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Conservation Laws and Symmetry Properties of Scalar-Tensor Gravitational Theories*

Harold B. Hart†
Department of Physics, Western Illinois University, Macomb, Illinois 61455

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Scalar-tensor (ST) analogs of the Einstein pseudotensor, the von Freud superpotential, the Møller superpotential, and the Komar vector are derived and used to form conservation laws appropriate to any version of the ST gravitational theory. When applied to a ST central-mass field, the conservation laws are found to yield correct values for total energy and momentum. The scalar field is shown to affect, in general, not only the form of the conservation laws but, through its symmetry properties, the identification of physically significant conserved quantities. It is pointed out that ST-conserved complexes have units of energy and momentum only in certain versions of the theory (e.g., the Brans-Dicke version).

I. INTRODUCTION

Since Einstein first published his theory of gravitation, a number of modifications have been proposed which purport to incorporate into the theory certain "desirable" features lacking in the original formulation. The most seriously considered of these modifications to date has been the scalar-tensor (ST) gravitational theory proposed in various forms by Kaluza,[1] Einstein and Bergmann,[2] Jordan,[3] Thiry,[4] and Brans and Dicke.[5] During the past decade, extensive examination of certain aspects of the ST theory (e.g., cosmology, gravitational red shift, deflection of light, planetary motion, etc.) has revealed significant points of similarity and contrast in ST and Einsteinian gravitation. Work in the important area of ST conservation laws, however, has been meager,[6,7] and the results obtained were either limited in applicability or incapable of yielding correct global conserved quantities when applied to simple ST fields, or both. It thus seems appropriate to consider here the derivation of correct ST conservation laws applicable to any version of the ST theory and the related problem of the symmetry properties of a ST field.

In Sec. II a generalized version of the ST theory, encompassing the various forms mentioned above, is introduced and ST differential identities are obtained. In Sec. III ST "double-index" differential conservation laws (laws involving conserved quantities with two indices) are derived and the ST analogs of the Einstein pseudotensor,[8] the von Freud superpotential,[9] and the Møller superpotential[10] are deduced. "Single-index" differential conservation laws (laws involving conserved quantities with one index) are discussed in Sec. IV, the ST Komar generator is determined, and symmetry properties of the ST theory are examined. Section V deals with integral conservation laws and the application of the results of Secs. III and IV to the spherically symmetric ST gravitational field of a central mass.

II. A GENERALIZED SCALAR-TENSOR THEORY

All versions of the ST theory are based on the introduction of a scalar field variable $\phi(x)$ (where x represents the four generalized coordinates x^1, x^2, x^3, and x^0) into general relativity (GR); this scalar field together with the metric tensor $g_{ij}(x)$ (Latin indices take on the values 0, 1, 2, 3; Greek indices take on the values 1, 2, 3) forms the ST gravitational field. The actual incorporation of the scalar field into the theory is accomplished by replacing Einstein's gravitational Lagrangian density

$$L_G=(-g)^{1/2}g^{ij}R_{ij} \tag{1}$$

with an appropriate ST Lagrangian density. We shall consider a generalized form suggested by Bergmann,[11]

$$\overline{L}_G=f_G(\phi)L_G+f_S(\phi)S \tag{2}$$

(bars will be used to indicate ST analogs of GR expressions), where L_G is given by Eq. (1), S is a function of the metric tensor and the first derivatives of the scalar field,

$$S=(-g)^{1/2}g^{ij}\phi_{,i}\,\phi_{,j}$$

(commas denote ordinary derivatives; semicolons denote covariant derivatives), and $f_G(\phi)$ and $f_S(\phi)$ are arbitrary scalar functions of ϕ. The form of S is determined by the condition that the field equations be of no higher than the second differential order and contain the highest derivative linearly. It is possible to reduce $f_G(\phi)$ and/or $f_S(\phi)$ to constant terms by suitable redefinition of the scalar and/or metric-tensor fields, but in the interest of generality we shall not do so.

Matter and other nongravitational fields may be included in the theory by introducing a term

$$\overline{L}_M = (-g)^{1/2} f_M(\phi) M$$

into the Lagrangian density. The term M is a phenomenological representation of the nongravitational fields, and the variational derivative of $(-g)^{1/2}M$ with respect to the metric tensor yields the stress-energy tensor density of all nongravitational contributions, $(-g)^{1/2}T^i{}_j$. The total ST Lagrangian density

$$\overline{L}_T = \overline{L}_G + \overline{L}_M = f_G(\phi)L_G + f_S(\phi)S + f_M(\phi)(-g)^{1/2}M \quad (3)$$

can be specialized to give various versions of the theory by appropriate choices of f_G, f_S, and f_M. It should be noted here that $f_M(\phi)$ is an arbitrary function except for the restriction that in the general-relativistic limit f_M/f_G must approach $8\pi G/c^4$ (G is the gravitational constant). The ST field equations are obtained by varying $\overline{L}_T$ with respect to g_{ij} and ϕ and setting these expressions equal to zero; i.e.,

$$\frac{\delta \overline{L}_T}{\delta g_{ij}} = 0 \quad (4)$$

and

$$\frac{\delta \overline{L}_T}{\delta \phi} = 0, \quad (5)$$

where the above expressions represent variational derivatives.

As is the case in Einstein's theory, it is possible to split the ST gravitational Lagrangian density into two parts,

$$\overline{L}_G = \overline{A} + \overline{B}^i{}_{,i}, \quad (6)$$

neither of which contains derivatives of g_{ij} and ϕ of higher than the first order. The explicit forms of $\overline{A}$ and $\overline{B}^i$ are

$$\overline{A} = f_G A + f_S S - f_{G,i} B^i \quad (7)$$

and

$$\overline{B}^i = f_G B^i, \quad (8)$$

where

$$A = (-g)^{1/2} g^{ij} (\Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{li}\Gamma^l_{jk})$$

and

$$B^i = (-g)^{1/2} (g^{ij}\Gamma^l_{jl} - g^{jl}\Gamma^i_{jl})$$

are the corresponding parts of the Einstein Lagrangian density

$$L_G = A + B^i{}_{,i}.$$

ST differential identities analogous to the Bianchi identities of GR are derived[12] by considering an arbitrary function of the coordinates, the field variables, and the first and second derivatives of the field variables

$$D(x^i, y_A(x), y_{A,j}(x), y_{A,jk}(x)) \equiv D(x; y_A)$$

(y_A represents the field variables g_{ij} and ϕ) which transforms as a scalar density under the group of general space-time coordinate transformations. Under an arbitrary infinitesimal coordinate transformation

$$x^{i\prime} = x^i + \xi^i(x) \quad (|\xi^i(x)| << 1), \quad (9)$$

the scalar density will satisfy the equation

$$D(x'; y_A'(x')) d^4x' = D(x; y_A(x)) d^4x,$$

where d^4x is an element of four-volume. Expanding the above and using the notation $\delta' y_A \equiv y_A'(x) - y_A(x)$ (the Lie derivative of y_A), we find

$$\frac{\delta D}{\delta y_A}\delta' y_A + t^m{}_{,m} = 0, \quad (10)$$

where

$$\frac{\delta D}{\delta y_A} = \frac{\partial D}{\partial y_A} - \left(\frac{\partial D}{\partial y_{A,p}}\right)_{,p} + \left(\frac{\partial D}{\partial y_{A,pq}}\right)_{,pq} \quad (11)$$

and

$$t^m = \left[\frac{\partial D}{\partial y_{A,m}} - \left(\frac{\partial D}{\partial y_{A,mn}}\right)_{,n}\right]\delta' y_A + D\delta x^m + \left(\frac{\partial D}{\partial y_{A,mn}}\right)\delta' y_{A,n}. \quad (12)$$

The y_A can be shown to transform according to the law

$$\delta' y_A = \xi^k \gamma_{Ak} - \xi^k{}_{,l}\gamma^l_{Ak}, \quad (13)$$

with γ_{Ak} and γ^l_{Ak} given by

$$\gamma_k = -\phi_{,k}, \quad \gamma^l_k = 0, \quad \gamma_{ijk} = -g_{ij,k},$$

and (14)

$$\gamma_{ijk}^{\ \ l} = \delta^l_i g_{kj} + \delta^l_j g_{ki}.$$

Equation (10) can then be expressed as

$$\xi^k\left[\frac{\delta D}{\delta y_A}\gamma_{Ak} + \left(\frac{\delta D}{\delta y_A}\gamma^l_{Ak}\right)_{,l}\right] + C^m{}_{,m} = 0, \quad (15)$$

with C^m a function of the y_A, the ξ^k, and their derivatives,

$$C^m = t^m - \frac{\delta D}{\delta y_A}\gamma^m_{Ak}\xi^k. \quad (16)$$

Integrating Eq. (15) over an arbitrary region R of space-time and applying Gauss's theorem, we obtain

$$\int_R d^4x\, \xi^k\left[\frac{\delta D}{\delta y_A}\gamma_{Ak} + \left(\frac{\delta D}{\delta y_A}\gamma^l_{Ak}\right)_{,l}\right] + \oint_{\text{HS}} dS_m C^m = 0,$$

where HS is the hypersurface bounding R. If we now require that the ξ^k and their derivatives vanish on HS while remaining arbitrary inside R, we obtain the differential identities

$$\frac{\delta D}{\delta y_A}\gamma_{Ak} + \left(\frac{\delta D}{\delta y_A}\gamma^l_{Ak}\right)_{,l} \equiv 0. \tag{17}$$

Letting $D = \overline{L}_G$ and expanding the above by means of Eq. (14) yields the four differential identities of the ST theory,

$$2g_{jk}\left(\frac{\delta \overline{L}_G}{\delta g_{ij}}\right)_{;i} - \left(\frac{\delta \overline{L}_G}{\delta \phi}\right)\phi_{,k} \equiv 0. \tag{18}$$

III. DOUBLE-INDEX CONSERVATION LAWS

In deriving double-index conservation laws it is clear that, instead of using the Bianchi identities (as was suggested by Brans[6]), one should employ the differential identities (18) of the ST theory. Expanding the covariant derivative in Eq. (18) we find

$$\left(g_{kj}\frac{\delta \overline{L}_G}{\delta g_{ij}}\right)_{,i} - \tfrac{1}{2}g_{ij,k}\left(\frac{\delta \overline{L}_G}{\delta g_{ij}}\right) - \tfrac{1}{2}\phi_{,k}\left(\frac{\delta \overline{L}_G}{\delta \phi}\right) = 0. \tag{19}$$

By using the fact that the variational derivative of a total divergence vanishes it is clear that

$$\frac{\delta \overline{L}_G}{\delta g_{ij}} = \frac{\delta \overline{A}}{\delta g_{ij}}$$

and

$$\frac{\delta \overline{L}_G}{\delta \phi} = \frac{\delta \overline{A}}{\delta \phi}.$$

Equation (19) can then be written as

$$\left(g_{kj}\frac{\delta \overline{A}}{\delta g_{ij}}\right)_{,i} - \tfrac{1}{2}g_{ij,k}\left(\frac{\delta \overline{A}}{\delta g_{ij}}\right) - \tfrac{1}{2}\phi_{,k}\left(\frac{\delta \overline{A}}{\delta \phi}\right) = 0.$$

It is easily verified that

$$\tfrac{1}{2}g_{ij,k}\left(\frac{\delta \overline{A}}{\delta g_{ij}}\right) + \tfrac{1}{2}\phi_{,k}\left(\frac{\delta \overline{A}}{\delta \phi}\right) = \frac{1}{2}\left(\overline{A}g^i{}_k - \frac{\partial \overline{A}}{\partial g_{lm,i}}g_{lm,k} - \frac{\partial \overline{A}}{\partial \phi_{,i}}\phi_{,k}\right)_{,i},$$

and defining the ST Einstein gravitational pseudotensor[8] as

$$\overline{t}^i_{Ek} \equiv \tfrac{1}{2}(f_M)^{-1}(-g)^{-1/2}\left(\frac{\partial \overline{A}}{\partial g_{lm,i}}g_{lm,k} + \frac{\partial \overline{A}}{\partial \phi_{,i}}\phi_{,k} - \overline{A}g^i{}_k\right), \tag{20}$$

we obtain the differential conservation law

$$\left(g_{kj}\frac{\delta \overline{A}}{\delta g_{ij}} + (-g)^{1/2}f_M\overline{t}^i_{Ek}\right)_{,i} = 0.$$

When the field equations for the combined ST and nongravitational fields are valid, the above becomes

$$[(-g)^{1/2}f_M(T^i{}_j + \overline{t}^i_{Ej})]_{,i} = 0 \quad (\text{letting } k \to j). \tag{21}$$

The conserved quantity

$$\overline{\Theta}^i_{Ej} \equiv (-g)^{1/2}f_M(T^i{}_j + \overline{t}^i_{Ej}) \tag{22}$$

is the ST analog of the Einstein energy-momentum complex[8]; however, we should note carefully that while it resembles closely the energy-momentum complexes of GR it has appropriate dimensions only in the special case where f_M is dimensionless. If f_M is constant (not necessarily dimensionless), we can define a new conserved complex

$$\overline{\Theta}'^i_{Ej} = (f_M)^{-1}\overline{\Theta}^i_{Ej}$$

which has the correct dimensions. These comments will be seen to apply to all of the ST complexes derived in this paper.

In determining the ST analog of the von Freud superpotential[9] we shall follow an approach first used by Møller[13] in GR and applied to the Jordan version of the ST theory by Just.[7] By means of Eqs. (13), (6), (12), and (16) we can rewrite Eq. (15) (with D replaced by $\overline{L}_G$) as

$$\xi^k X_k - (Y^m_k\xi^k + B^{ma}_k\xi^k{}_{,a} + M^{mab}_k\xi^k{}_{,ab})_{,m} = 0, \tag{23}$$

where

$$X_k = \frac{\delta \overline{L}_G}{\delta y_A}\gamma_{Ak} + \left(\frac{\delta \overline{L}_G}{\delta y_A}\gamma^a_{Ak}\right)_{,a},$$

$$Y^m_k = \frac{\delta \overline{L}_G}{\delta y_A}\gamma^m_{Ak} - \gamma_{Ak}\frac{\partial \overline{A}}{\partial y_{A,m}} + \overline{B}^m{}_{,k} - \delta^m_k\overline{A} - \delta^m_k\overline{B}^i{}_{,i},$$

$$B^{ma}_k = \frac{\partial \overline{A}}{\partial y_{A,m}}\gamma^a_{Ak} + \gamma^a_{Ak}\frac{\partial \overline{B}^m}{\partial y_A} - \gamma_{Ak}\frac{\partial \overline{B}^m}{\partial y_{A,a}} + \gamma^a_{Ak,j}\left(\frac{\partial \overline{B}^m}{\partial y_{A,j}}\right), \tag{24}$$

and

$$M^{mab}_k = \gamma^a_{Ak}\left(\frac{\partial \overline{B}^m}{\partial y_{A,b}}\right).$$

Expanding the divergence term and recombining, we find

$$\xi^k(X_k - Y^a_{k,a}) - \xi^k{}_{,a}(B^{ba}_k{}_{,b} + Y^a_k) - \xi^k{}_{,ab}(M^{cab}_k{}_{,c} + B^{ba}_k) - \xi^k{}_{,abc}(M^{cab}_k) = 0.$$

Since the ξ^k are arbitrary, each coefficient must vanish separately. The first term [by Eq. (17)] yields a differential conservation law

$$Y^i_{j,i} = X_j \equiv 0.$$

From the remaining conditions we find that

$$Y^i_j = U^{[ik]}_j{}_{,k}$$

(where the brackets indicate antisymmetry in the indices i,k) and

$$U^{[ik]}_j = \tfrac{1}{3}(M^{lik}_j - M^{ilk}_j)_{,l} + B^{ik}_j.$$

$U_j^{[ik]}$ is thus seen to be a superpotential for the conserved complex Y^i_j. Using Eqs. (24), (20), (14), (8), and (7), the above expressions become

$$Y^i_j = 2[(-g)^{1/2} f_M \bar{t}^i_{Ej} + (-g)^{1/2} f_M T^i{}_j] + (\bar{B}^i \delta^k_j - \bar{B}^k \delta^i_j)_{,k}$$

and

$$U_j^{[ik]} = \left[(-g)^{1/2}\left(2\frac{\partial f_G}{\partial\phi}(\delta^i_j \phi^{,k} - \delta^k_j \phi^{,i}) + f_G(g^{il,k} g_{lj} - g^{kl,i} g_{lj})\right)\right].$$

If we define the ST von Freud superpotential, $\bar{U}_{Ej}{}^{[ik]}$, by the equation

$$(\bar{U}_{Ej}{}^{[ik]})_{,k} = (-g)^{1/2} f_M (T^i{}_j + \bar{t}^i_{Ej}) = \bar{\Theta}^i_{Ej}, \tag{25}$$

it is clear from the previous expressions that

$$\begin{aligned}\bar{U}_{Ej}^{[ik]} &= \tfrac{1}{2}[U_j^{[ik]} + (\bar{B}^k \delta^i_j - \bar{B}^i \delta^k_j)] \\ &= \tfrac{1}{2}\left((-g)^{-1/2} g_{jl}[f_G g(g^{kl} g^{mi} - g^{il} g^{mk})]_{,m} + (-g)^{1/2} g_{jl}\frac{\partial f_G}{\partial\phi}\phi_{,m}(g^{il} g^{mk} - g^{kl} g^{mi})\right). \end{aligned} \tag{26}$$

New complexes, pseudotensors, and superpotentials can be formed by noting that the addition of a quantity $W_j^{[ik]}{}_{,k}$ to $\bar{\Theta}^i_{Ej}$ yields a new differential conservation law

$$\bar{\Theta}'^i{}_{j,i} = 0,$$

where

$$\bar{\Theta}'^i{}_j = \bar{\Theta}^i_{Ej} + W_j^{[ik]}{}_{,k}.$$

The new pseudotensor and superpotential become, respectively,

$$\bar{t}'^i{}_j = \bar{t}^i_{Ej} + W_j^{[ik]}{}_{,k}$$

and

$$\bar{U}'_j{}^{[ik]} = \bar{U}_{Ej}{}^{[ik]} + W_j^{[ik]}.$$

If we choose (in analogy with Møller's work in GR[10])

$$W_j^{[ik]} = \bar{U}_{Ej}^{[ik]} - \delta^i_j \bar{U}_{El}^{[lk]} + \delta^k_j \bar{U}_{El}^{[li]},$$

we obtain the ST analog of the Møller superpotential

$$\bar{U}_{Mj}^{[ik]} = (-g)^{1/2} g^{il} g^{km}[(g_{jm} f_G)_{,l} - (g_{jl} f_G)_{,m}]. \tag{27}$$

IV. SINGLE-INDEX CONSERVATION LAWS AND SYMMETRY PROPERTIES

A different approach to GR conservation laws, initiated by the work of Bergmann[14] and based on Noether's theorem,[15] indicates that every infinitesimal coordinate transformation [i.e., every choice of ξ^i in Eq. (9)] leads to a differential conservation law. The conserved quantities in this case have only a single index (they are thus current or flux vectors) and seem to be more appropriate to generally invariant theories than the double-index quantities of Einstein, Møller, etc. One form in particular, the locally covariant Komar vector,[16] seems to possess the most satisfactory qualities of any conserved quantity yet discovered.

ST single-index conservation laws can be derived by considering Eq. (10) with D replaced by $\bar{L}_G$,

$$\frac{\delta \bar{L}_G}{\delta y_A}\delta' y_A + t^i{}_{,i} = 0.$$

By Eq. (13) this becomes

$$t^i{}_{,i} + \frac{\delta \bar{L}_G}{\delta y_A}\xi^j \gamma_{Aj} - \frac{\delta \bar{L}_G}{\delta y_A}\xi^j{}_{,i}\gamma_{Aj}^{\ i} = 0,$$

and by using the differential identities (17) we obtain a differential conservation law

$$\left(-t^i + \frac{\delta \bar{L}_G}{\delta y_A}\gamma_{Aj}^{\ i}\xi^j\right)_{,i} = 0.$$

Since the above is an identity, it can be related to a double-index superpotential through the equation

$$\left(-t^i + \frac{\delta \bar{L}_G}{\delta y_A}\gamma_{Aj}^{\ i}\xi^j\right) = \bar{U}^{[ij]}{}_{,j}.$$

Requiring the combined gravitational and non gravitational field equations to hold yields

$$[-t^i + 2 f_M (-g)^{1/2} T^i_j \xi^j] = \bar{U}^{[ij]}{}_{,j}.$$

By comparing this expression with the conservation laws of Sec. III it is clear that we can make the associations

$$t^i = -2 f_M \xi^j (-g)^{1/2} \bar{t}^i{}_j$$

and

$$\bar{U}^{[ij]} = 2\xi^k \bar{U}^{[ij]}_k,$$

with $\bar{t}^i{}_j$ and $\bar{U}^{[ij]}_k$ identified as ST pseudotensors and three-index superpotentials, respectively. The conserved vector quantity

$$\bar{\Theta}^i = (\xi^k \bar{U}^{[ij]}_k)_{,j} = (-g)^{1/2} f_M \xi^j (T^i{}_j + \bar{t}^i{}_j)$$

is clearly dependent upon the choice of superpotential, and if, in particular, we use Møller's superpotential (27), we find

$$\begin{aligned}\bar{\Theta}^i_M &= (\xi^k \bar{U}_M{}^{[ij]}_k)_{,j} \\ &= \xi^k (-g)^{1/2} g^{il} g^{jm}[(g_{mk} f_G)_{,l} - (g_{lk} f_G)_{,m}]_{,j}.\end{aligned}$$

The ST Komar vector, $\bar{\Theta}^i_K$, is obtained by adding a quantity

$$W^{[ij]}{}_{,j} = (-g)^{1/2} g^{il} g^{jm}[(g_{mk} f_G \xi^k{}_{,l}) - (g_{lk} f_G \xi^k{}_{,m})]_{,j}$$

to $\bar{\Theta}^i_M$. Straightforward expansion gives

$$\begin{aligned}\bar{\Theta}^i_K &= (\bar{U}^{[ij]}_K)_{,j}\\ &= (-g)^{1/2} g^{il} g^{jm} [(g_{mk} f_G \xi^k)_{,l} - (g_{lk} f_G \xi^k)_{,m}]_{,j}\\ &= (-g)^{1/2} [(f_G \xi^j)^{;i} - (f_G \xi^i)^{;j}]_{,j}\\ &= (-g)^{1/2} [(f_G \xi^j)^{;i} - (f_G \xi^i)^{;j}]_{;j} .\end{aligned} \tag{28}$$

Note that $\bar{\Theta}^i_K$ is locally covariant and reduces to the appropriate GR form when $\phi \to$ const.

Since the above conserved vectors depend on the choice of ξ^i, there will be an infinity of conserved quantities corresponding to the infinite group of general coordinate transformations. It is generally thought, however, that physically significant conserved quantities are generated only by those infinitesimal transformations which represent intrinsic symmetry properties of the gravitational field[17] (when they exist). In GR, symmetry transformations are those which leave the metric tensor invariant and thus satisfy the equation

$$\delta' g_{ij} = 0 .$$

This expression can be expanded in terms of ξ^i to give Killing's equations

$$\xi^{i;j} + \xi^{j;i} = 0 .$$

Since a ST field depends on both the metric tensor and the scalar field, ST symmetry transformations are those which simultaneously satisfy Killing's equations and the scalar symmetry condition

$$\delta' \phi = \phi_{,j} \xi^j = 0 . \tag{29}$$

To illustrate the restriction imposed by the scalar symmetry condition, consider the Brans-Dicke ST DeSitter solution[18] in Cartesian coordinates:

$$ds^2 = -(dx^0)^2 + a^2(x^0)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] ,$$

$$\phi = \phi_0 (x^0/x^0_0)^K \quad (x^0_0, K = \text{constants}).$$

Solution of Killing's equations yields ten independent Killing vectors having the following components:

$$\xi^0 = A + B_\alpha x^\alpha ,$$

$$\xi^\alpha = C^\alpha{}_\beta x^\beta + B^\alpha \int \frac{dx^0}{a^2(x^0)} + D^\alpha - \frac{\dot{a}}{a} x^\alpha (A + B_\beta x^\beta) ,$$

where $\dot{a} \equiv da/dx^0$, $-C^\alpha{}_\beta = C^\beta{}_\alpha$, $B^\alpha = B_\alpha$. The ten parameters A, B_α, $C^\alpha{}_\beta$, D^α correspond to temporal translations, generalized Lorentz transformations, spatial rotations, and spatial translations, respectively. Application of the ST symmetry condition (29) gives

$$\phi_{,0} \xi^0 = 0 ,$$

which requires that $\xi^0 = 0$. We thus obtain "restricted Killing vectors" with components

$$\xi^0 = 0$$

and

$$\xi^\alpha = C^\alpha{}_\beta x^\beta + D^\alpha .$$

The temporal translations and generalized Lorentz transformations are not symmetry transformations of this ST field and any conserved quantities generated by them will have no simple physical interpretations. The scalar field thus affects not only the functional form of conserved ST quantities but, through its symmetry properties, plays a major role in determining which conserved quantities are to be considered physically meaningful.

V. INTEGRAL CONSERVATION LAWS AND APPLICATIONS

Integral conservation laws for isolated gravitating systems are obtained directly from the differential conservation laws of Secs. III and IV by integration over a region of space-time enclosing the system and application of Gauss's theorem. The resulting conserved quantities

$$\bar{P}_{Ej} = \int \bar{\Theta}^i_{Ej} \, dV_i ,$$

$$\bar{P}_{Mj} = \int \bar{\Theta}^i_{Mj} \, dV_i , \tag{30}$$

and

$$\bar{P}_K(\xi) = \int \bar{\Theta}^i_K(\xi) dV_i$$

(where the integrals are taken over open timelike hypersurfaces) are clearly related to the total energy and momentum of the systems involved but, as mentioned in Sec. III, have dimensions appropriate to these quantities only if f_M is a dimensionless quantity. In versions of the ST theory where f_M is a constant (not necessarily dimensionless), we can form the conserved quantities

$$\bar{P}'_{Ej} = (f_M)^{-1} \bar{P}_{Ej} ,$$

$$\bar{P}'_{Mj} = (f_M)^{-1} \bar{P}_{Mj} , \tag{31}$$

and

$$\bar{P}'_K(\xi) = (f_M)^{-1} \bar{P}_K(\xi) ,$$

which have appropriate dimensions. Thus, in general, the total conserved quantities of the ST theory are not total energy and momentum but are closely related quantities whose actual physical meaning depends on the particular version of the theory being considered.

In the BD version of ST theory, the exact static spherically symmetric vacuum gravitational field of a body of mass M is given in isotropic Cartesian coordinates by[5]

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}(dx^2 + dy^2 + dz^2),$$

$$\phi = \phi_0\left(\frac{1 - B/r}{1 + B/r}\right)^{C/\lambda},$$

where

$$r = (x^2 + y^2 + z^2)^{1/2},$$

$$e^{2\alpha} = e^{2\alpha_0}\left(\frac{1 - B/r}{1 + B/r}\right)^{2/\lambda},$$

and

$$e^{2\beta} = e^{2\beta_0}\left(\frac{1 - B/r}{1 + B/r}\right)^{2(\lambda - C - 1)/\lambda}(1 + B/r)^4.$$

The quantity λ is given by

$$\lambda^2 = (C + 1)^2 - C(1 - \tfrac{1}{2}\omega C),$$

where ω is a numerical parameter satisfying $\omega \geq 6$; α_0, β_0, B, ϕ_0, and C are constants which must take on the values

$$\alpha_0 = \beta_0 = 0, \quad C \simeq -1/(2 + \omega),$$

$$B \simeq \frac{M}{2c^2\phi_0}\left(\frac{2\omega + 4}{2\omega + 3}\right)^{1/2},$$

and

$$\phi_0 = \frac{1}{G}\left(\frac{4 + 2\omega}{3 + 2\omega}\right)$$

in order to agree with the weak-field limiting solution[5]

$$\phi = \phi_0 + \frac{2M}{(3 + 2\omega)c^2 r},$$

$$ds^2 = \left(-1 + \frac{2(4 + 2\omega)M}{c^2(3 + 2\omega)\phi_0 r}\right)dt^2 + \left(1 + \frac{4(1 + \omega)M}{c^2(3 + 2\omega)\phi_0 r}\right)(dx^2 + dy^2 + dz^2).$$

A previous calculation of the total energy of this system,[6] employing a conservation law derived from the GR differential identities, gave

$$\left(\frac{2\omega + 2}{2\omega + 3}\right)Mc^2,$$

a result related to but certainly not equal to the total energy Mc^2. In order to calculate the total system energy and momentum using the ST conservation laws derived above, we must work with Eqs. (31) rather than (30) since the BD version assumes a constant but not dimensionless f_M. The superpotentials (26), (27), and (28) can be employed to reduce the hypersurface integrals in Eqs. (31) to ordinary surface integrals,

$$\overline{P}'_{Ej} = (f_M)^{-1}\oint \overline{U}^{[0l]}_{Ej}\,dS_{0l},$$

$$\overline{P}'_{Mj} = (f_M)^{-1}\oint \overline{U}^{[0l]}_{Mj}\,dS_{0l}, \tag{32}$$

$$\overline{P}'_{K}(\xi) = (f_M)^{-1}\oint \overline{U}^{[0l]}_{K}\,dS_{0l}.$$

A straightforward calculation yields, in each case, zero total momentum and total energy equal to Mc^2.

The conservation laws are thus capable of giving the correct values for the total energy and momentum of a central-mass gravitational field, thereby satisfying one of the minimum requirements of any useful conservation law.

VI. CONCLUSION

The conservation laws derived above appear to be the natural extensions of the work of Einstein, Møller, Bergmann, and Komar to the ST theory. Although they have been applied in only one simple case, it seems reasonable to assume that their range of validity and applicability is as wide as that of the corresponding GR conservation laws in Einstein's theory and that ST investigations analogous to those done in GR can profitably be carried out with their assistance. It also seems likely that problems unique to the ST theory (e.g., scalar radiation) can be successfully treated.

The concept of restricted Killing vectors [i.e., Killing vectors restricted by the symmetry condition on the scalar field given in Eq. (29)] and its relationship to ST conservation laws requires further investigation. In particular, the question of whether or not the scalar symmetry condition actually restricts the Killing vectors of a gravitational field in all cases needs to be answered. The effect of the scalar restriction on conservation laws in physically closed systems should also be examined since the possibility of long-range scalar fields and time-dependent spherically symmetric fields exists.

If we accept conservation of energy and momentum as fundamental physical principles, the fact that total ST conserved quantities have units of energy and momentum only in certain versions of the theory has important consequences. ST theories which treat f_M as a constant and/or dimensionless function must then be considered more acceptable than other versions. This observation lends further support to the widely accepted notion that the BD version is the most physically reasonable formulation of the ST theory.

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Evanescent Waves in Potential Scattering from Regular Lattices*

S. Twareque Ali and Marjan Baće

Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627

and

Joseph H. Eberly†

Physics Department and Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

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A recently developed approach to scattering by regular structures is applied to an investigation of diffracted evanescent waves. The scattered field is expressed exactly, in the near or far field, by a sum of "plane lattice wave" modes. Attention is directed at the diffraction conditions obeyed by these modes when the scatterer is a finitely thick three-dimensional nonorthogonal (triclinic) lattice of individual scattering centers.

I. INTRODUCTION

Recent work has shown clearly that the angular spectrum of plane waves and especially the evanescent modes of the angular spectrum can play a helpful role in the study of a wide range of electrodynamic phenomena.[1] Within the past several years, the angular spectrum has been brought to play in new approaches to Čerenkovian effects[2] and inverse scattering,[3] for example. In addition, it has figured in a study of source-free fields,[4] a formulation of a diffraction theory of holography,[5] and the quantization of an electromagnetic wave field in an infinite space half filled with dielectric.[6]

It is important to realize that there are really two aspects to the use of the evanescent modes of a radiation field. In the first place, there may be no unique set of evanescent modes associated with a given field. This is due simply to the fact that evanescent modes are characterized by exponential decay in one direction and plane-wave propagation in the transverse directions, and the direction in which the exponential decay occurs may be undetermined. Thus one must approach with great caution the task of assigning physical significance to an evanescent wave. This lack of unique or clear physical meaning need not detract, of course, from the power of mathematical mode expansions which include evanescent modes.

On the other hand, there are physical problems for which a given direction is already singled out. It may happen that this quasi-one-dimensionality suggests the introduction of a mode expansion involving evanescent modes in a particular way. Individual evanescent modes, in such situations, may have a very direct physical interpretation. For example, the wave field outside of a totally internally reflecting dielectric decays away from the dielectric surface. Lalor and Wolf[7] have treated the problem of reflection and refraction at such an interface by the use of the physically suggested mode decomposition of the transmitted field, and Carniglia and Mandel[6] have found triads of evanescent and nonevanescent modes at a dielectric in-