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Comments on the $\pi\pi$ Model of Kang and Lee

E. P. Tryon*

Columbia University, New York, New York 10027

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The model for low-energy $\pi\pi$ amplitudes proposed recently by Kang and Lee is analyzed and shown to violate analyticity and/or crossing symmetry, and also Regge sum rules for S- and P-wave scattering lengths. Possible reasons for these violations are discussed, and the outline is given of an argument which indicates that the ρ meson is not a dynamically bound state of two π mesons.

Recently Kang and Lee (henceforth KL) proposed a model[1] for the low-energy $\pi\pi$ interaction based on analyticity, unitarity, the crossing conditions of Roskies,[2] the inequalities of Martin,[3] and the current-algebra predictions of Weinberg.[4] KL also assumed that the S waves and P wave can be decomposed into ratios N/D such that both N and D satisfy once-subtracted dispersion relations, and D contains no poles. The model succeeds in generating ρ and σ mesons as dynamically bound states of the $\pi\pi$ system, and is in qualitative agreement[5] with experiment.

In this paper, we demonstrate that the model of KL is severely inconsistent with analyticity and/or crossing symmetry, and with Regge sum rules for S- and P-wave scattering lengths. We suggest several possible reasons for these difficulties, and outline an argument which indicates that the ρ meson is *not* a dynamically bound state of two π mesons.

Let us denote the $\pi\pi$ partial-wave amplitudes by $A^{(l)I}(s)$, and use units wherein $m_\pi=\hbar=c=1$ (except where MeV is explicitly stated). We normalize the $A^{(l)I}$ such that

$$A^{(l)I}(s)=\left(\frac{s}{s-4}\right)^{1/2}e^{i\delta_l^I}\sin\delta_l^I$$

for $s>4$.

Analyticity and crossing symmetry imply that if $-32<s\leq 0$, then

$$\operatorname{Im}A^{(l)I}(s)=\frac{2}{s-4}\int_4^{4-s}dt\,P_l\left(1+\frac{2t}{s-4}\right)\sum_{I'=0}^{2}\beta_{II'}\sum_{l'=0}^{\infty}(2l'+1)\operatorname{Im}A^{(l')I'}(t)P_{l'}\left(1+\frac{2s}{t-4}\right), \tag{1}$$

where β denotes the SU(2) crossing matrix.[6] The Legendre series on the right-hand side of Eq. (1) diverges over part of the range of integration if $s<-32$, but may provide an asymptotic expansion over some interval of s to the left of -32.

In the model of KL, the left-hand cuts of the S- and P-wave N functions are represented by a few poles: three for the $I=0$ S wave, two for the $I=2$ S wave, and three for the P wave. These poles correspond to the $\operatorname{Im}A^{(l)I}$ being represented by sums of δ functions for negative s:

$$\operatorname{Im}A^{(l)I}(s)=\sum_n\gamma_n^{(l)I}\delta(s-s_n^{(l)I}), \tag{2}$$

which serves to define the parameters $\gamma_n^{(l)I}$ and $s_n^{(l)I}$.

It is obvious that one could insert experimental S- and P-wave phase shifts into the right-hand side of Eq. (1) to obtain a guide in selecting the $s_n^{(l)I}$ and $\gamma_n^{(l)I}$ intended to simulate the left-hand cuts of physical $A^{(l)I}$. This procedure was in fact used by KL to guide them in their selection of pole positions. However, instead of using Eq. (1) to estimate reasonable values for the $\gamma_n^{(l)I}$, and then formulating the N/D equations in terms of these parameters, KL followed a different procedure. They chose to formulate the N/D equations in terms of the residues of the poles in N, which are initially unknown parameters. They then determined these residues by simultaneously imposing the crossing conditions of Roskies,[2] the inequalities of Martin,[3] and the current-algebra predictions of Weinberg,[4] and by requiring the generation of ρ and σ resonances.

The positions and residues of the poles in the N functions of KL are presented in Ref. 1. We have computed the corresponding D functions at the positions of these poles, and have thereby determined the residues of the resulting poles in $A^{(l)I}$.

We have also established that the denominator functions $D^{(0)0}(s)$ and $D^{(1)1}(s)$ contain no zeros on the physical sheet, while $D^{(0)2}(s)$ contains precisely one

TABLE I. Values for the parameters representing left-hand cuts in the model of KL [see Eq. (2)].

$(l)I$	$s_1^{(l)I}$	$s_2^{(l)I}$	$s_3^{(l)I}$	$\gamma_1^{(l)I}$	$\gamma_2^{(l)I}$	$\gamma_3^{(l)I}$
(0)0	−0.56	−22	−70	0.0153	−91.8	811
(0)2	−1.28	−16	-4.61×10^4	−0.0542	−23.8	4.27×10^5
(1)1	−14.8	−34	−76	65.7	−742	1870

such zero. This zero occurs at $s=-4.61\times10^4$, and generates a pole in $A^{(0)2}$, whose residue we have determined.

In Table I, we present the $s_n^{(l)I}$ and $\gamma_n^{(l)I}$ corresponding to the poles in the S waves and P wave of KL. Note that five of the nine $s_n^{(l)I}$ lie on the interval $-32<s<0$, where Eq. (1) is exact, while three of the four remaining $s_n^{(l)I}$ lie on the interval $-76 \leqslant s<-32$.

In Fig. 1, we present the $\mathrm{Im}A^{(l)I}$ for $s<0$ obtained by inserting the physical-region S- and P-wave absorptive parts of KL into the right-hand side of Eq. (1). Since the f_0 resonance occurs at $s=84$, inclusion of the D waves would not significantly alter the $\mathrm{Im}A^{(l)I}$ displayed in Fig. 1 for $-s<80$.[7] However, Eq. (1) is not strictly valid for $-s>32$, so it is an open question whether the $\mathrm{Im}A^{(l)I}$ in Fig. 1 are good approximations for $32<-s<80$.

Observe in Fig. 1 that $\mathrm{Im}A^{(0)0}$ is negative for $0\leqslant -s<14$, whereas $\gamma_1^{(0)0}$, which represents $\mathrm{Im}A^{(0)0}$ in the vicinity of $-s=0.56$, is *positive*. Observe also that $\mathrm{Im}A^{(0)0}$ is positive for $14<-s\leqslant 80$, whereas $\gamma_2^{(0)0}$, which represents $\mathrm{Im}A^{(0)0}$ in the vicinity of $-s=22$, is large and *negative*.

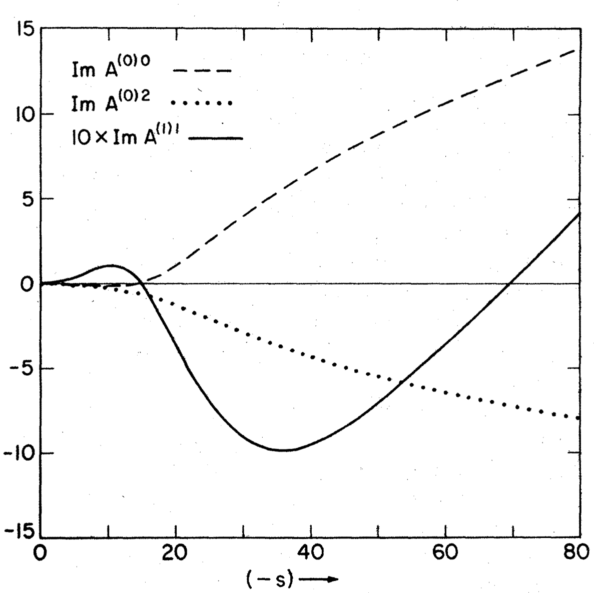

FIG. 1. $\mathrm{Im}A^{(l)I}$ for negative s obtained by inserting the physical-region S and P waves of KL into the right-hand side of Eq. (1).

Turning to the $I=2$ S wave, we see that $\gamma_1^{(0)2}$ and $\gamma_2^{(0)2}$ are in qualitative agreement with Fig. 1. However, the values of $s_1^{(0)2}$ and $s_2^{(0)2}$ do not appear to be especially well suited for representing $\mathrm{Im}A^{(0)2}$. As for the very distant pole resulting from the zero in $D^{(0)2}$, the relative magnitudes of $\gamma_3^{(0)2}$ and $s_3^{(0)2}$ are such that the corresponding pole term in $A^{(0)2}$ is nearly constant over the low-energy region.[8] Since the value of $A^{(0)2}$ is fixed by assumption at a subtraction point near threshold,[9] this very distant pole plays no significant role in the low-energy dynamics of $A^{(0)2}$, and we shall not discuss it further.

Turning now to the P wave, we see that $\mathrm{Im}A^{(1)1}$ is positive and quite small for $0\leqslant -s<15$, but is negative and much larger for $15<-s<70$. However, $\gamma_1^{(1)1}$, which represents $\mathrm{Im}A^{(1)1}$ in the vicinity of $-s=15$, is *positive* and *very large*. We also note that $\gamma_2^{(1)1}$ and $\gamma_3^{(1)1}$, which represent $\mathrm{Im}A^{(1)1}$ in the vicinity of $-s=34$ and 76, respectively, are *enormous*. In fact, $\gamma_1^{(1)1}$ and $\gamma_2^{(1)1}$ are both at least ten times larger than is consistent with Fig. 1, while the enormous value of $\gamma_3^{(1)1}$ requires that $\mathrm{Im}A^{(1)1}$ must become very large in the region just above that displayed in Fig. 1, if yet another inconsistency is to be avoided.

From the preceding analysis, it is evident that several of the pole parameters of KL are severely inconsistent with the $\mathrm{Im}A^{(l)I}$ of Fig. 1, even on the interval $0<-s<32$, where Eq. (1) is exact.[7] Since Eq. (1) is a consequence of analyticity and crossing symmetry for the full amplitudes $A^I(s,t)$, we conclude that the amplitudes of KL are severely inconsistent with the analyticity and/or crossing symmetry of physical $A^I(s,t)$.

We remark that the five conditions of Roskies which involve only the S waves and P wave are sufficient to ensure crossing symmetry, in the sense that any S waves and P wave which satisfy these conditions are the partial waves of *some* set of fully crossing-symmetric functions.[2] However, the crossing-symmetric functions of which the S waves and P wave are a part may lack the analyticity of physical $A^I(s,t)$.[2] Since the S waves and P wave of KL satisfy the five conditions of Roskies to good approximation, we conclude that it is primarily the analyticity of physical $A^I(s,t)$ which is violated by the model of KL. This violation is present even though the inequalities of Martin are well satisfied. Thus it is clear that the combined conditions of Roskies and Martin are *not* sufficient to ensure that models for low-order partial waves are consistent, even roughly, with the analyticity and crossing symmetry of physical $A^I(s, t)$.

We wish to emphasize that the inconsistencies between Eq. (1) and the pole parameters of KL are greatest in the P wave and in the $I=0$ S wave. Although the model of KL does succeed in generating ρ and σ mesons as dynamically bound states, the forces of the model in these channels are quite inconsistent with the forces occurring in nature.[10]

Another interesting test of the model consists of comparing the S- and P-wave scattering lengths with the $I=1$ Regge sum rules for them.[11] These sum rules are obtained by evaluating the following dispersion relation at appropriate values of s (Ref. 12):

$$A_F^1(s) = \frac{s-4}{\pi} \int_4^\infty \frac{ds'}{s'(s'+s-4)} \operatorname{Im}\left[T_F^1(s') + \frac{2s(s'-2)}{(s'-4)(s'-s)} A_F^1(s')\right], \tag{3}$$

where A_F^I denotes the forward amplitude with isospin I in the s channel, and

$$T_F^1 \equiv \tfrac{1}{3}A_F^0 + \tfrac{1}{2}A_F^1 - \tfrac{5}{6}A_F^2$$

denotes the combination of forward amplitudes with $I=1$ in the t channel.

Upon evaluating Eq. (3) at $s=0$ and noting that crossing symmetry implies[12]

$$A_F^1(0) = -\tfrac{1}{3}A_F^0(4) + \tfrac{5}{6}A_F^2(4),$$

one obtains a sum rule for the S-wave scattering lengths[13] $a_I = A_F^I(4)$, $I=0, 2$:

$$2a_0 - 5a_2 = \frac{24}{\pi} \int_4^\infty \frac{ds}{s(s-4)} \operatorname{Im} T_F^1(s). \tag{4}$$

Evaluating A_F^1 near $s=4$, one obtains the P-wave scattering length a_1:

$$a_1 = \frac{4}{3\pi} \int_4^\infty \frac{ds}{s^2} \operatorname{Im}\left[T_F^1(s) + 8\frac{s-2}{(s-4)^2} A_F^1(s)\right], \tag{5}$$

where

$$a_1 \equiv \lim_{s\to 4} \left(\frac{4}{s-4} A^{(1)1}(s)\right).$$

In the model of KL, $2a_0 - 5a_2 = 0.63$.[13] However, upon substituting the S- and P-wave absorptive parts of KL into the right-hand side of Eq. (4), one finds that $\operatorname{Im}A^{(0)0}$ contributes 0.46, $\operatorname{Im}A^{(0)2}$ contributes -0.04, and $\operatorname{Im}A^{(1)1}$ contributes 0.67, for a combined value of 1.07. The net contribution to the right-hand side of Eq. (4) from partial waves with $l \geq 2$ is usually estimated to be about 0.2,[14] and is universally regarded as positive. Thus we see that the left- and right-hand sides of Eq. (4) disagree by a factor of about 2 for the model of KL.

The KL value for a_1 is 0.038. However, one finds upon substituting the absorptive parts of KL into the right-hand side of Eq. (5) that $\operatorname{Im}A^{(0)0}$ contributes 0.017, $\operatorname{Im}A^{(0)2}$ contributes -0.002, and $\operatorname{Im}A^{(1)1}$ contributes 0.053, for a combined value of 0.068. The net contribution from partial waves with $l \geq 2$ is usually estimated to be about 0.01,[14] and is certainly positive. Thus the left- and right-hand sides of Eq. (5) also disagree by a factor of about 2 for the model of KL.

We shall now suggest and briefly discuss several possible reasons for the preceding difficulties with the model of KL.

A. The Number of Poles Used to Simulate Left-Hand Cuts May Be Too Small

Two of the five conditions of Roskies, and all the inequalities of Martin, test second- and/or higher-order derivatives[15] of the amplitudes. These derivatives are quite sensitive to the details of nearby singularities, and it may be that the number of poles used by KL is simply not large enough to accommodate all the constraints imposed by KL, unless some of the residues are given unphysical values.

In fact KL remark that when the nearest pole positions are varied by about $|\Delta s|=1$, the resonance masses vary by amounts up to nearly 100 MeV. Since the distance between the two poles nearest $s=0$ exceeds $\Delta s=14$ in each of the three channels, there is no evident reason why one should not consider variations of the nearest pole positions by amounts up to $|\Delta s|=5$ or more. In that case, it would appear from the aforementioned remark of KL that the resonance masses would vary by amounts greatly exceeding 100 MeV. This extreme sensitivity of the results to the pole positions is quite consistent with our suggestion that too few poles are being used to represent the left-hand cuts, when so many constraints are being imposed on second- and higher-order derivatives.

B. Physical $\pi\pi$ S Waves May Not Satisfy Once-Subtracted Dispersion Relations

It appears likely that Veneziano $\pi\pi$ partial waves with $I=0$ and $I=2$ grow exponentially along any ray to infinity in the left half of the s plane.[16] Even if physical S waves were to differ from their Veneziano counterparts by satisfying polynomial bounds at infinity, one subtraction might not be sufficient to result in the convergence of dispersion integrals.

C. Physical $\pi\pi$ S Waves May Contain Zeros at Complex s Which Require Poles in the Corresponding D Functions

It has recently been conjectured that physical $\pi\pi$ partial waves with $I=0$ and $I=2$ contain infinitely many zeros at complex s on the physical sheet.[16] Thus it may be that if one decomposes the S waves into ratios N/D wherein both N and D satisfy once-subtracted dispersion relations, then poles are required in the D functions to generate some of the conjectured zeros at complex s.

D. The ρ and/or σ Mesons May *Not* Be Dynamically Bound States of Two π Mesons

This possibility is an especially interesting one, and we shall now outline an argument which indicates that the ρ meson is indeed not a bound state of two π mesons.

Veneziano $\pi\pi$ partial waves with $I=1$ satisfy unsubtracted dispersion relations,[17] so it is reasonable to assume that the physical $A^{(1)1}$ satisfies a once-subtracted dispersion relation, namely,

$$A^{(1)1}(s)=\frac{s-4}{\pi}\int_{-\infty}^{\infty}ds'\frac{\mathrm{Im}A^{(1)1}(s')}{(s'-4)(s'-s)}. \tag{6}$$

Although Eq. (1) can be used to determine $\mathrm{Im}A^{(1)1}(s)$ for $-32<s<0$, we presently have no means at our disposal for computing $\mathrm{Im}A^{(1)1}(s)$ for $s<-32$. However, Eq. (6) implies that

$$a_1=\frac{4}{\pi}\int_{-\infty}^{\infty}ds'\frac{\mathrm{Im}A^{(1)1}(s')}{(s'-4)^2}, \tag{7}$$

so a great deal can be inferred about the left-hand cut of $A^{(1)1}$ by comparing Eq. (7) for a_1 with Eq. (5). In fact the value of

$$\tilde{a}_1\equiv\frac{4}{\pi}\int_{-\infty}^{-32}ds'\frac{\mathrm{Im}A^{(1)1}(s')}{(s'-4)^2}$$

can be determined strictly from physical-region absorptive parts, simply by comparing Eq. (7) with Eq. (5), while using Eq. (1) over the interval $-32<s<0$. Since a knowledge of $\tilde{a}_1$ is equivalent to a knowledge of the effective strength of the distant left-hand cut when the latter is viewed from the low-energy region, this technique is quite useful.

If one uses the Froissart-Gribov representation to subtract out partial waves with $l\geqslant 3$ from the A_F^1 given by Eq. (3), one can obtain a new representation for $A^{(1)1}(s)$. This new representation can be used to evaluate $\mathrm{Re}A^{(1)1}$ over the interval $-32<s<0$, strictly in terms of physical-region absorptive parts.[18] By comparing this new representation for $A^{(1)1}$ with Eq. (6), it is possible to derive formulas for the second and all higher negative moments of $\mathrm{Im}A^{(1)1}$ over the interval $-\infty<s<-32$, strictly in terms of physical-region absorptive parts. An investigation of this kind has been carried out[18] and indicates that the left-hand cut of $A^{(1)1}$ is too weak to generate the ρ resonance, even when the possibility of substantial inelasticity is taken roughly into account. Details of this investigation will be reported in a forthcoming publication.

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*Present address: Department of Physics and Astronomy, Hunter College of CUNY, New York, New York 10021.

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[2]R. Roskies, Phys. Letters 30B, 42 (1969); Nuovo Cimento 65A, 467 (1970).

[3]A. Martin, Nuovo Cimento 47A, 265 (1967); 58A, 303 (1968); 63A, 167 (1969).

[4]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[5]Quantitative agreement is lacking in the P wave. Although the ρ resonance of Ref. 1 has a mass of 770 MeV, the left "half-width" is 150 MeV, and δ_1^1 returns very slowly to 0° after reaching 90°, instead of rising to 180°. The total width between half-maxima of the P-wave cross section is greater than 4 GeV.

[6]The matrix β is given in Ref. 1. We remark that a factor of 2 has been omitted from Eq. (2.7) of Ref. 1, which corresponds to Eq. (1) of the present paper. However, the method of Ref. 1 does not rely upon this equation in any way which would be affected by the omission of the factor of 2.

[7]If one assumes $m(f_0)=1250$ MeV, $\Gamma(f_0\to\pi\pi)=125$ MeV, and represents the f_0 contribution to the $I=0$ D wave by a Breit-Wigner formula with correct threshold behavior [L. A. P. Balázs, Phys. Rev. 129, 872 (1963)], then the f_0 contribution to $\mathrm{Im}A^{(1)1}$ is less than 0.001 for $0<-s<32$, is less than 0.005 for $32<-s<48$, and is less than 0.05 for $48<-s<66$. The contribution reaches 0.10 at $-s=70$, and equals 0.40 at $-s=80$. One might suppose that the *net* contribution of all higher partial waves is appreciable somewhere on the interval $0<-s<32$, even though the D-wave contributions are extremely small there. However, this seems quite unlikely. For example, in the single-term Veneziano model for $\pi\pi$ amplitudes, the D-wave contribution to $\mathrm{Re}A^0$ is larger than the net contribution of all higher partial waves between

threshold and 1.4 GeV ($s = 100$). The same should be true to an even greater extent for the absorptive parts of unitary amplitudes.

[8]The derivative with respect to s of this pole term in $A^{(0)2}$ is given by

$$\frac{1}{\pi}\frac{\gamma_3^{(0)2}}{(s_3^{(0)2}-s)^2},$$

which equals 6.31×10^{-5} at threshold.

[9]Each S wave is subtracted at $s_0 = \frac{4}{3}$, and the value predicted by Weinberg from current algebra is imposed there. The P wave is subtracted at threshold, where it is assumed to vanish. These values are imposed in a way which does not constrain the singularities.

[10]Since the ρ resonance of KL is too broad (Ref. 5), the $\mathrm{Im}A^{(l)I}$ displayed in Fig. 1 are not in good quantitative agreement with those which occur in nature. However, the left-hand cuts of KL disagree even more with the $\mathrm{Im}A^{(l)I}$ which occur in nature than they do with the $\mathrm{Im}A^{(l)I}$ displayed in Fig. 1.

[11]There are also $I = 2$ Regge sum rules for low-energy parameters, but these converge only because of extreme cancellations, and cannot be evaluated with sufficient reliability at present to be of any practical use. Cf. E. P. Tryon, Phys. Rev. Letters 22, 110 (1969).

[12]Cf. M. G. Olsson, Phys. Rev. 162, 1338 (1967).

[13]Our S-wave scattering lengths are defined with opposite signs from those of KL.

[14]For example, in the single-term Veneziano model for $\pi\pi$ amplitudes [C. Lovelace, Phys. Letters 28B, 264 (1968)], the contribution to the right-hand side of Eq. (4) from partial waves with $l \geq 2$ is 0.21, and the corresponding contribution to the right-hand side of Eq. (5) is 0.012, assuming one uses Lovelace's values for the Regge parameters and sets $\Gamma(\rho\to\pi\pi) = 125$ MeV.

[15]If one expands the S and P waves as power series in $(s-2)$ over the interval $0 \leq s \leq 4$, then the integrals in two of the five conditions of Roskies are independent of constant and linear terms, and the integrals in one of these conditions are independent of quadratic terms as well. It is also straightforward to verify that if the S waves were linear in s over the interval $0 \leq s \leq 4$, and satisfied the two conditions of Roskies which involve only S waves, then the Martin inequalities would all be marginally satisfied as equalities.

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Factorization of Multi-Regge Amplitudes. II*

J. H. Weis

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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Some general comments are made on the relationship between the factorization of multi-Regge amplitudes and the singularity structure of multiparticle amplitudes. A close relationship is found to exist between the validity of factorization and the absence of simultaneous discontinuities in overlapping variables suggested by the Steinmann relations.

Multi-Regge factorization of the full amplitude and its total-energy discontinuity have recently been proven in several models: the dual-resonance model,[1] ladder graphs in ϕ^3 perturbation theory,[2,3] and Gribov's hybrid model.[3,4] The procedure of taking into proper account the cuts due to singularities in variables which are dependent owing to nonlinear Gram-determinant constraints was found to play a crucial role in obtaining factorization.[1,2]

Here we make some general, model-independent, comments on the relationship between multi-Regge factorization and the singularity structure of multiparticle amplitudes. In particular, we find that the singularity structure suggested by the Steinmann relations[5] has a very intimate relationship to factorization.

The Steinmann relations state that the full amplitude has no simultaneous discontinuity in energy variables of overlapping channels[6] in the physical region. Here we wish to make the assumption that the full amplitude can be expressed in terms of generalized multi-Froissart-Gribov signatured amplitudes (see Eq. (1.1) of Paper I [(I 1.1)]). The Steinmann relation can then be applied to the signatured amplitude itself, since simultaneous discontinuities in overlapping subenergies in the signatured amplitude would lead to corresponding discontinuities in the full amplitude (within a given physical region, the full amplitude has discontinuities coming from only one term in Eq. (1.1) of Paper I [(I 1.1)]). Therefore, whereas the signatured amplitude for the process $p_a+p_b\to p_0+p_1+\cdots+p_{n-1}$ in general has discontinuities in the subenergies of all groups of adjacent outgoing particles