion fields because for a zero- or one-loop calculation in which we are interested, and for the two- and four-pion Green's functions, the other terms will not contribute.

Setting $c_0=\mu_0{}^2F_0$, where μ_0 is the unrenormalized pion mass, we replace the Lagrangian (2.1) by the effective Lagrangian

$$\mathcal{L}_{\text{nonlinear}}=\frac{1}{2}(\partial_\mu\vec{\pi})^2-\frac{1}{2}\mu_0{}^2\vec{\pi}^2-\frac{\mu_0{}^2}{8F_0{}^2}\vec{\pi}^4+\frac{1}{2F_0{}^2}(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2-\frac{\mu_0{}^2}{16F_0{}^4}\vec{\pi}^6+\frac{1}{2}\frac{\vec{\pi}^2(\vec{\pi}\cdot\partial_\mu\vec{\pi})^2}{F_0{}^2}. \tag{2.2}$$

We shall represent the pion or σ propagator by $D_\pi(s)$ or $D_\sigma(s)$ and call F the vacuum expectation value of the renormalized σ field. We shall represent the connected amputated four-pion Green's function by

$$T_{ijkl}(p_1,p_2,p_3,p_4)=\delta_{ij}\delta_{kl}A(p_1,p_2,p_3,p_4)+\delta_{ik}\delta_{jl}A(p_1,-p_3,-p_2,p_4)+\delta_{il}\delta_{jk}A(p_1,-p_4,p_3,-p_2), \tag{2.3}$$

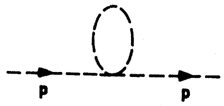

FIG. 1. One-loop contribution to the pion progagator in the nonlinear model.

FIG. 2. One-loop contribution to the four-pion Green's function in the nonlinear model.

the amplitude A being symmetric in the interchange $p_3 \leftrightarrow p_4$ and also $p_1 \leftrightarrow p_3$ simultaneously with $p_2 \leftrightarrow p_4$.

To distinguish between the Green's functions of the linear and nonlinear models, we shall put a bar on the second ones. We can immediately derive the tree approximation or Born terms for the two Lagrangians. For the linear model we obtain

$$iD_\pi^{-1}(s) = s - m_\pi^2, \tag{2.4}$$

$$iD_\sigma^{-1}(s) = s - \sigma^2 \text{ with } \sigma^2 \equiv M_\sigma^2 = -iD_\sigma^{-1}(0), \tag{2.5}$$

$$A(p_1, p_2, p_3, p_4) = -\frac{\sigma^2 - m_\pi^2}{F^2}\,\frac{s - m_\pi^2}{s - \sigma^2} \tag{2.6}$$

with

$$s = (p_1 + p_2)^2,$$

$$V_{\sigma\pi\pi}(p_1; p_2, p_3) = \frac{\sigma^2 - m_\pi^2}{F}, \tag{2.7}$$

while, for the nonlinear model, we have

$$i\bar{D}_\pi^{-1}(s) = s - m_\pi^2, \tag{2.8}$$

$$\bar{A}(p_1, p_2, p_3, p_4) = \frac{s - m_\pi^2}{F^2}, \tag{2.9}$$

with

$$s = (p_1 + p_2)^2.$$

We see immediately that the barred quantities are obtained from the previous ones by letting $\sigma \to \infty$.

III. ONE-LOOP CALCULATION OF THE NONLINEAR MODEL

For the pion propagator, we have only one graph represented in Fig. 1. An important point is that this kind of graph cannot be ignored as in usual renormalizable theory.[6] Actually here we shall find for it a well-defined finite value. This graph gives clearly a polynomial of the form

$$\bar{D}_\pi^{-1}(s) = \frac{\gamma}{16\pi^2 F^2}(s - m_\pi^2)^2, \tag{3.1}$$

where γ is to be fixed in some way.

For the four-pion Green's function we have the graphs represented in Fig. 2, plus the crossed graphs in the u and t channels. A straightforward calculation leads to

$$\bar{A}(s, t, u, p_1^2, p_2^2, p_3^2, p_4^2) = \frac{1}{16\pi^2 F^4}(\bar{A}_F + \bar{A}_{IF}), \tag{3.2}$$

where

$$\begin{aligned}\bar{A}_F = {} & \frac{I(s)}{2}(s - m_\pi^2)(2s + t + u - 3m_\pi^2) \\ & + \frac{I(t)}{3}\left[-\frac{st}{2} - tu - m_\pi^2(s-u) - \frac{m_\pi^2}{t}(p_1^2 - p_3^2)(p_2^2 - p_4^2) + (p_1^2p_2^2 + p_3^2p_4^2) + \tfrac{1}{2}(p_1^2p_4^2 + p_2^2p_3^2)\right] \\ & + \frac{I(u)}{3}\left[-\frac{su}{2} - tu - m_\pi^2(s-t) - \frac{m_\pi^2}{u}(p_1^2 - p_4^2)(p_2^2 - p_3^2) + (p_1^2p_2^2 + p_3^2p_4^2) + \tfrac{1}{2}(p_1^2p_3^2 + p_2^2p_4^2)\right].\end{aligned} \tag{3.3}$$

$I(s)$ is the Mandelstam function defined by

$$I(-s) = 2 - 2\left(\frac{4m_\pi^2 + s}{s}\right)^{1/2} \ln\frac{(s)^{1/2} + (4m_\pi^2 + s)^{1/2}}{2m_\pi}. \tag{3.4}$$

Here s, t, and u are the usual invariants.

The second piece $\bar{A}_{IF}$ can be written, taking into account the symmetry properties of $\bar{A}$, as

$$\bar{A}_{IF} = As^2 + B(t^2 + u^2) + Dtu + Cs(t+u) + Em_\pi^2 s + Fm_\pi^2(t+u) + Gm_\pi^4 + I(p_1^2 + p_2^2)(p_3^2 + p_4^2) + H(p_1^2p_2^2 + p_3^2p_4^2). \tag{3.5}$$

Of course only the "bubble" graph of Fig. 2 contributes to $\bar{A}_F$, while the other graph contributes only to the "infinite" part of $\bar{A}$, that is $\bar{A}_{IF}$.

These new nine parameters $A, B, \ldots, H$ as well as γ must be fixed in some way. We could think to use the relations among those parameters imposed by the Adler[11] and the Weinberg[12] conditions. However, these conditions are not sufficient to determine all the parameters. We propose to fix these constants by using the linear σ model as re-

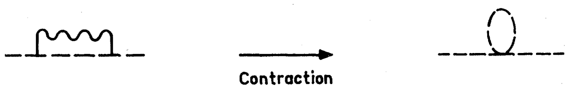

FIG. 3. Contraction of the propagator.

gulator of the nonlinear model, and obtain them from the limit of an infinite σ mass in the linear model.

IV. THE ONE-LOOP GRAPHS AND THEIR LIMIT IN THE LINEAR MODEL

To each graph of the linear model, we shall associate a graph of the nonlinear model, obtained from the previous one, by reducing to a point any internal σ propagator. We shall call such an operation a "contraction."

For the propagator we have only one graph (Fig. 3). For the four-pion Green's function we have the correspondence (Fig. 4) (we do not draw the graphs deduced by crossing lines). In taking the limit $\sigma^2 \to \infty$ of the linear σ model, important cancellations occur, principally between the one-particle irreducible graphs and the reducible ones. The result is the following (see Appendixes A and B for the demonstration):

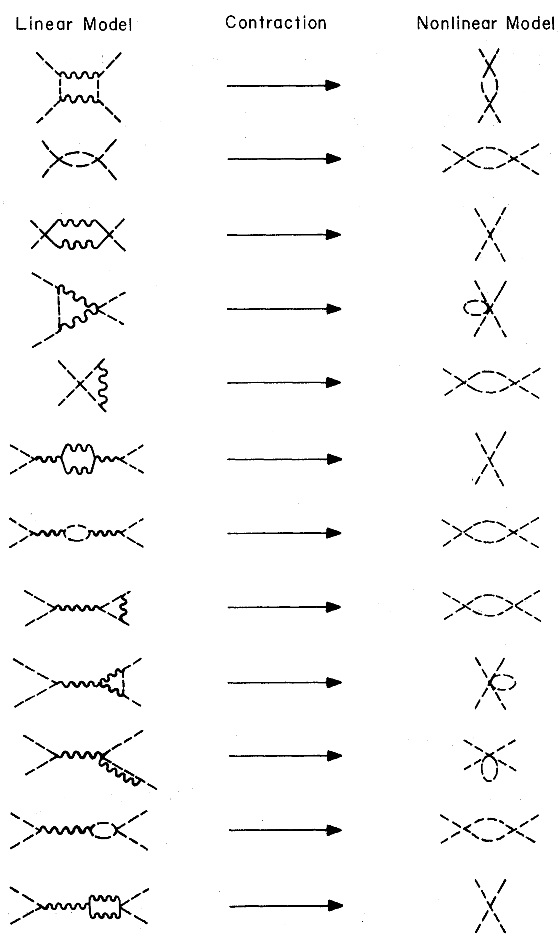

FIG. 4. Contractions of the four-pion Green's function.

$$\gamma = \tfrac{1}{6}, \tag{4.1}$$

$$A = \tfrac{7}{6} - \tfrac{1}{2}\ln(\sigma^2/m_\pi^2) - \tfrac{3}{2}I(-a^2), \tag{4.2}$$

$$B = 0, \tag{4.3}$$

$$D = \tfrac{16}{9} - \tfrac{2}{3}\ln(\sigma^2/m_\pi^2), \tag{4.4}$$

$$C = -\tfrac{5}{9} + \tfrac{1}{3}\ln(\sigma^2/m_\pi^2), \tag{4.5}$$

$$E = -\tfrac{4}{3} + \tfrac{1}{2}\ln(\sigma^2/m_\pi^2) + 3I(-a^2), \tag{4.6}$$

$$F = 1 - \tfrac{1}{2}\ln(\sigma^2/m_\pi^2), \tag{4.7}$$

$$G = \tfrac{1}{6} - \tfrac{3}{2}I(-a^2), \tag{4.8}$$

$$I = -\tfrac{4}{9} + \tfrac{1}{6}\ln(\sigma^2/m_\pi^2), \tag{4.9}$$

$$H = -\tfrac{16}{9} + \tfrac{2}{3}\ln(\sigma^2/m_\pi^2). \tag{4.10}$$

Before going forward we shall make two remarks: first, that the σ^4, $\sigma^2\ln\sigma^2$, and σ^2 terms are absent. This absence allows us to hope that it is possible to renormalize the model at all orders. Second, we remark that a "new" parameter $-a^2$ has appeared. $s = -a^2$ is the point where the σ propagator has been subtracted in the linear model. These terms involving $I(-a^2)$ are *absolutely* necessary if one wants to avoid infrared divergences when the pion mass goes to zero.

Now we shall give a simple way to reconstruct the subtraction polynomial $\overline{A}_{IF}$, as well as a simple explanation for the absence of the σ^4, $\sigma^2\ln\sigma^2$, and σ^2 terms.

V. SIMPLIFIED RECONSTRUCTION OF THE SUBTRACTION TERMS IN THE ONE-LOOP APPROXIMATION OF THE NONLINEAR MODEL

A. The Propagators

Since the pion and the σ propagators involve only simple functions in the one-loop approximation for the linear model, it is very easy to take the limit $\sigma \to \infty$. We set

$$i\overline{D}_\pi^{-1}(s) = s - m_\pi^2 - \overline{\Sigma}_\pi(s), \tag{5.1}$$

$$i\overline{D}_\sigma^{-1}(s) = s - \sigma^2 - \overline{\Sigma}_\sigma(s). \tag{5.2}$$

We recall that in the linear model, the σ propagator was defined using the convention of renormalization:

$$iD_\sigma^{-1}(-a^2) = -(\sigma^2 + a^2), \tag{5.3}$$

where a^2 is any positive number. (To avoid infrared divergences when the pion mass tends to zero, it is necessary that a^2 be different from zero.) In the nonlinear model, $-a^2$ represents the point where the connected two-point $\sigma(x)$ function vanishes when $\sigma(x)$ is replaced by $[F^2-\vec{\pi}^2(x)]^{1/2}$. Using the results of Appendix A, we find

$$\bar{\Sigma}_\pi(s)=-\frac{(s-m_\pi{}^2)^2}{96\pi^2F^2} \tag{5.4}$$

and

$$\bar{\Sigma}_\sigma(s)=\frac{3}{2}\,\frac{\sigma^4}{16\pi^2F^2}[I(s)-I(-a^2)]+O(\sigma^2). \tag{5.5}$$

In particular, using (5.4) we obtain the value of f_π:

$$f_\pi\equiv-\frac{F}{m_\pi{}^2}iD_\pi{}^{-1}(0)=F\left(1-\frac{m_\pi{}^2}{F^2}\,\frac{1}{96\pi^2}\right). \tag{5.6}$$

We see that the difference between f_π and F is extremely small.

B. The Ward Identities

We recall that for the linear model, we have two Ward identities (see Ref. 10), which read

$$FA(p_2{}^2,p_3{}^2,p_4{}^2,0,p_2{}^2,p_3{}^2,p_4{}^2)=-V_{\sigma\pi\pi}(p_2{}^2;p_3{}^2,p_4{}^2)\,\frac{iD_\pi{}^{-1}(p_2{}^2)}{iD_\sigma{}^{-1}(p_2{}^2)}, \tag{5.7}$$

$$FV_{\sigma\pi\pi}(q^2;q^2,0)=iD_\pi{}^{-1}(q^2)-iD_\sigma{}^{-1}(q^2). \tag{5.8}$$

By combining (5.7) and (5.8), we can derive the relation

$$F^2A(q^2,q^2,0,0,q^2,q^2,0)=iD_\pi{}^{-1}(q^2)-\frac{[iD_\pi{}^{-1}(q^2)]^2}{iD_\sigma{}^{-1}(q^2)}. \tag{5.9}$$

Setting $q^2=-a^2$ in (5.9) and letting $\sigma\to\infty$, we then obtain

$$F^2\bar{A}(-a^2,-a^2,0,0,-a^2,-a^2,0)=i\bar{D}_\pi{}^{-1}(-a^2). \tag{5.10}$$

Taking the derivative of (5.9) with respect to q^2, and setting $q^2=m_\pi{}^2$ we also recover the Weinberg conditions:

$$F^2\,\frac{d}{dq^2}A(q^2,q^2,0,0,q^2,q^2,0)\Big|_{q^2=m_\pi{}^2}=Z_\pi{}^{-1}=+1. \tag{5.11}$$

The Adler condition is obtained from (5.7) for $p_2{}^2=m_\pi{}^2$:

$$A(m_\pi{}^2,p_3{}^2,p_4{}^2,0,m_\pi{}^2,p_3{}^2,p_4{}^2)=0,\quad\forall\,p_3{}^2,p_4{}^2. \tag{5.12}$$

Being independent of the σ mass, the relations (5.11) and (5.12) remain true when $\sigma\to\infty$, and so we can write them with a bar on A.

Finally we shall use Eq. (5.7) for $p_2{}^2=0$. We have

$$FA(0,q^2,q^2,0,0,q^2,q^2)=-iD_\pi{}^{-1}(0)\,\frac{V_{\sigma\pi\pi}(0;q^2,q^2)}{iD_\sigma{}^{-1}(0)} \tag{5.13}$$

or in the one-loop approximation,

$$FA^{1\,\text{loop}}(0,s,s,0,0,s,s)=-\frac{m_\pi{}^2}{\sigma^2}V^{1\,\text{loop}}_{\sigma\pi\pi}(0;s,s)+\left[\frac{m_\pi{}^2}{\sigma^2}\Sigma^{1\,\text{loop}}_\pi(0)-\Sigma^{1\,\text{loop}}_\sigma(0)\right]\frac{(\sigma^2-m_\pi{}^2)}{\sigma^2F} \tag{5.14}$$

taking into account that

$$V^{0\,\text{loop}}_{\sigma\pi\pi}=(\sigma^2-m_\pi{}^2)/F; \tag{5.15}$$

we shall use Eq. (5.14) for $\sigma\to\infty$.

C. Application

Being independent of the σ mass, Eqs. (5.11) and (5.12) must be true separately for each term of the double expansion in the σ mass and the number of loops. On the other hand, the expansion of $A(s,t,u)$ at the one loop is of the form

$$\frac{\sigma^4}{F^4}\left(a_1+b_1\ln\frac{\sigma^2}{m_\pi{}^2}\right)+\frac{\sigma^2}{F^4}\left(a_2P_1+b_2Q_1\ln\frac{\sigma^2}{m_\pi{}^2}\right)+\frac{1}{F^4}\left(P_2+Q_2\ln\frac{\sigma^2}{m_\pi{}^2}\right)+\bar{A}_F(s,t,u)+C\left(\frac{\ln(\sigma^2/m_\pi{}^2)}{\sigma^2}\right). \tag{5.16}$$

By only dimensional considerations we see that a_1, b_1, a_2, b_2 are constants independent of the external momenta p_i, while P_1 and Q_1 are first-degree polynomials and P_2 and Q_2 are second-degree polynomials in the invariants s, t, u, and $p_i{}^2$. The reason for this is that the pion mass cannot occur in the denominators of a_1, a_2, b_1, b_2 (no infrared divergences).

P_1 and Q_1 are of the form

$$\alpha s+\beta(t+u)+\delta m_\pi{}^2. \tag{5.17}$$

The Adler condition then tells us that

$$a_1=b_1=0, \tag{5.18}$$

$$\beta=0 \text{ and } \alpha+\delta=0, \tag{5.19}$$

while the Weinberg condition states for the one-loop contribution

$$\alpha+\beta=0. \tag{5.20}$$

Therefore $\alpha=\beta=\delta=0$, and we are left with only the logarithmic and finite terms.

We now determine the polynomials P_2 and Q_2. They are best represented in the form (3.5). Applying the Adler condition, we derive the set of relations

$$B=0,\quad D+H=0,\quad F+I+C=0,\quad A+E+G=0. \tag{5.21}$$

The Weinberg condition implies

$$2(A+B+C+I)+E+F=0, \tag{5.22}$$

while the equation (5.10) gives

$$(A+B+C+I)a^4-(E+F)m_\pi^2a^2+Gm_\pi^4+(a^2+m_\pi^2)^2\tfrac{3}{2}I(-a^2)=\tfrac{1}{6}(a^2+m_\pi^2)^2. \tag{5.23}$$

Combining (5.21) – (5.23) we obtain a first system:

$$\begin{aligned}&B=0,\\&D+H=0,\\&E+F=-\tfrac{1}{3}+3I(-a^2),\\&A=-(E+G),\\&I+C=-F,\\&G=\tfrac{1}{6}-\tfrac{3}{2}I(-a^2).\end{aligned} \tag{5.24}$$

We shall now determine the value of F by using Eq. (5.14) for σ mass going to infinity. A short calculation (see Appendix C) gives finally the identity

$$(D+H)s^2+2Fm_\pi^2s+Gm_\pi^4\equiv 2m_\pi^2s[1-\tfrac{1}{2}\ln(\sigma^2/m_\pi^2)]+\tfrac{3}{2}m_\pi^4I(-a^2)+\tfrac{1}{6}m_\pi^4. \tag{5.25}$$

Using this relation, we can find the coefficient F:

$$F=1-\tfrac{1}{2}\ln(\sigma^2/m_\pi^2). \tag{5.26}$$

All coefficients have been fixed up, except D and H, and C and I, *for which we know only the sum*. It is evident that the most general Ward identities cannot help us to fix these last constants because when one momentum is zero, D and H, as well as I and C, always appear only through their sum. Only the complete calculation can give these coefficients.

VI. THE GOLDSTONE MODE

When we want to take the limit of an exactly chiral nonlinear σ model, we must send the pion mass equal to zero. We know that there must not occur infrared divergences, the pion being only emitted in pairs.[2,13] By asking that no infrared divergences be present, we can completely fix their logarithmic contribution in D, H, I, and C. In fact, when $m_\pi^2\to 0$, we have

$$I(s)\sim+\ln m_\pi^2. \tag{6.1}$$

To avoid infrared divergences, we must have

$$\begin{aligned}&A\sim\ln(\sigma^2/m_\pi^2),\\&B\sim 0,\\&C\sim\tfrac{1}{3}\ln(\sigma^2/m_\pi^2),\\&D\sim-\tfrac{2}{3}\ln(\sigma^2/m_\pi^2),\\&H\sim\tfrac{2}{3}\ln(\sigma^2/m_\pi^2),\\&I\sim\tfrac{1}{6}\ln(\sigma^2/m_\pi^2).\end{aligned} \tag{6.2}$$

One sees from Eqs. (4.2)–(4.10) that the conditions (6.2) are fulfilled. In particular, we see the necessity of the presence of the term $\frac{3}{2}I(-a^2)$ in (4.2) to compensate part of the infrared divergence: Without this term (which would be the case if one chose $a^2=0$), a spurious infrared divergence would appear because $I(s)$ would have been subtracted at a singular point (the point $s=0$ is now singular for the zero-mass case).

VII. ANOTHER POSSIBLE RENORMALIZATION

One could think of renormalizing the σ propagator not at an arbitrary point $-a^2$, but at a point associated with the physical σ mass. But because σ^2 becomes positive and large, we cannot require the σ-mass operator to vanish at its physical mass because it is now complex in the second sheet.

A possible choice which will not violate unitarity is

$$\operatorname{Re}D_\sigma^{-1}(\sigma^2)=0. \tag{7.1}$$

Now, using (5.9) it is easy to compute the change in the four-pion Green's functions. It is found that one must change $I(-a^2)$ as follows:

$$I(-a^2)\to\lim_{\sigma^2\to\infty}[\operatorname{Re}I(\sigma^2)-\tfrac{1}{3}+3(2-\tfrac{1}{3}\pi\sqrt{3})]; \tag{7.2}$$

that is,

$$I(-a^2)\to-\ln(\sigma^2/m_\pi^2)+\tfrac{23}{3}-\pi\sqrt{3}. \tag{7.3}$$

VIII. CONCLUSION

We have presented a method for regularizing and renormalizing the nonlinear σ model. Using the well-known fact that in the limit of an infinite σ mass (or the infinite coupling constant) the tree graphs of the linear σ model tend towards the tree graphs of the nonlinear σ model, we have proposed considering the linear σ model as the regularization of the nonlinear model. The advantages of the method are clear. The regularization in this way respects at each order the symmetry structure under chiral transformations.

We then studied explicitly the limit of an infinite σ mass in the one-loop approximation for the propagator and the four-pion Green's function. To do this, we used the Ward-Takahashi identities in order to renormalize the linear σ model and then we took the limit for an infinite σ mass. The propagator has a finite limit, and the four-pion Green's function diverges only logarithmically with the σ mass. This result holds also for the pion-nucleon and nucleon-nucleon amplitudes (see a forthcoming paper).[14] This generalizes a result on the pion-nucleon vertex previously obtained.[15]

We showed also how to obtain rapidly some coefficients of the second-degree subtraction polynomial of the four-pion Green's function, by using the limit of the Ward identities of the linear σ model. (They contain clearly the Adler self-consistency relation and the Weinberg conditions.)

However, we showed that two coefficients of the nine remain arbitrary, and can only be fixed by our procedure. Also, the pion propagator depends on an orbitrary parameter which is determined in our method.

These results show that, in the one-loop approximation, and due to our limiting procedure, the nonlinear σ model depends on the same number of parameters as the linear σ model. Only one counterterm is needed, in place of the σ mass of the linear model. We do not know if this result generalizes for any order. In order to be able to answer to this question it is necessary to find an efficient way to take the limits in the linear σ model, different from the direct and tedious calculation. But what is sure is that the method we propose reduces in a sensitive way the number of subtraction parameters in the nonlinear σ model and respects all the current-algebra requirements.

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APPENDIX A: EXPANSION OF THE PION AND σ PROPAGATORS AND THE σ-π-π VERTEX IN TERMS OF THE σ MASS

We use the results of Ref. 10, slightly modified by the following. In Ref. 10 the pion and the σ propagator were renormalized by giving their values, as well as those of their derivatives at zero external momenta. Such a procedure is inconvenient for two reasons: First, when the σ mass tends to infinity, we shall not obtain the most general solution; and second, when the pion mass tends to zero we shall encounter infrared divergences due to an improper choice of the point of subtraction in such a case.

Therefore we shall renormalize the pion at its physical mass by the conditions

$$iD_\pi^{-1}(m_\pi^2)=0,\qquad \frac{d}{ds}iD_\pi^{-1}(s)\bigg|_{s=m_\pi^2}=+1. \tag{A1}$$

The σ propagator is defined by the Ward identity

$$iD_\sigma^{-1}(s)=iD_\pi^{-1}(s)-FV_{\sigma\pi\pi}(s;s,0); \tag{A2}$$

since $V_{\sigma\pi\pi}$ is logarithmically divergent, we need an extra condition to fix $iD_\sigma^{-1}(s)$. We impose its value in a point we choose to be $-a^2$:

$$iD_\sigma^{-1}(-a^2)=-(\sigma^2+a^2). \tag{A3}$$

σ is a parameter of the theory, which will be sent equal to infinity, to obtain the nonlinear model.

We first give the formulas which permit us to go, for the linear σ model, from the case $a^2=0$ to $a^2\neq 0$. For the one-loop approximation in which we are interested a short calculation gives

$$\Sigma_\pi^{(a)}(s)=\Sigma_\pi^{(a=0)}(s)-\Sigma_\pi^{(a=0)}(m_\pi^2)-(s-m_\pi^2)\frac{d}{ds}\Sigma_\pi^{(a=0)}(s)\bigg|_{s=m_\pi^2}, \tag{A4}$$

$$\Sigma_\sigma^{(a)}(s)=\Sigma_\sigma^{(a=0)}(s)-\Sigma_\sigma^{(a=0)}(-a^2)-(s+a^2)\frac{d}{ds}\Sigma_\pi^{(a=0)}(s)\bigg|_{s=m_\pi^2}, \tag{A5}$$

where $\Sigma(s)$ is the mass operator:

$$iD_\pi^{-1(a)}(s)=s-m_\pi^2-\Sigma_\pi^{(a)}(s), \tag{A6}$$

$$iD_\sigma^{-1(a)}(s)=s-\sigma^2-\Sigma_\sigma^{(a)}(s). \tag{A7}$$

The vertex undergoes the transformation

$$V^{(a)}_{\sigma\pi\pi}(p_1^2;p_2^2,p_3^2)=V^{(a=0)}_{\sigma\pi\pi}(p_1^2;p_2^2,p_3^2)+\frac{1}{F}\left[\Sigma_\pi^{(a=0)}(-a^2)-\Sigma_\sigma^{(a=0)}(-a^2)-(a^2+m_\pi^2)\frac{d}{ds}\Sigma_\pi^{(a=0)}(s)\Big|_{s=m_\pi^2}\right], \tag{A8}$$

while the one-particle irreducible four-pion amplitude is expressed by

$$A^{\mathrm{IR}(a)}(s,t,u,p_1^2,p_2^2,p_3^2,p_4^2)=A^{\mathrm{IR}(a=0)}(s,t,u,p_1^2,p_2^2,p_3^2,p_4^2)-A^{\mathrm{IR}(a=0)}(-a^2,-a^2,0,0,-a^2,-a^2)+\frac{\Sigma_\pi^{(a)}(-a^2)}{F^2}. \tag{A9}$$

Finally we give the expression for the reducible four-pion amplitude:

$$A^{R(a)}(s,t,u,p_1^2,p_2^2,p_3^2,p_4^2)=-\frac{(\sigma^2-m_\pi^2)^2}{F^2}\frac{\Sigma_\sigma^{(a)}(s)}{(s-\sigma^2)^2}+\frac{1}{F}\frac{(\sigma^2-m_\pi^2)}{s-\sigma^2}[V^{(a)}_{\sigma\pi\pi}(s,p_1^2,p_2^2)+V^{(a)}_{\sigma\pi\pi}(s,p_3^2,p_4^2)]. \tag{A10}$$

Using Eqs. (X.25) and (X.26) of Ref. 10, we have

$$iD_\pi^{-1}(s)|_{a=0}=s-\mu^2-\frac{(\sigma^2-\mu^2)^2}{F^2}[I_{\sigma\mu}(s)-sI'_{\sigma\mu}(0)] \tag{A11}$$

and

$$iD_\sigma^{-1}(s)|_{a=0}=s-\sigma^2-\frac{(\sigma^2-\mu^2)^2}{F^2}[\tfrac{3}{2}I_{\mu\mu}(s)+\tfrac{9}{2}I_{\sigma\sigma}(s)-sI'_{\sigma\mu}(0)], \tag{A12}$$

where $\mu^2\equiv m_\pi^2$ and

$$I_{\sigma\mu}(s)=\frac{s}{\pi}\int_{(\sigma+\mu)^2}^{\infty}-\frac{1}{16\pi}\{[s'-(\sigma+\mu)^2][s'-(\sigma-\mu)^2]\}^{1/2}\frac{ds'}{s'^2(s'-s)}. \tag{A13}$$

The expansions for $I_{\sigma\mu}$ and $I_{\sigma\sigma}$ for large σ are

$$I_{\sigma\mu}(s)=-\frac{s}{32\pi^2}\frac{1}{\sigma^2}+\frac{s\mu^2}{16\pi^2}\frac{\ln\sigma^2}{\sigma^4}-\frac{s}{16\pi^2}(\tfrac{3}{2}\mu^2+\mu^2\ln\mu^2+\tfrac{1}{6}s)\frac{1}{\sigma^4}+\cdots, \tag{A14}$$

$$I_{\sigma\sigma}(s)=-\frac{s}{16\pi^2}\frac{1}{6\sigma^2}\left[1+\frac{s}{10\sigma^2}+\cdots\right]. \tag{A15}$$

It is easy, setting

$$I(s)\equiv-16\pi^2I_{\mu\mu}(s), \tag{A16}$$

to obtain Eqs. (5.4) and (5.5) through a trivial calculation.

APPENDIX B: EXPANSION OF THE FOUR-PION GREEN'S FUNCTION IN TERMS OF THE σ MASS

As previously, we first do the calculation for $a^2=0$ and connect it to $a^2\neq0$ by Eqs. (A9), (A8), (A4), and (A5).

The contributions of the various graphs of the one-loop approximation are given in Table I (see Fig. 5) for the irreducible part of the amplitude. We see that we need the expansion of the following functions:

$$\tilde{D}_{\sigma\mu}(s,t,u,p_1^2,p_2^2,p_3^2,p_4^2)=-16\pi^2[D_{\sigma\mu}(s,t,p_1^2,p_2^2,p_3^2,p_4^2)-D_{\sigma\mu}(0,0,0,0,0,0)], \tag{B1}$$

$$\tilde{V}_\mu(s,p_1^2,p_2^2)=-16\pi^2[V_\mu(s,p_1^2,p_2^2)-V_\mu(0,0,0)], \tag{B2}$$

$$\tilde{V}_\sigma(s,p_1^2,p_2^2)=-16\pi^2[V_\sigma(s,p_1^2,p_2^2)-V_\sigma(0,0,0)], \tag{B3}$$

where

$$D_{\sigma\mu}=-\frac{1}{16\pi^2}\int_0^1\frac{\prod_1^4 d\alpha_i\,\delta\left(1-\sum_1^4\alpha_i\right)}{C_{\sigma\mu}^2} \tag{B4}$$

and

$$C_{\sigma\mu}\equiv-\alpha_2\alpha_3s-\alpha_1\alpha_4t-\alpha_1\alpha_3p_1^2-\alpha_1\alpha_2p_2^2-\alpha_3\alpha_4p_3^2-\alpha_2\alpha_4p_4^2+\sigma^2(\alpha_2+\alpha_3)+\mu^2(\alpha_1+\alpha_4), \tag{B5}$$

$$V(p_1^2;p_2^2,p_3^2)=-\frac{1}{16\pi^2}\int\frac{\prod_1^3 d\alpha_i\,\delta\left(1-\sum_1^3\alpha_i\right)}{\alpha_2\alpha_3p_1^2+\alpha_3\alpha_1p_2^2+\alpha_1\alpha_2p_3^2-\alpha_1m_1^2-\alpha_2m_2^2-\alpha_3m_3^2}. \tag{B6}$$

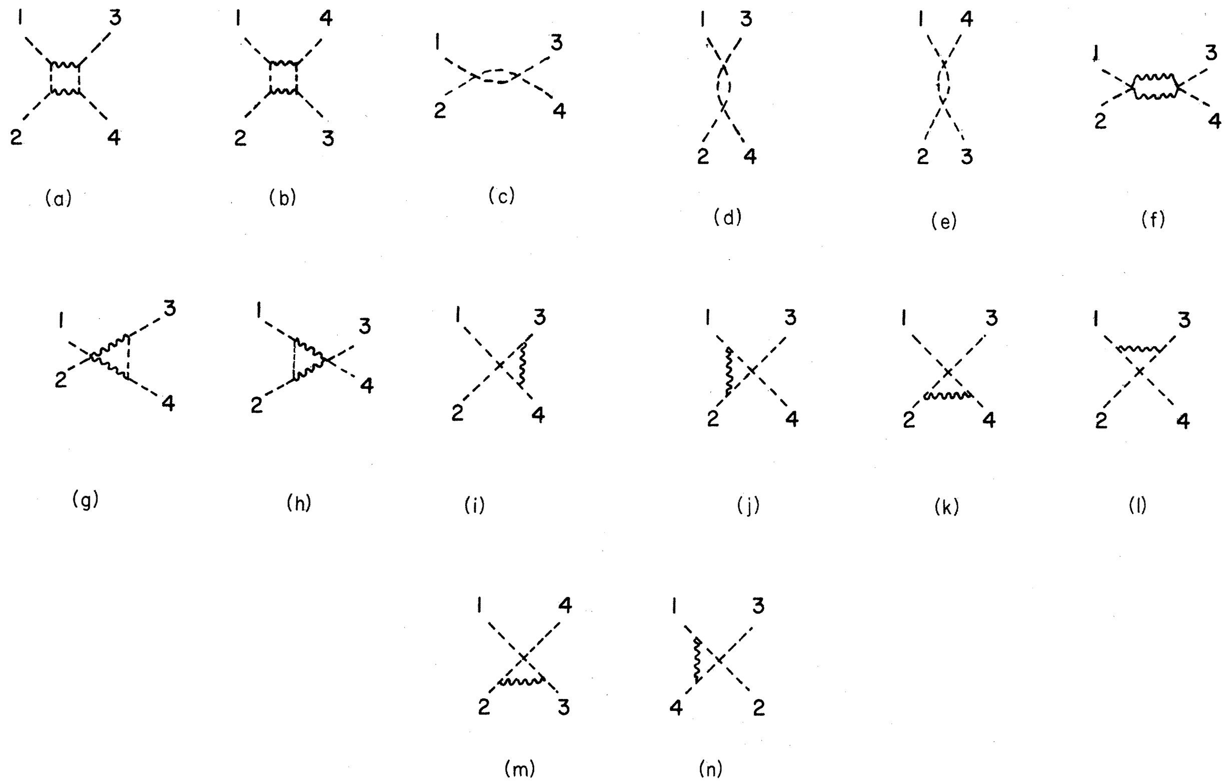

FIG. 5. The four-pion one-particle irreducible graphs at the one loop in the linear σ model.

In (B6) we have either $m_1=m_2=\mu$ and $m_3=\sigma(V_\sigma)$ or $m_1=m_2=\sigma$ and $m_3=\mu(V_\mu)$. By applying standard methods[16] for expanding multiple integrals in terms of a large parameters, and setting

$$\rho=p_1^2+p_2^2+p_3^2+p_4^2, \tag{B7}$$

$$\bar\rho=p_1^4+p_2^4+p_3^4+p_4^4+p_1^2p_2^2+p_1^2p_3^2+p_2^2p_4^2+p_3^2p_4^2+\tfrac12 p_1^2p_4^2+\tfrac12 p_2^2p_3^2, \tag{B8}$$

$$\bar\rho'=p_1^4+p_2^4+p_3^4+p_4^4+p_1^2p_2^2-2p_1^2p_3^2-2p_2^2p_4^2+p_3^2p_4^2-p_1^2p_4^2-p_2^2p_3^2, \tag{B9}$$

we get to order $(\ln\sigma^2)/\sigma^{10}$

$$\begin{aligned}\tilde D(s,t,p_1^2,p_2^2,p_3^2,p_4^2)={}&\frac{I(t)}{\sigma^4}+(\tfrac12\rho-t)\,\frac{\ln(\sigma^2/\mu^2)}{\sigma^6}+\frac{1}{\sigma^6}\{I(t)[\tfrac12\rho-t+2\mu^2]+\tfrac16 s+2t-\tfrac32\rho\}\\&+\frac{\ln(\sigma^2/\mu^2)}{\sigma^8}[t^2+\tfrac16 st-t(\rho+6\mu^2)-s\mu^2+\tfrac92\mu^2\rho+\tfrac13\bar\rho]\\&+\frac{1}{\sigma^8}\bigg[I(t)\bigg(t^2+\tfrac16 st-t(\rho+4\mu^2)-\tfrac23\mu^2 s+3\mu^4+\tfrac52\mu^2\rho+\tfrac13\bar\rho-\frac{\mu^2}{3t}\bar\rho'\bigg)\\&\qquad+\tfrac1{60}s^2-\tfrac12 st-2t^2-s(\tfrac1{12}\rho+\tfrac73\mu^2)+t(\tfrac52\rho+8\mu^2)-\tfrac{15}{2}\mu^2\rho-\tfrac{11}{9}\bar\rho+\tfrac1{18}\bar\rho'\bigg].\end{aligned} \tag{B10}$$

In the same way, we obtain for $\tilde V_\mu(s,p_1^2,p_2^2)$

$$\begin{aligned}\tilde V_\mu(s,p_1^2,p_2^2)={}&-\frac{1}{4\sigma^4}(\tfrac13 s+p_1^2+p_2^2)+\mu^2(p_1^2+p_2^2)\,\frac{\ln(\sigma^2/\mu^2)}{\sigma^6}\\&+\frac{1}{\sigma^6}\{-2\mu^2(p_1^2+p_2^2)+\tfrac16\mu^2 s-\tfrac19[p_1^4+p_2^4+p_1^2p_2^2+\tfrac14 s(p_1^2+p_2^2)+\tfrac1{10}s^2]\}+O\bigg(\frac{\ln\sigma^2}{\sigma^8}\bigg),\end{aligned} \tag{B11}$$

and to the same order,

TABLE I. The four-pion one-particle irreducible contributions to the one-loop diagrams of the linear σ model (for $a^2=0$).

Figure	Contribution
5(a)	$-\frac{(\sigma^2-\mu^2)^4}{F^4}[D_{\sigma\mu}(s,t,p_1^2,p_2^2,p_3^2,p_4^2)-D_{\sigma\mu}(0,0,0,0,0,0)]$
5(b)	$-\frac{(\sigma^2-\mu^2)^4}{F^4}[D_{\sigma\mu}(s,u,p_1^2,p_2^2,p_3^2,p_4^2)-D_{\sigma\mu}(0,0,0,0,0,0)]$
5(c)	$-\frac{(\sigma^2-\mu^2)^2}{F^4}\frac{7}{2}I_{\mu\mu}(s)$
5(d)	$-\frac{(\sigma^2-\mu^2)^2}{F^4}I_{\mu\mu}(t)$
5(e)	$-\frac{(\sigma^2-\mu^2)^2}{F^4}I_{\mu\mu}(u)$
5(f)	$-\frac{(\sigma^2-\mu^2)^2}{F^4}\frac{1}{2}I_{\sigma\sigma}(s)$
5(g)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\mu(s,p_3^2,p_4^2)-V_\mu(0,0,0)]$
5(h)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\mu(s,p_1^2,p_2^2)-V_\mu(0,0,0)]$
5(i)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\sigma(s,p_3^2,p_4^2)-V_\sigma(0,0,0)]$
5(j)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\sigma(s,p_1^2,p_2^2)-V_\sigma(0,0,0)]$
5(k)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\sigma(t,p_2^2,p_4^2)-V_\sigma(0,0,0)]$
5(l)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\sigma(t,p_1^2,p_3^2)-V_\sigma(0,0,0)]$
5(m)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\sigma(u,p_2^2,p_3^2)-V_\sigma(0,0,0)]$
5(n)	$-\frac{(\sigma^2-\mu^2)^3}{F^4}[V_\sigma(u,p_1^2,p_4^2)-V_\sigma(0,0,0)]$

$$\begin{aligned}\tilde{V}_\sigma(s,p_1^2,p_2^2)=&-\frac{I(s)}{\sigma^2}+\tfrac{1}{2}(s-p_1^2-p_2^2)\frac{\ln(\sigma^2/\mu^2)}{\sigma^4}+\frac{1}{\sigma^4}[I(s)(\tfrac{1}{2}s-\tfrac{1}{2}p_1^2-\tfrac{1}{2}p_2^2-\mu^2)-\tfrac{3}{4}s+\tfrac{5}{4}(p_1^2+p_2^2)]\\&+\frac{\ln(\sigma^2/\mu^2)}{\sigma^6}[-\tfrac{1}{3}s^2+\tfrac{2}{3}s(p_1^2+p_2^2+3\mu^2)-3\mu^2(p_1^2+p_2^2)-\tfrac{1}{3}(p_1^4+p_2^4+p_1^2p_2^2)]\\&+\frac{1}{\sigma^6}\Big\{I(s)\Big[-\tfrac{1}{3}s^2+\tfrac{2}{3}s(p_1^2+p_2^2)+\tfrac{4}{3}\mu^2s-\tfrac{5}{3}\mu^2(p_1^2+p_2^2)-\tfrac{1}{3}(p_1^4+p_2^4+p_1^2p_2^2)-\mu^4+\frac{\mu^2}{3s}(p_1^2-p_2^2)^2\Big]\\&+\tfrac{5}{9}s^2-s[\tfrac{13}{9}(p_1^2+p_2^2)+2\mu^2]+4\mu^2(p_1^2+p_2^2)+\tfrac{1}{18}(19p_1^4+19p_2^4+22p_1^2p_2^2)\Big\}+O\left(\frac{\ln\sigma^2}{\sigma^8}\right). \qquad \text{(B12)}\end{aligned}$$

It is then easy to obtain the irreducible part of the amplitude. For the reducible part, we give in Table II (see Fig. 6) the contributions of the various graphs to $V_{\sigma\pi\pi}$. The functions implied are the same as previously.

By adding all contributions, we first get the amplitude for $a^2=0$, and then for $a^2\neq 0$, by the use of formulas of Appendix A.

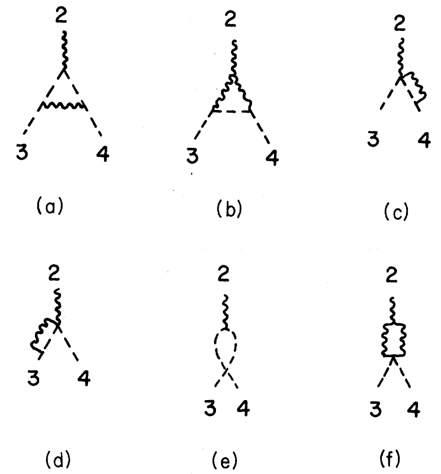

FIG. 6. The one-loop σ-π-π vertex graphs in the linear σ model.

TABLE II. The one-loop contributions to the vertex σ-π-π in the linear σ model (for $a^2=0$).

Figure	Contribution
6(a)	$-\frac{(\sigma^2-\mu^2)^3}{F^3}[V_\sigma(p_2^2,p_3^2,p_4^2)-V_\sigma(0,0,0)]$
6(b)	$-\frac{3(\sigma^2-\mu^2)^3}{F^3}[V_\mu(p_2^2,p_3^2,p_4^2)-V_\mu(0,0,0)]$
6(c)	$-\frac{(\sigma^2-\mu^2)^2}{F^3}I_{\sigma\mu}(p_4^2)$
6(d)	$-\frac{(\sigma^2-\mu^2)^2}{F^3}I_{\sigma\mu}(p_3^2)$
6(e)	$-\frac{5}{2}\frac{(\sigma^2-\mu^2)^2}{F^3}I_{\mu\mu}(p_2^2)$
6(f)	$-\frac{3}{2}\frac{(\sigma^2-\mu^2)^2}{F^3}I_{\sigma\sigma}(p_2^2)$

APPENDIX C: CALCULATION OF THE COEFFICIENT OF $m_\pi^2\,(t+u)$ IN THE SUBTRACTION POLYNOMIAL

To make use of Eq. (5.14) in the limit $\sigma\to\infty$ we first need to give the expression for $V_{\sigma\pi\pi}(0;s,s)$. It turns out that this is just given by

$$V_{\sigma\pi\pi}(0;s,s)=\frac{(\sigma^2-\mu^2)^2}{F^3}\left\{2I_{\sigma\mu}(s)+(\sigma^2-\mu^2)\left[\frac{\partial}{\partial\mu^2}I_{\sigma\mu}(s)+3\frac{\partial}{\partial\sigma^2}I_{\sigma\mu}(s)\right]\right\}. \tag{C1}$$

Therefore, differentiating the expansion (A14) with respect to μ^2 and σ^2 we obtain the expansion for large σ of $V_{\sigma\pi\pi}(0;s,s)$ and finally through a straightforward calculation we get the relation (5.25).

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