PHYSICAL REVIEW D VOLUME 5, NUMBER 6 15 MARCH 1972

Expansions of the Unequal-Mass Scattering Amplitude in Terms of Poincaré Representations and Complex Angular Momentum at Zero Energy

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(Received 26 October 1971)

The expansion of the unequal-mass scattering amplitude in terms of Poincaré-group representations was considered for positive and zero values of s, the squared total four-momentum. The usual singularity problem at $s=0$ was avoidable, but it turned out that the relevant variable is not j, the total angular momentum, but a quantity nonsingularly related to the Poincaré-invariant $W_\mu W^\mu$ even at $s=0$. The notion of complex angular momentum and signature was reexamined, and some modification of the old formalism seemed useful. The results are perfectly compatible with dispersion relations and with the requirements of Regge behavior. In an appendix a theorem is proved for the expansion of a class of functions which are not square-integrable, but have Regge behavior with respect to unitary E(2) representations (that is, for Fourier-Bessel expansions).

I. INTRODUCTION

The difficulties of Regge-pole theory at zero energy in the case of unequal-mass scattering have inspired many authors, and many different approaches have been proposed to solve the problem. The general attitude is to take for granted the presence of unpleasant singularities in the Watson-Sommerfeld transformed form of the unequal-mass scattering amplitude, and the task is to discover how to remove the singularities. On the other hand, one must realize that even the presence of these singularities is questionable. What actually happens in the Reggeization procedure is that some formulas, well defined in the s channel, are extrapolated to new regions, into the t or u channel. It is far from trivial that, although the starting situation is very similar, everything must be learned from the equal-mass case. Instead, Fourier analysis on the Poincaré group is probably the "magic word" one is to remember in the Reggeization procedure.

Many authors have investigated the connection between the forms of the scattering amplitude obtained by Watson-Sommerfeld transformations and from direct group-theoretic expansions, mostly for spacelike total four-momentum, $s<0$.[1-4] The present paper is mainly devoted to the problems at $s=0$ in the unequal-mass case. Some steps of our approach were made in Refs. 5 and 6, but our results go far beyond theirs.

We deal both with the $s=0$ limit of the Watson-Sommerfeld transform and with the connection of this limit with the group-theoretical expansion in terms of lightlike Poincaré-representation matrix elements. These investigations lead to the following conclusion: The appropriate variable at $s=0$ is not j, but w, the eigenvalue of the Poincaré-invariant $W_\mu W^\mu$, W_μ being the Pauli-Lubanski operator. As is well known, at $s=0$ real positive values of w correspond to unitary Poincaré representations (infinite-spin representations), they are sufficient to expand a square-integrable scattering amplitude. Complex values of w correspond to nonunitary representations, and a complex-angular-momentum theory is to be formulated in terms of functions of the complex variable w. Obviously, when s is not zero, one may equally well use w or j. On the other hand, one cannot provide a (Poincaré) group-theoretical interpretation to a theory which uses the variable j at $s=0$. (Our way of looking at the problems with unequal-mass scattering is very strongly supported by Hermann.[7]) In other words our suggestion is that s and j are not the "most economical" variables to formulate a complex-angular-momentum theory, but s and w are. (Also, Feldman and Matthews have suggested that the correct variable to be used is not j but w.[6] See also Ref. 8.) The undesirable singularity at $s=0$ is a consequence only of the uneconomical choice of variables. (The analog of this phenomenon is well known in the context of the singularities which arise when using the variables s and $\cos\theta_s$ instead of the "most economical" pair s, t.) In arriving at this conclusion, group-theoretical interpretability is only a hint rather than a necessary condition.

In this paper the scattering of two spinless particles with masses m and μ (pion-nucleon-type kinematics) will be examined. In Sec. II some remarks on Poincaré representations are presented (for a detailed discussion see Ref. 9), which are of basic importance in the subsequent investigations. In Sec. III the Watson-Sommerfeld representation of the scattering amplitude is given, and our modifications of the complex angular momentum are described in comparison with the conventional treatments. In Sec. IV the $s=0$ limit is calculated, and in Sec. V a comparison is made between the Sommerfeld-Watson representation and the expansion with respect to Poincaré-representation matrix elements. In Sec. VI some details of our approach are discussed, and in two appendixes mathematical statements made in the previous sections are proved.

II. REMARKS ON POINCARÉ REPRESENTATIONS

If one takes the standpoint that the Regge-Watson-Sommerfeld representation of the scattering amplitude is nothing but essentially a group-theoretical expansion in terms of Poincaré representations (this is supported, e.g., by the fact that resonances are classified by putting them on Regge trajectories), then the $s=0$ problem of unequal-mass scattering can be, at least in part, transferred to the representation theory of the Poincaré group. Namely, the question arises of whether the representations of the Poincaré group can be described in a form that is continuous in the Casimir-operator eigenvalue $P_\mu{}^2=s$ at $s=0$ when the four-momentum P_μ becomes lightlike. This problem was thoroughly investigated in Ref. 9, and we summarize its most important points here.

The Poincaré group has been represented on a sufficiently large function space, and explicit functions in this space could be found with the following properties:

(1) They are eigenfunctions of the four-momentum P_μ, with arbitrary real eigenvalues p_μ; of $W_\mu W^\mu$, with arbitrary complex eigenvalues $sj(j+1) = w^2-\frac{1}{4}s$, where $s=p_\mu{}^2$; and of W_0 with eigenvalue $p\lambda$, where p is the magnitude of the three-momentum $\vec{p}$ and λ is the helicity. That is, the functions with given s and w form an irreducible set for representing the Poincaré group in a helicity basis.

(2) They are continuous functions of the four-momentum p_μ, and consequently of s as well. Appropriate normalization is essential to achieve continuity at $s=0$. (The point $p_\mu=0$ is a very peculiar one,[9] and is unimportant in this paper. Hereafter $s=0$ will always be associated with lightlike four-vectors.)

After obtaining the basis functions, representation matrix elements of the Poincaré group have been calculated. The result is of the following form:

$$\langle p_\mu, w, \lambda|(a,\Lambda)|p'_\mu, w', \lambda'\rangle = N(s,w,w')\delta^4(p_\mu - \Lambda p'_\mu)D^w_{\lambda\lambda'}(\phi,\theta,\psi)\exp(-ip_\mu a_\mu)\,, \tag{2.1}$$

where $N(s,w,w')$ is a continuous function of $p_\mu{}^2=s$ when w and w' are fixed. The functions $D^w_{\lambda\lambda'}$ denote the familiar representation functions of the groups SU(2), SU(1, 1), or E(2) depending on whether s is positive, negative, or zero, respectively.[10] (In the cases where $s\neq 0$, more conventionally the label j is used instead of w.) The Euler angles ϕ, θ, ψ in the $D^w_{\lambda\lambda'}$ function are functions of the six parameters of the homogeneous-Lorentz-group element Λ and of the four components of p_μ. The method of determining the functions $\phi(\Lambda,p_\mu)$, $\theta(\Lambda,p_\mu)$, and $\psi(\Lambda,p_\mu)$ is well known since they are the Euler angles of the Wigner rotation $L_{p_\mu}{}^{-1}\Lambda L_{\Lambda^{-1}p_\mu}$, where L_{p_μ} and $L_{\Lambda^{-1}p_\mu}$ are boosts which transform the four-vector $(\sqrt{s},0,0,0)$ (for $s>0$), or $(0,0,0,\sqrt{-s})$ (for $s<0$) to p_μ and $(\Lambda p)_\mu$, respectively. It can be checked again that the functions $D^w_{\lambda\lambda'}(\phi,\theta,\psi)$ are continuous functions of the components of p_μ when w is fixed. This might be surprising since a similar statement is not true for $\theta(\Lambda,p_\mu)$. Namely, $\lim_{s\to 0}\theta(\Lambda,p_\mu)\equiv 0$ (p_μ becomes lightlike) independently of Λ. On the other hand, if we calculate the matrix element (2.1) directly for lightlike representations [that is, also the Euler angles of the "Wigner rotation" $\hat{L}_{p_\mu}{}^{-1}\Lambda\hat{L}_{\Lambda^{-1}p_\mu}$ with boosts $\hat{L}_{p_\mu}$ and $\hat{L}_{\Lambda^{-1}p_\mu}$ transforming a four-vector $(p,0,0,p)$ to p_μ and $(\Lambda p)_\mu$, respectively], we find that $\theta(\Lambda,p_\mu)\in(0,\infty)$.

This discrepancy can be very easily eliminated by reinterpreting the function $D^w_{\lambda\lambda'}$ in the following manner: It is the representation matrix element $D^w_{\lambda\lambda'}(\phi^v,\theta^v,\psi^v)$ [$\equiv D^w_{\lambda\lambda'}(\phi,\theta,\psi)$] of the little group of the four-vector $(p_0,0,0,p)$, $p_0{}^2-p^2=p_\mu{}^2=s$, the Euler angles of which being those of $\tilde{L}_{p_\mu}{}^{-1}\Lambda\tilde{L}_{\Lambda^{-1}p_\mu}$ where $\tilde{L}_{p_\mu}$ and $\tilde{L}_{\Lambda^{-1}p_\mu}$ are boosts transforming the four-vector $(p_0,0,0,p)$ to p_μ and $(\Lambda p)_\mu$, respectively. It is easy to verify that

$$\phi^v\equiv\phi\,,\quad \psi^v\equiv\psi\,,\qquad \theta^v=\left(\frac{|p_0|+p}{\big||p_0|-p\big|}\right)^{1/2}\theta\,, \tag{2.2}$$

and θ^v is now a continuous function of s even at $s=0$. (For the representations of little groups for four-vectors like $(p_0,0,0,p)$ see, for example, Ref. 11.)

The significance of choosing Euler variables which are continuous functions of s becomes clear when we come to the next relevant point, the orthogonality relations of the matrix elements:

$$I_1=\int d^4a\,d\mu(\Lambda)\langle p_\mu,w,\lambda|(a,\Lambda)|p'_\mu,w',\lambda'\rangle \times\langle p''_\mu,w'',\lambda''|(a,\Lambda)|p'''_\mu,w''',\lambda'''\rangle^*\,, \tag{2.3}$$

where the integration goes over the translation and the homogeneous-Lorentz-group part of the Poincaré group. [Concerning the measure $d\mu(\Lambda)$ on the Lorentz group see, e.g., Ref. 10.] After performing trivial integrations one obtains

$$I_1 = N(s,w,w')N(s,w'',w''')\delta(p_\mu{}^2 - p_\mu'^2) \times \delta^4(p_\mu' - p_\mu''')\delta^4(p_\mu - p_\mu'')I_2 , \qquad (2.4)$$

where

$$I_2 = \int d\mu(\phi^v, \theta^v, \psi^v) D^w_{\lambda\lambda'}(\phi^v, \theta^v, \psi^v) D^{w''*}_{\lambda''\lambda'''}(\phi^v, \theta^v, \psi^v) = (|p_0| + p)^2 N(s,w,w'')\delta_{\lambda\lambda''}\delta_{\lambda'\lambda'''} . \qquad (2.5)$$

In the expression (2.5), $d\mu(\phi^v, \theta^v, \psi^v)$ is the measure for the little group of $(p_0, 0, 0, p)$. Lengthy but straightforward calculation gives, e.g., for $s > 0$

$$d\mu(\phi^v, \theta^v, \psi^v) = \left(\frac{|p_0| + p}{|p_0| - p}\right)^{1/2} \sin\left[\left(\frac{|p_0| - p}{|p_0| + p}\right)^{1/2} \theta^v\right] d\phi^v d\theta^v d\psi^v = \frac{|p_0| + p}{|p_0| - p} d(\cos\theta) d\phi d\psi \equiv \frac{|p_0| + p}{|p_0| - p} d\mu(\phi, \theta, \psi) \qquad (2.6)$$

and

$$N(s,w,w'') = \frac{1}{(2j+1)s}\delta_{jj''} , \qquad (2.7)$$

where

$$w^2 = sj(j+1) + \tfrac{1}{4}s, \quad w''^2 = sj''(j''+1) + \tfrac{1}{4}s .$$

We call attention to the fact that in the integral (2.5) the measure $d\mu(\phi^v, \theta^v, \psi^v)$ has appeared, rather than $d\mu(\phi, \theta, \psi)$. This is strongly correlated with the singular behavior of the angle θ at $s = 0$.

The formulas (2.5)–(2.7) make it possible to write down the partial-wave expansion, that is, the expansion with respect to irreducible, timelike Poincaré representations for an unequal-mass scattering amplitude, in a form which we expect, after Reggeization, to have nice analytic properties even at zero energy:

$$\begin{aligned}\langle p_3, s_3, \lambda_3; p_4, s_4, \lambda_4 | T | p_1, s_1, \lambda_1; p_2, s_2, \lambda_2\rangle &= (2\pi)^4 F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t)\delta^4(p_1 + p_2 - p_3 - p_4) \\ &= (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)\frac{s}{(|p_0| + p)^2} \\ &\quad \times \sum_j F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,j)(2j+1)d^j_{\lambda\mu}(\theta_s) ,\end{aligned} \qquad (2.8)$$

where $\lambda = \lambda_3 - \lambda_4$, $\mu = \lambda_1 - \lambda_2$, and θ_s is the scattering angle in the center-of-mass (c.m.) frame for the s channel. The partial-wave amplitudes are defined as follows:

$$F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,j) = \frac{(|p_0| + p)^2}{s} \times \int_{-1}^{1} d(\cos\theta_s) F_{\lambda_1\lambda_2\lambda_3\lambda_4}(s,t) d^j_{\mu\lambda}(\theta_s) . \qquad (2.9)$$

The symbol $d^j_{\mu\lambda}$ denotes the familiar Wigner d functions.

III. COMPLEX ANGULAR MOMENTUM

In this section we describe a complex-angular-momentum theory for unequal-mass scattering which, on the one hand, is related to that for equal-mass scattering as strongly as possible, but, on the other hand, makes use of the remarks of the previous section. In other words, first, the scattering amplitude is to be expanded in terms of Poincaré representations in a frame in which the total four-momentum $P_\mu = p_{1\mu} + p_{2\mu}$ is of the form $(p_0, 0, 0, p)$. Second, the appropriate variable to be used in a complex-angular-momentum theory is w rather than j. (Of course, this distinction is irrelevant when $s \neq 0$, and we shall use the variable j until we do not want to go to $s = 0$.)

The crucial points of conventional complex-angular-momentum theory (see Refs. 12 and 13) are the following:

(1) Using Carlson's theorem, one defines two functions over the complex j plane from the s-channel partial-wave amplitudes.

(2) By a Watson-Sommerfeld transformation one casts the partial-wave series into an integral along a curve of the j plane from $-\frac{1}{2} - i\infty$ to $-\frac{1}{2} + i\infty$.

(3) After analytic continuation in the s and t Mandelstam variables one obtains the crossed-channel scattering amplitude represented by the background integral (along the line $\mathrm{Re}j = -\frac{1}{2}$) and the residues of poles appearing on the half plane $\mathrm{Re}j > -\frac{1}{2}$. (Cuts will not be considered in this paper.) It is assumed that the contribution of the integral along the infinite half-circle is still negligible.

Now we consider the elastic scattering of two spinless particles with masses m and μ, $p_1{}^2 = p_3{}^2 = m^2$, $p_2{}^2 = p_4{}^2 = \mu^2$. The Mandelstam variables are

$$\begin{aligned} s &= (p_1 + p_2)^2 = (p_3 + p_4)^2 , \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2 , \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2 , \end{aligned} \qquad (3.1)$$

$$\cos\theta_s = 1 + \frac{2st}{\Delta(s, m^2, \mu^2)} , \qquad (3.2)$$

where θ_s is the s-channel scattering angle in the c.m. frame, and

$$\Delta(s, m^2, \mu^2) = [s - (m+\mu)^2][s - (m-\mu)^2] .$$

In the s channel the partial-wave series for the scattering amplitude $F(s, t)$ looks like

$$F(s,t) = \frac{m\mu s}{(m^2 - \mu^2 + \Delta^{1/2})^2 - s^2} \times \sum_{j=0}^{\infty} F(s,j)(2j+1)P_j(\cos\theta_s) , \quad (3.3)$$

where the partial-wave amplitudes $F(s, j)$ are defined as follows:

$$F(s,j) = \frac{(m^2 - \mu^2 + \Delta^{1/2})^2 - s^2}{2m\mu s} \times \int_{-1}^{1} d(\cos\theta_s) F(s,t) P_j(\cos\theta_s) . \quad (3.4)$$

In the spinless case the d functions of (2.8) and (2.9) are the familiar Legendre polynomials $P_j(z)$. The kinematical factor

$$\frac{s m\mu}{(m^2 - \mu^2 + \Delta^{1/2})^2 - s^2}$$

in (3.3) and (3.4) corresponds to the $s/(|p_0| + p)^2$ of (2.8) and (2.9). It could have been included into $F(s, j)$, but it has significance when we go to $s = 0$; therefore we prefer to write it explicitly. In the equal-mass case it is only a numerical factor $-\frac{1}{4}$.

We assume $F(s, t)$ to satisfy an unsubtracted dispersion relation in the variable t at fixed s:

$$F(s,t) \equiv F_t(s,t) + F_u(s,t) = \frac{1}{\pi}\int_{4m^2}^{\infty} dt' \frac{A_t(s,t')}{t'-t} + \frac{1}{\pi}\int_{-\infty}^{(m-\mu)^2 - s} dt' \frac{A_u(s,t')}{t'-t} . \quad (3.5)$$

Correspondingly, we define

$$F(s,j) \equiv F_t(s,j) + F_u(s,j)$$

and obtain

$$F_t(s,j) = \frac{2}{\pi} \frac{(m^2 - \mu^2 + \Delta^{1/2})^2 - s^2}{m\mu\Delta} \times \int_{4m^2}^{\infty} dt' A_t(s,t') Q_j\left(1 + \frac{2st'}{\Delta(s, m^2, \mu^2)}\right), \quad (3.6)$$

$$F_u(s,j) = \frac{2}{\pi} \frac{(m^2 - \mu^2 + \Delta^{1/2})^2 - s^2}{m\mu\Delta} \times \int_{-\infty}^{(m-\mu)^2 - s} dt' A_u(s,t') Q_j\left(1 + \frac{2st'}{\Delta} - i0\right) . \quad (3.7)$$

In (3.7) the real, < -1 argument of the Q_j function is to be understood as a limit from the lower half plane.

It is usual at this stage to introduce complex angular momentum. As is well known, the mathematical problem of defining an analytic function having prescribed values at non-negative integer values of j involves an essential nonuniqueness. The tradition in Regge-pole theory is to look for analytic continuations satisfying the conditions of Carlson's theorem, and this leads to the signatured functions

$$F_\pm(s,j) = F_t(s,j) \mp F_u(s,j) e^{i\pi j} . \quad (3.8)$$

We are not going to follow this tradition, but rather we define complex angular momentum directly through (3.6) and (3.7). Some problems arising from the use of $F_{t,u}(s, j)$ instead of $F_\pm(s, j)$ will be discussed at the end of this section. The merits of our choice will become clear only in the subsequent sections.

Now, still in the s channel, we can write the integral

$$F_{t,u}(s,t) = \frac{1}{2i} \frac{s m\mu}{(m^2 - \mu^2 + \Delta^{1/2})^2 - s^2} \times \int_C dj \frac{2j+1}{\sin\pi j} F_{t,u}(s,j) P_j\left(-1 - \frac{2st}{\Delta}\right) \quad (3.9)$$

on the j plane instead of writing the original partial-wave series. The contour C encircles the positive real half axis. Until we are in the s channel all the poles of the integrand in (3.9) are due to the zeros of $\sin\pi j$ at integer values of j. After analytic continuation into the t or u channel also the functions $F_{t,u}(s, j)$ have poles at real $j = \alpha(s)$ values. Then also the contribution of these poles is to be included in the expression (3.9). The basic assumption of Regge-pole phenomenology is that the contribution of these latter poles dominates over the remainder, the contribution of the poles due to $\sin\pi j$. The usual "proof" for this is to deform the contour C into a straight line along $\mathrm{Re}j = -\frac{1}{2}$ and an infinite half circle on the right half plane. If one assumes that the integral along this half circle is zero, it is easy to see from the asymptotic expressions for the $P_j(z)$ functions that, for large values of $\cos\theta_s$, the background integral is reasonably neglected in comparison with the Regge-pole contributions. Obviously, this "proof" relies very strongly on the appropriate asymptotic behavior of the $F(s, j)$ functions in the variable j. In the t or u channel this cannot be justified simply by looking at the integrand of (3.6) and (3.7), and it cannot be done either for the signatured functions $F_\pm(s, j)$ even in the more familiar equal-mass case.[12, 13] On the other hand, the successes of Regge-pole phenomenology serve as a justification of the assumption. In our treatment, $F_t(s, j)$

and $F_u(s,j)$ must be well behaved, instead of $F_\pm(s,j)$. This assumption may very well be compatible with phenomenology since, although there is an $\exp(i\pi j)$ factor present in (3.8), $F_{t,u}(s,j)$ and $F_\pm(s,j)$ may have the suitable properties even simultaneously.

There is only one thing we certainly lost by using $F_{t,u}(s,j)$ instead of $F_\pm(s,j)$. Namely, in the case of signatured functions, the analogs of expressions (3.6) and (3.7) make it possible to prove that in the s channel the functions $F_\pm(s,j)$ decrease [since the functions $Q_j(z)$ do so] fast enough for the contribution of the infinite half circle to be zero. This is not the case with $F_u(s,j)$. However, one must note that in the s channel this problem has no particular significance. The contour integral has no advantages over the partial-wave series; either the contribution of the half circle must or need not be kept.

Our final formulas for $F_{t,u}(s,t)$ are

$$F_{t,u}(s,t)=F^b_{t,u}(s,t)+F^p_{t,u}(s,t)\,, \tag{3.10}$$

where $F^b_{t,u}(s,t)$ is the background term:

$$F^b_{t,u}(s,t)=\frac{1}{2i}\,\frac{sm\mu}{(m^2-\mu^2+\Delta^{1/2})^2-s^2}\times\int_{-1/2-i\infty}^{-1/2+i\infty}dj\,\frac{2j+1}{\sin\pi j}\,F_{t,u}(s,j)P_j\left(-1-\frac{2st}{\Delta}\right), \tag{3.11}$$

and $F^p_{t,u}(s,t)$ denotes the Regge-pole part:

$$F^p_{t,u}(s,t)=\frac{1}{2i}\,\frac{sm\mu}{(m^2-\mu^2+\Delta^{1/2})^2-s^2}\times\sum_{\text{poles}}\oint_{C_i}dj\,\frac{2j+1}{\sin\pi j}\,F_{t,u}(s,j)P_j\left(-1-\frac{2st}{\Delta}\right). \tag{3.12}$$

As was discussed, we assume the representation (3.10)–(3.12) of the functions $F_{t,u}(s,t)$ to be valid in the t and u channels.

IV. COMPLEX ANGULAR MOMENTUM IN THE $s=0$ LIMIT

Having fixed our definitions for a complex-angular-momentum theory at $s\neq 0$, we investigate its $s=0$ limit, which is a physical point for the u channel. We make use of the fact that there is a finite piece of the u-channel physical region above $s=0$, and in the present paper we restrict ourselves to reaching the point $s=0$ through positive values of s. That is, we consider the formulas (3.6), (3.7), (3.11), and (3.12) for $s-i0$, $u+i0$, $0<s<(m-\mu)^2$, $(m+\mu)^2<u<(m^2-\mu^2)^2/s$, and, keeping u fixed, we let s go to zero. It is worth remarking that still we are on the lower edge of the cut of the Q_j function in (3.7).

In the usual treatments the limit $s=0$ is taken at fixed values of j, and the singularity problem arises due to the singularity of $P_j(z)$ at $z=-1$ and of $Q_j(z)$ at $z=1$. In our approach w is the fundamental variable, and we calculate the limit keeping w fixed. Indeed, first we introduce a (dimensionless) variable ϵ, instead of j, which is not singularly connected with w even at $s=0$:

$$w^2=sj(j+1)+\tfrac{1}{4}s=\frac{(m^2-\mu^2)^2}{4mu}\epsilon^2. \tag{4.1}$$

The most economical way to calculate the limit of the Legendre functions is to use the following integral representations[14]:

$$P_j(z)=\frac{1}{\pi}\int_0^\pi[z+(z^2-1)^{1/2}\cos\phi]^j d\phi,\qquad |\arg z|<\tfrac{1}{2}\pi \tag{4.2}$$

$$Q_j(z)=\int_0^\infty[z+(z^2-1)^{1/2}\cosh t]^{-j-1}dt,\qquad |\arg(z\pm1)|<\pi. \tag{4.3}$$

At the end of the calculations one recognizes Bessel functions of the first and third kind in the following forms[15]:

$$J_0(z)=\frac{1}{\pi}\int_0^\pi\exp(iz\cos\Phi)d\Phi, \tag{4.4}$$

$$K_0(z)=\int_0^\infty\exp(-z\cosh t)dt,\quad |\arg z|<\tfrac{1}{2}\pi. \tag{4.5}$$

Also the relations between Hankel's functions and the K function are useful:

$$H_j^{(1)}(z)=J_j(z)+iY_j(z)=-(2i/\pi)\exp(-ij\tfrac{1}{2}\pi)K_j(z\exp(-i\tfrac{1}{2}\pi)), \tag{4.6}$$

$$H_j^{(2)}(z)=J_j(z)-iY_j(z)=i(2/\pi)\exp(ij\tfrac{1}{2}\pi)K_j(z\exp(i\tfrac{1}{2}\pi)). \tag{4.7}$$

Here $Y_j(z)$ stands for the Bessel functions of the second kind.

First we deal with the limits of the functions $F_{t,u}(s,j)$:

$$\lim_{s\to0}F_t(s,j)\equiv F_t(0,\epsilon)=\frac{8}{\pi}\,\frac{1}{m\mu}\int_{4m^2}^\infty dt'A_t(0,t')K_0(\epsilon(t'/m\mu)^{1/2}), \tag{4.8}$$

$$\lim_{s\to 0} F_u(s,j) \equiv F_u(0,\epsilon)$$
$$= -\frac{4i}{m\mu}\int_{-\infty}^{0} dt' A_u(0,t') H_0^{(2)}(\epsilon(-t'/m\mu)^{1/2}) + \frac{8}{\pi}\frac{1}{m\mu}\int_0^{(m-\mu)^2} dt' A_u(0,t')\, K_0(\epsilon(t'/m\mu)^{1/2}) . \tag{4.9}$$

Next we calculate the limit of the background integral term (3.11). It is easy to see that

$$\lim_{s\to 0}\frac{1}{\sin\pi j} P_j(-1-2st/\Delta) = \begin{cases} iH_0^{(1)}(\epsilon(-t/m\mu)^{1/2}) & \text{for } \mathrm{Im}\epsilon > 0 \\ -iH_0^{(2)}(\epsilon(-t/m\mu)^{1/2}) & \text{for } \mathrm{Im}\epsilon < 0. \end{cases} \tag{4.10}$$

This yields

$$F^b_{t,u}(0,t) = \frac{1}{8\pi}\int_0^{i\infty} F_{t,u}(0,\epsilon) H_0^{(1)}(\epsilon(-t/m\mu)^{1/2})\epsilon d\epsilon - \frac{1}{8\pi}\int_{i\infty}^{0} F_{t,u}(0,\epsilon) H_0^{(2)}(\epsilon(-t/m\mu)^{1/2})\epsilon d\epsilon . \tag{4.11}$$

If $F_{t,u}(0,\epsilon)$ behaves at most like a polynomial in the right ϵ plane for $|\epsilon| \to \infty$, Eq. (4.11) can be written also as

$$F^b_{t,u}(0,t) = \frac{1}{4\pi}\int_0^{\infty} \epsilon d\epsilon\, F_{t,u}(0,\epsilon) J_0(\epsilon(-t/m\mu)^{1/2}) . \tag{4.12}$$

This last expression looks exactly like an expansion with respect to lightlike, unitary representations of the Poincaré group. (A similar result was obtained also in Ref. 6.) Our assumption about the asymptotic behavior of the function $F_t(0,\epsilon)$ is obviously correct. The situation is more complicated in the case of $F_u(0,\epsilon)$. The integral representation (4.9) defines it only for $\mathrm{Im}\epsilon < 0$, where our assumption about its asymptotic behavior can be again verified. For $\mathrm{Im}\epsilon > 0$ it remains unverified, just as for $s \neq 0$. It will be shown later, however, that the assumptions we made are reasonable.

The calculation of the pole terms leads to an interesting result, if one supposes that at $s=0$ the poles are placed at real $\epsilon_i(s=0)=\epsilon_i$ points. Due to (4.10), the contour integrals of (3.12) must be evaluated not by the theorem of residues, but by applying the formula

$$\frac{1}{x \pm i0} = \frac{\mathrm{P}}{x} \mp i\pi\delta(x).$$

The result is

$$F^p_{t,u}(0,t) = \sum_{\text{poles}} \beta_{t,u}(\epsilon_i) Y_0(\epsilon_i(-t/m\mu)^{1/2}), \tag{4.13}$$

where $\beta_{t,u}(\epsilon_i)$ denotes the residues of the functions $F_{t,u}(0,\epsilon)$. It is remarkable that the second kind of function Y_0 has appeared in (4.13).

All the calculations of this section were performed by changing the order of integrations and limiting in s. Obviously, had we not used the functions $F_{t,u}(s,j)$ instead of $F_\pm(s,j)$, we should have obtained meaningless results. On the other hand, the limit of the functions $F_\pm(s,j)$ may very well exist, even if the limit of the integrands does not. [We remind the reader of theorems about the existence of

$$\lim_{\mu\to\infty}\int_a^b f(x)\sin\mu x\, dx$$

for example.] However, simply making the assumption that $\lim F_\pm(s,j)$ exists would lead to uncontrollable expressions.

V. SELF-CONSISTENCY AND COMPATIBILITY WITH DISPERSION RELATIONS

This section is devoted to the examination of two problems. The first is related to the connection of complex-angular-momentum theory and expansions with respect to Poincaré representations. Our concept (in fact, it is due to Hermann[7]) is that complex angular momentum is important even if the scattering amplitude is square-integrable: It is a tool for continuing into each other the Poincaré expansions of the scattering amplitude for total four-momenta of different character. This interpretation makes use of the fact that those unitary representations of the Poincaré group which appear to be relevant for the expansion of square-integrable functions in the timelike, lightlike, and spacelike cases can be characterized by the eigenvalues of one and the same Casimir operator $W_\mu W^\mu$ (beyond, of course, $P_\mu^2 = s$). It is not *a priori* obvious that there exists an analytic function $F(s,w)$ which, at the appropriate values of s and w, takes on the values of the expansion coefficients for the previous three expansions. [It is very difficult to say anything about the effect of non-square-integrability, beyond that it presumably corresponds to certain w singularities of $F(s,w)$.]

The second question is independent of group theory, and is probably more important from the point of view of theories based on the well-established analytic properties of the scattering amplitude. Namely, the question arises of whether our prescription for the $s=0$ limit is compatible with dispersion relations we assumed to be valid also for the u-channel amplitude.

To answer the first question we compare the formulas for the u channel obtained by the analytic continuation of the Watson-Sommerfeld transformed form of the s-channel scattering amplitude [that is, the formulas (3.6), (3.7), (3.10), (3.11), and (4.8), (4.9), (4.11), and (4.12)] and the appropriate crossed-channel expansions we are going to write down assuming a square-integrable (in $\cos\theta_s$) scattering amplitude also in the u channel. Clearly, the main task is to cast the inverse formulas for these latter expansions.

$$F_{t,u}(s,j)=\frac{(m^2-\mu^2+\Delta^{1/2})^2-s^2}{2sm\mu}\int_{-1}^{1}dc\,F_{t,u}(s,t)P_j(c) \tag{5.1}$$

and

$$F_{t,u}(0,\epsilon)=\int_0^\infty \xi d\xi\, F_{t,u}(0,t)J_0(\epsilon\xi) \tag{5.2}$$

into a form comparable to those previously obtained in Secs. III and IV. Our notation is

$$c\equiv\cos\theta_s=1+2st/\Delta(s,m^2,\mu^2), \tag{5.3}$$

$$\xi=(-t/m\mu)^{1/2}. \tag{5.4}$$

For this purpose, at $s\neq 0$, we should apply the identity

$$Q_j(x)=\tfrac{1}{2}\int_{-1}^{1}dc\frac{P_j(c)}{x-c}=\tfrac{1}{2}\int_{-\Delta/s}^{0}\frac{P_j(1+2st/\Delta)}{t'-t}dt, \tag{5.5}$$

where

$$x=1+2st'/\Delta(s,m^2,\mu^2). \tag{5.6}$$

There was no problem with (5.5) in the s channel, where we needed it only for $t'-t\neq 0$, $-1<c<1$, $|z|>1$. When we are in the u channel, in the region $0<s<(m-\mu)^2$, $(m+\mu)^2<u<(m^2-\mu^2)^2/s$, the situation changes, and can be summarized as follows. From a detailed study of the original Cauchy integral one can see that the dispersion relation (3.5) is to be rewritten as

$$F(s,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}dt'\frac{A(s,t')}{t'-t+i0}, \tag{5.7}$$

where

$$A(s,t')=\begin{cases}A_t(s,t') & \text{if}\quad t'>4m^2\\ 0 & \text{if}\quad (m-\mu)^2-s<t'<4m^2\\ A_u(s,t') & \text{if}\quad t'<(m-\mu)^2-s,\end{cases} \tag{5.8}$$

and $1/(t'-t+i0)$ denotes the generalized function[16] $P/(t'-t)-i\pi\delta(t'-t)$. The plus sign before $i0$ in (5.8) comes from the $i\epsilon$ prescription of S-matrix theory. The condition $-1<c<1$ remains true, but it is easy to see that now we need (5.5) also for values of z on the real axis between -1 and $+1$, when (5.5) fails to be valid in the sense of classical functions. It remains, however, true in the sense of generalized functions. Namely, it is shown in Appendix A that

$$Q_j(x\pm i0)=\tfrac{1}{2}\int_{-1}^{1}dc\frac{P_j(c)}{x-c\pm i0} \tag{5.9}$$

is true for $-1<c<1$, and for any value of x. Then it is obvious that the formulas (3.6) and (3.7) appear for the expansion coefficients also in the u channel. This shows that, starting either from the s or the u channel, one can define one and the same complex angular momentum. It is clear, moreover, that no simple trick (like the introducing of signatured functions in the s-channel) makes it possible to define an analytic continuation satisfying Carlson's theorem. In fact, complex angular momentum functions satisfying Carlson's theorem in the u channel would be incompatible with the ones defined in the s channel.

In the case of $s=0$ the basic formula one must apply is[15]

$$\int_{-\infty}^{0}dt\frac{J_0(\epsilon(-t/m\mu)^{1/2})}{t'-t}=K_0(\epsilon(t'/m\mu)^{1/2}), \tag{5.10}$$

which is valid in classical sense for $|\arg t'|<\pi$. However, it is shown in Appendix A that, for $\arg t'=\pm\pi$, Eq. (5.12) remains true, and it is to be understood as

$$\int_{-\infty}^{0}dt\frac{J_0(\epsilon(-t/m\mu)^{1/2})}{t'-t\pm i0}=\mp i\tfrac{1}{2}\pi H_0^{(1,2)}(\epsilon(t'/m\mu)^{1/2}). \tag{5.11}$$

These relations assure that the formulas of the crossed-channel expansion and the ones obtained in Sec. IV from the s-channel expansion coincide also at $s=0$.

The problem of compatibility with dispersion relations, mentioned at the beginning of this section, can be formulated in the following manner. The expansion procedure followed in the previous sections consists, first, in giving the kernel of (5.7) the form

$$\frac{1}{t'-t+i0}=\frac{2s}{\Delta}\sum_{j=0}^{\infty}(2j+1)P_j(c)Q_j(x+i0)=\frac{s}{i\Delta}\int_C d_j\frac{2j+1}{\sin\pi j}P_j(-c)Q_j(x+i0), \tag{5.12}$$

or, for $s=0$,

$$\frac{1}{t'-t+i0}=\frac{1}{m\mu}\int_0^\infty \epsilon d\epsilon J_0(\epsilon(-t/m\mu)^{1/2}) \times K_0(\epsilon[(t'+i0)/m\mu]^{1/2}). \quad (5.13)$$

Second, substituting the right-hand side of (5.12) or (5.13) into (5.7), and changing the order of integrations, one gets

$$F_t(s,t)=\frac{1}{i\pi}\frac{s}{\Delta}\int_C dj\frac{2j+1}{\sin\pi j}P_j(-c)\left[\int_{4m^2}^\infty A_t(s,t')Q_j\left(1+\frac{2st'}{\Delta}\right)dt'\right], \quad (5.14)$$

or, when $s=0$,

$$F_t(0,t)=\frac{1}{\pi}\frac{1}{m\mu}\int_0^\infty \epsilon d\epsilon J_0(\epsilon(-t/m\mu)^{1/2})\left[\int_{4m^2}^\infty A_t(0,t')K_0(\epsilon(t'/m\mu)^{1/2})dt'\right]. \quad (5.15)$$

[The reader can easily write down the corresponding expressions for $F_u(s,t)$.] We consider a limiting prescription compatible with the dispersion relation if the limit of the expression for $1/(t'-t+i0)$ given at $s>0$ is identical with the expression given at $s=0$. Obviously, our prescription has this property.

Another problem arising is the question of whether changing the order of integrations is a legal step. In fact, this is the question of the convergence of our expansions. We are not going to discuss this delicate problem for $s\neq 0$, where we have the more or less familiar, old formulas. For $s=0$ and the function $F_t(0,t)$ we state the following theorem: If the function $F_t(0,t)$ can be represented for $t<0$ by an integral

$$F_t(0,t)=\frac{1}{\pi}\int_{4m^2}^\infty dt'\frac{A_t(0,t')}{t'-t},$$

where the discontinuity $A_t(0,t')$ is integrable in any finite interval of $(4m^2,\infty)$, and behaves like t'^α for $t'\to\infty$, then the equality (5.15) is true. (Obviously, this theorem makes it possible to write a Fourier-Bessel integral for a non-square-integrable class of functions.) The proof of this theorem is given in Appendix B. To get a corresponding theorem also for $F_u(0,t)$ we need further work.

VI. DISCUSSION

In the previous sections we described the basic ideas for calculating the limit of the Watson-Sommerfeld transformed scattering amplitude to $s=0$ in the unequal-mass case. We discuss here some characteristics of our result:

$$F(s=0,t)=\frac{1}{4\pi}\int_0^\infty \epsilon d\epsilon[F_t(0,\epsilon)+F_u(0,\epsilon)] \times J_0(\epsilon(-t/m\mu)^{1/2}) + \sum_{\text{poles}}\beta(\epsilon_i)Y_0(\epsilon_i(-t'/m\mu)^{1/2}), \quad (6.1)$$

where

$$F_t(0,\epsilon)=\frac{8}{\pi}\frac{1}{m\mu}\int_{4m^2}^\infty dt'A_t(0,t')K_0(\epsilon(t'/m\mu)^{1/2}), \quad (6.2)$$

and

$$F_u(0,\epsilon)=-\frac{4i}{m\mu}\int_{-\infty}^0 dt'A_u(0,t')H_0^{(2)}(\epsilon(-t'/m\mu)^{1/2}) + \frac{8}{\pi}\frac{1}{m\mu}\int_0^{(m-\mu)^2}dt'A_u(0,t')K_0(\epsilon(t'/m\mu)^{1/2}). \quad (6.3)$$

Our first observation is that the pole terms do not exhibit the t^α- power behavior for $(-t)\to\infty$, since the Y_0 functions behave like

$$Y_0(z)\underset{|z|\to\infty}{\sim}(2/\pi z)^{1/2}\sin(z-\tfrac{1}{4}\pi). \quad (6.4)$$

On the other hand, the theorem stated at the end of the previous section indicates that the first "background" term of (6.1) is probably sufficient to expand a scattering amplitude with Regge behavior. [In fact, we proved it only for $F_t(0,t)$, but similar statements seem to be valid also for $F_u(0,t)$.] It follows that, if we believe in the t^α asymptotic behavior, the usual rule concerning the dominance of pole terms over the "background" integral does not apply at $s=0$. The formulas (6.2) and (6.3) indicate that the pole terms of (6.1) are probably not present at all. It is easy to see that the usual assumptions

$$A_t(0,t')\underset{t'\to\infty}{\sim}t'^\alpha,\quad A_u(0,t')\underset{-t'\to\infty}{\sim}(-t')^\alpha,\quad \alpha<0,$$

do not lead to any singularity on the half plane $\mathrm{Re}\,\epsilon>0$.

To see some details we assume a very simple model:

$$A_t(0,t')=\begin{cases}a_t t'^\alpha & \text{if } t'>0\\ 0 & \text{if } t'<0,\end{cases} \quad (6.5)$$

$$A_u(0,t')=\begin{cases}0 & \text{if } t'>0\\ a_u(-t')^\alpha & \text{if } t'<0,\end{cases} \tag{6.6}$$

where $-\frac{1}{2}<\alpha<0$ and a_t and a_u are real constants. The integrals corresponding to (6.2) and (6.3) yield

$$F_t(0,\epsilon)=\frac{8}{\pi}\frac{a_t}{m\mu}\int_0^\infty dt'\,t'^\alpha K_0(\epsilon(t'/m\mu)^{1/2})$$
$$=\frac{16a_t}{\pi}(4m\mu)^\alpha\Gamma^2(\alpha+1)\epsilon^{-2(\alpha+1)}, \tag{6.7}$$

$$F_u(0,\epsilon)=-\frac{4ia_u}{m\mu}\int_{-\infty}^0 dt'(-t')^\alpha H_0^{(2)}(\epsilon(-t'/m\mu)^{1/2})$$
$$=-\frac{16a_u}{\pi}e^{-i\pi\alpha}(4m\mu)^\alpha\Gamma^2(\alpha+1)\epsilon^{-2(\alpha+1)}. \tag{6.8}$$

We remind the reader that the integral (6.3) defines $F_u(0,\epsilon)$ only for $\mathrm{Im}\epsilon<0$. After evaluating the integral in this region the result can be extended also for $\mathrm{Im}\epsilon>0$. [Equation (6.8) is just an example of this.] Finally, the integral (6.1) gives the expected result for the scattering amplitude:

$$F(0,t)=\frac{1}{\sin\pi\alpha}[a_t(-t)^\alpha-a_u t^\alpha]. \tag{6.9}$$

One could examine more-complicated models (with more-complicated A_t, A_u functions, but with the previous asymptotic behavior), but the following features of this simple model would remain unchanged. There are no poles of the functions $F_{t,u}(0,\epsilon)$ in the ϵ right-half plane. Instead, one always finds a branch point at $\epsilon=0$ with the characteristic power $\epsilon^{-2(\alpha+1)}$. For large values of $|t|$ the dominant contribution to the integral (6.1) comes from the lower end of the integration path, and, asymptotically, the form (6.9) is always reproduced.

It is worth noticing how nicely these results correspond to the Lorentz pole picture, the usual solution of the singularity problem at $s=0$. First, we have seen that the "cause" of the t^α behavior is "concentrated" at the $\epsilon=0$ point, which is the image of the j plane (due to the singular mapping at $s=0$). That is, the power behavior is something deeply connected with the j plane. Second, it is not very hard to imagine that the infinite sequence of the (j-plane) daughters accumulates (on the ϵ plane) when $s=0$, and forms a branch point at $\epsilon=0$. Of course, it is difficult to guess the nature of the branch point. Just as in the case of considering all the conspiring daughters, we did not find here any singularity at $s=0$, only the t^α behavior was reproduced.

Our last remark concerns signature. In the previous sections it was important that we did not introduce signatured functions. Of course, Eq. (3.8) always makes it possible to restore the old formalism with signature if $s\neq 0$. For $s=0$, Eq. (3.8) becomes singular. However, introducing the quantities

$$a_\pm=\tfrac{1}{2}(a_t\mp a_u), \tag{6.10}$$

Eq. (6.9) can be rewritten as follows:

$$F(0,t)=\frac{1+\exp(-i\pi\alpha)}{\sin\pi\alpha}a_+(-t)^\alpha$$
$$+\frac{1-\exp(-i\pi\alpha)}{\sin\pi\alpha}a_-(-t)^\alpha. \tag{6.11}$$

Of course, this form follows for $F(0,t)$ from the assumptions of power behavior and symmetry between the t and u channels. It is more remarkable that our formalism is compatible with it without superimposing the formulas [notice the factor $\exp(-i\pi\alpha)$ in (6.8)].

ACKNOWLEDGMENTS

The authors are pleased to thank Professor G. F. Chew, Professor M. Jacob, Professor A. Martin, and Professor Ya. A. Smorodinsky for useful discussions and suggestions which helped them very much in clarifying several problems.

APPENDIX A

In Sec. V we stated the equalities

$$Q_j(x\pm i0)=\tfrac{1}{2}\int_{-1}^1 dc\frac{P_j(c)}{x-c\pm i0} \tag{A1}$$

and

$$\sum_{j=0}^\infty(2j+1)P_j(c)Q_j(x\pm i0)=\frac{1}{x-c\pm i0} \tag{A2}$$

for $-1<c<+1$. Their proof is straightforward by using the identity

$$\frac{1}{x-c\pm i0}=\frac{\mathrm{P}}{x-c}\mp i\pi\delta(x-c),$$

the formula 15.3(6) of Ref. 17:

$$\mathrm{P}\int_{-1}^1 dc\,\frac{P_j(c)}{x-c}=\tfrac{1}{2}Q_j(x)$$

(P denotes the principal value of the integral), and 3.4(9) of Ref. 14:

$$Q_j(x\pm i0)=Q_j(x)\mp i\tfrac{1}{2}\pi P_j(x).$$

Next we investigate the expression

$$\frac{m\mu}{t'-t\pm i0}=\int_0^\infty \epsilon\, d\epsilon\, J_0(\epsilon(-t/m\mu)^{1/2})$$
$$\times K_0(\epsilon[(t'\pm i0)/m\mu]^{1/2}), \tag{A3}$$

where both t and t' are negative. We apply the regularization technique of Ref. 16, and define the integral (A3) as follows:

$$\frac{m\mu}{t'-t\pm i0}=\mp i\tfrac{1}{2}\pi\lim_{s\to 2}\int_0^\infty \epsilon^{s-1}J_0(\epsilon(-t/m\mu)^{1/2})\times H_0^{(1,2)}(\epsilon(-t'/m\mu)^{1/2})d\epsilon . \tag{A4}$$

It was found in Ref. 11 that

$$\lim_{s\to 2}\int_0^\infty d\epsilon\,\epsilon^{s-1}J_0(\epsilon(-t/m\mu)^{1/2})J_0(\epsilon(-t'/m\mu)^{1/2}) = 2m\mu\delta(t-t') . \tag{A5}$$

The remaining task is to calculate the quantity

$$\lim_{s\to 2}\int_0^\infty d\epsilon\,\epsilon^{s-1}J_0(\epsilon(-t/m\mu)^{1/2})Y_0(\epsilon(-t'/m\mu)^{1/2}) .$$

For this purpose one must use the formulas 6.8 (37, 38, 47) of Ref. 18:

$$\int_0^\infty x^{s-1}J_0(bx)Y_0(ax)dx = \pi^{-1}2^{s-1}a^{-s}\sin\tfrac{1}{2}\pi(s-1)\,\Gamma^2(\tfrac{1}{2}s)F(\tfrac{1}{2}s,\tfrac{1}{2}s;1;b^2/a^2) ,$$

if $a>b>0$, $0<\mathrm{Re}\,s<2$;

$$\int_0^\infty x^{s-1}J_0(bx)Y_0(ax)dx = -\int_0^\infty x^{s-1}J_0(ax)Y_0(bx)dx - \frac{4}{\pi^2}\cos\tfrac{1}{2}\pi(s-1)\int_0^\infty x^{s-1}K_0(bx)K_0(ax)dx ,$$

if $b>a>0$, $0<\mathrm{Re}\,s<2$;

$$\int_0^\infty x^{s-1}K_0(ax)K_0(bx)dx = \frac{2^s a^{-s}}{\Gamma(s)}\Gamma^4(\tfrac{1}{2}s)F(\tfrac{1}{2}s,\tfrac{1}{2}s;s;1-b^2/a^2) ,$$

if $\mathrm{Re}(a+b)>0$, $\mathrm{Re}\,s>0$. The result is

$$-i\tfrac{1}{2}\pi\lim_{s\to 2}\int_0^\infty \epsilon^{s-1}J_0(\epsilon(-t/m\mu)^{1/2})Y_0(\epsilon(-t'/m\mu)^{1/2})d\epsilon = m\mu\mathrm{P}/(t'-t) , \tag{A6}$$

which completes the proof of (A3).

APPENDIX B

In this appendix we prove the theorem stated at the end of Sec. V. The theorem is as follows:

If

$$F(t)=\int_{t_0}^\infty \frac{A(t')}{t'-t}dt' , \tag{B1}$$

where $A(t')$ is integrable in any finite interval of (t_0,∞), and

$$|A(t')| \underset{t'\to\infty}{\sim} t'^{\alpha} ,$$

then

$$F(t)=\int_{t_0}^\infty A(t')\left(\int_0^\infty \epsilon J_0(\epsilon\sqrt{-t})K_0(\epsilon\sqrt{t'})d\epsilon\right)dt' = \int_0^\infty J_0(\epsilon\sqrt{-t})\epsilon\left(\int_{t_0}^\infty A(t')K_0(\epsilon\sqrt{t'})dt'\right)d\epsilon . \tag{B2}$$

The proof will be performed in two steps. First we prove that

$$\int_{t_0}^\infty A(t')\left(\int_0^\infty \epsilon J_0(\epsilon\sqrt{-t})K_0(\epsilon\sqrt{t'})d\epsilon\right)dt' = \lim_{\Omega\to 0}\int_{t_0}^\infty A(t')\left(\int_0^{1/\Omega} \epsilon J_0(\epsilon\sqrt{-t})K_0(\epsilon\sqrt{t'})d\epsilon\right)dt' . \tag{B3}$$

Second, we show that

$$\int_{t_0}^\infty A(t')\left(\int_0^{1/\Omega} \epsilon J_0(\epsilon\sqrt{-t})K_0(\epsilon\sqrt{t'})d\epsilon\right)dt' = \int_0^{1/\Omega} J_0(\epsilon\sqrt{-t})\epsilon\left(\int_{t_0}^\infty A(t')K_0(\epsilon\sqrt{t'})dt'\right)d\epsilon . \tag{B4}$$

Combining (B3) and (B4) we just obtain (B2).

The proof is based on two well-known theorems of mathematical analysis:

(1) If the function $f(x,y)$ can be written as $f(x,y)=g(x,y)k(y)$, where $k(y)$ is integrable in any finite interval of $\tilde\alpha<y<+\infty$, $g(x,y)$ is continuous in $a\le x\le b$, $\tilde\alpha\le y\le\infty$, and the integral

$$F(x)=\int_{\tilde\alpha}^\infty f(x,y)dy$$

converges uniformly in $[a,b]$, then the function $F(x)$ is continuous in $[a,b]$, and, under the same assumptions, the following equality is true:

$$\int_a^b\left(\int_{\tilde\alpha}^\infty f(x,y)dy\right)dx=\int_{\tilde\alpha}^\infty\left(\int_a^b f(x,y)dx\right)dy .$$

(2) The integral $\int_{\tilde\alpha}^\infty f(x,y)dy$ converges uniformly in $[a,b]$ if there exists a function $G(y)$ such that $|f(x,y)|\le G(y)$, for any $a\le x\le b$, $\tilde\alpha\le y$, and if the integral $\int_{\tilde\alpha}^\infty G(y)dy$ converges.

First we define two functions:

$$f_t(t',\Omega)\equiv\int_0^{1/\Omega} xJ_0(x)K_0(x(-t'/t)^{1/2})dx , \tag{B5}$$

and

$$F(t,\Omega)\equiv-\frac{1}{t}\int_{t_0}^\infty f_t(t',\Omega)A(t')dt' . \tag{B6}$$

Obviously,

$$F(t)=F(t,0)\equiv-\frac{1}{t}\int_{t_0}^\infty f_t(t',0)A(t')dt' . \tag{B7}$$

We show that $F(t,\Omega)$ is a continuous function of Ω in some $0\le\Omega\le\Omega_0$ interval, where Ω_0 is an arbitrary positive number. The function $A(t')$ is, by assumption, integrable in any finite interval of (t_0, ∞). It is easy to see that the integral in (B6) is uniformly convergent in $0 \le \Omega \le \Omega_0$. That is,

$$|f_t(t', \Omega)A(t')| \le |A(t')| \int_0^\infty |xK_0(x(-t'/t)^{1/2})|dx$$

$$= -\frac{t}{t'}|A(t')| \int_0^\infty |yK_0(y)|dy.$$

The last integral is convergent, thus we have the relation

$$-\frac{1}{t}|f_t(t', \Omega)A(t')| \le C\frac{|A(t')|}{t'}$$

valid for any Ω in $[0, \Omega_0]$ and t' in $[t_0, \infty)$. It is assumed that

$$|A(t')| \underset{t' \to \infty}{\sim} t'^{\alpha}, \quad \alpha < 0;$$

therefore also the integral

$$\int_{t_0}^\infty \frac{|A(t')|}{t'} dt'$$

converges. Consequently, the integral (B6) converges uniformly. Lengthy but straightforward calculation yields that the difference

$$|f_t(t'+\delta, \Omega+\omega) - f_t(t', \Omega)|$$
$$\le |f_t(t'+\delta, \Omega+\omega) - f_t(t'+\delta, \Omega)|$$
$$+ |f_t(t'+\delta, \Omega) - f_t(t', \Omega)|$$

can be made arbitrarily small for any $0 \le \Omega \le \Omega_0$, $t_0 \le t'$. It follows that $F(t, \Omega)$ is a continuous function of Ω in $0 \le \Omega \le \Omega_0$, that is,

$$\lim_{\Omega \to 0} F(t, \Omega) = F(t, 0).$$

The equality (B3) is proved.

Before starting with the proof of (B4), we introduce a new auxiliary function:

$$f(t', \Omega, E) = \int_E^{1/\Omega} xJ_0(x)K_0(x(-t'/t)^{1/2})dx, \tag{B8}$$

and consider

$$F(t, \Omega, E) \equiv -\frac{1}{t}\int_{t_0}^\infty A(t')f(t', \Omega, E)dt'. \tag{B9}$$

One can prove again that, at fixed t and Ω, the function $F(t, \Omega, E)$ is a continuous function of E in $0 \le E \le E_0$; that is,

$$F(t, \Omega, 0) \equiv F(t, \Omega) = (-1/t)\lim_{E \to 0}\int_{t_0}^\infty dt' A(t')\left(\int_E^{1/\Omega} xJ_0(x)K_0(x(-t'/t)^{1/2})dx\right). \tag{B10}$$

Repeating the same reasoning as previously, it is easy to check that

$$\int_{t_0}^\infty A(t')\left(\int_E^{1/\Omega} xJ_0(x)K_0(x(-t'/t)^{1/2})dx\right)dt' = \int_E^{1/\Omega} xJ_0(x)\left(\int_{t_0}^\infty A(t')K_0(x(-t'/t)^{1/2})dt'\right)dx. \tag{B11}$$

From Eqs. (B8), (B9), and (B3) the validity of (B2) follows.

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