Is There a Quantum Measurement Problem?*

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It is shown that the quantum-mechanical state vector correctly describes not only the probabilities for the outcomes of measurements, but also the correlations between the outcomes of successive measurements. In particular, von Neumann's axiom M is shown to be redundant. Consequently, no extra–quantum-mechanical "reduction" of the joint object-apparatus state vector is required for a full statistical description of a sequence of measurements. It is also shown that any attempt to determine experimentally whether or not a reduction of the joint state vector has taken place during a measurement is incompatible with the preservation of the outcome of that measurement.

INTRODUCTION

Recent papers have questioned the completeness and logical consistency of quantum mechanics on the grounds that the quantum-mechanical equations of motion do not yield a correct account of some aspects of observed phenomena. Thus, Fine[1] asserts that in general the end result of the quantum-mechanical description of a measurement is the "unacceptable" situation where "neither the object nor the apparatus has a definite state," which is taken to imply that "no laboratory observations can be cited in support of the quantum theory." Earlier, D'Espagnat[2] had claimed that "the idea that the pointer (on the measurement apparatus) is always left in some interval of the scale cannot be even approximately correct if quantum mechanics is strictly true," and he had emphasized that "the problem is a real one."

Wigner[3] has claimed that these difficulties arise from a philosophical attitude which ascribes an element of "objective reality" to the quantum-mechanical state vector and that the problem can be circumvented by regarding quantum mechanics merely as a calculational tool for the computation of correlations between the outcomes of successive measurements on a system. He appears to regard the choice between these two views as a matter of taste. Nevertheless he has argued on the basis of the "realistic" view "that this theory is not adequate for the description of life, including consciousness."[4]

I wish to show here that quantum mechanics correctly describes such observable phenomena as the position of the pointer on the measurement apparatus. The contrary views expressed by Fine and D'Espagnat and implied by many other authors appear to be unfounded. I will also show that quantum mechanics correctly describes the correlations between successive measurements on the same system. Accordingly, von Neumann's axiom M,[5] which independently postulates certain types of such correlations, is in fact derivable from the remaining postulates of quantum mechanics. This axiom has been interpreted as *requiring* a sudden, noncausal, nonunitary projection or "reduction" of the quantum-mechanical state vector when a measurement is made. The redundancy of axiom M removes any need for such an extra–quantum-mechanical process, and the widespread search for physical or nonphysical mechanisms that would either accomplish a reduction or approximate it becomes pointless.

von Neumann's own examples of state-vector "reduction" turn out to be wholly causal consequences of physical interactions with other systems, as is demonstrated by von Neumann's own mathematical results.[6]

The "reduced" state of the joint object-apparatus system has an interesting interpretation that is related to the logical structure of the measurement process.

The "classical" nature of the measurement apparatus plays no *essential* role in the quantum-mechanical description of the measurement process. It merely provides a convenient way of insuring the permanence of the record of a measurement.

In this paper I take quantum mechanics to be a statistical theory which, in general, describes the distributions of the outcomes of ensembles of measurements. Individual cases are treated as members of such ensembles.

This paper does *not* deal with any of the following questions: Is a probabilistic theory a sufficient description of phenomena that have only statistically reproducible properties? What is the precise meaning of probability? Should quantum mechanics provide instructions for the measurement of *all* observables?

THE POINTER POSITION

For simplicity I consider first the case of a von Neumann measurement[6] in which the measured-object system is initially in the pure state $\sum_n c_n\phi_n$, where ϕ_n are eigenstates with eigenvalues λ_n of the observable $\underline{O}$ to be measured. The measuring apparatus is initially in the pure state ξ_0, so that before the measuring interaction between object and apparatus the joint system is in the pure state,

$$\Psi^{(1)} = \sum_n c_n\phi_n \otimes \xi_0 \,. \tag{1}$$

The measurement interaction between object and apparatus that was considered by von Neumann produces a unitary transformation $\underline{U}_a$ of $\Psi^{(1)}$ with the property that if $c_n = \delta_{in}$, then the transformed state is $\phi_i \otimes \xi_i$, where ξ_i is an eigenstate of the observable $\underline{A}$ of the apparatus whose eigenvalue μ_i describes a certain pointer position. As a consequence, the state of the joint system after the measurement interaction is

$$\Psi^{(2)} = \underline{U}_a \Psi^{(1)} = \sum_n c_n \phi_n \otimes \xi_n \,. \tag{2}$$

It is this expression which has been criticized as being inconsistent with the observed fact that after each measurement the pointer has a definite position μ_i.[1,2] At the same time it is claimed that this observed fact is correctly described by the "reduced" mixture with density operator,

$$\underline{W} = \sum_n |c_n|^2 \underline{P}_{\phi_n \otimes \xi_n} \,, \tag{3}$$

which corresponds to the ensemble of "reduced states" $\phi_n \otimes \xi_n$ with weights $|c_n|^2$. Here $\underline{P}_{\phi_n\otimes\xi_n}$ is the projection operator corresponding to the pure state $\phi_n \otimes \xi_n$. It has been said that, in contrast to $\Psi^{(2)}$, the mixture $\underline{W}$ is in agreement with experience because each of its components is a state in which the pointer has a definite position μ_i. The question of whether, in more general circumstances, such a mixture $\underline{W}$ can arise in the measurement process has been called "the problem of measurement" by Fine[1] and has been shown to admit no exact solution.[1-3,7]

But what of the assertion that $\underline{W}$ specifies definite pointer positions and that $\Psi^{(2)}$ does not? The only correct and realistic way to test this assertion is to calculate the pointer positions specified by $\Psi^{(2)}$ and by $\underline{W}$ according to the rules of quantum mechanics.[5] The probability of finding the apparatus with pointer position μ_i when the joint object-apparatus system is in state $\Psi^{(2)}$ is given by $(\Psi^{(2)}, \underline{P}_{1\otimes\xi_i}\Psi^{(2)})$. When the system is in state $\underline{W}$, the same probability is given by $\mathrm{Tr}\underline{P}_{1\otimes\xi_i}\underline{W}$. The operator $\underline{P}_{1\otimes\xi_i}$ is the projection operator corresponding to the apparatus having pointer position μ_i. Evaluating these two expressions, we find that

$$\langle \underline{P}_{1\otimes\xi_i}\rangle_{\Psi^{(2)}} \equiv (\Psi^{(2)}, \underline{P}_{1\otimes\xi_i}\Psi^{(2)}) = |c_i|^2 \,, \tag{4a}$$

$$\langle \underline{P}_{1\otimes\xi_i}\rangle_{\underline{W}} \equiv \mathrm{Tr}\underline{P}_{1\otimes\xi_i}\underline{W} = |c_i|^2 \,. \tag{4b}$$

Clearly, therefore, $\Psi^{(2)}$ and $\underline{W}$ both specify that the probability of finding the pointer to have the position μ_i is $|c_i|^2$ and both are in complete agreement with experience in that regard. By the same token both $\Psi^{(2)}$ and $\underline{W}$ specify $|c_i|^2$ for the probability that the object system is found to be in state ϕ_i. Furthermore, both $\Psi^{(2)}$ and $\underline{W}$ specify that whenever the object is in state ϕ_i the pointer value is μ_i,

$$\frac{\langle \underline{P}_{\phi_i\otimes\xi_j}\rangle_{\Psi^{(2)}}}{\langle \underline{P}_{\phi_i\otimes 1}\rangle_{\Psi^{(2)}}} = \frac{\langle \underline{P}_{\phi_i\otimes\xi_j}\rangle_{\underline{W}}}{\langle \underline{P}_{\phi_i\otimes 1}\rangle_{\underline{W}}} = \delta_{ij} \,. \tag{5}$$

Equation (4a) demonstrates the fact that the pure state (2) specifies a definite probability for the occurrence of the stationary pointer position μ_i.[8] Using the same method, it is easily shown that a properly constructed apparatus ($\sum_i P_{\xi_i} = 1$ and $P_{\xi_i}P_{\xi_j} = \delta_{ij}P_{\xi_i}$) has the following two additional properties. After a measurement the pointer is left in one of the positions μ_i with certainty (probability one) and the pointer is never (probability zero) left jointly in two different positions μ_i and μ_j. Together these three statements imply that at the conclusion of any measurement the pure state (2) specifies that the pointer is definitely in one and only one of its possible positions μ_i.

CORRELATIONS BETWEEN POINTERS

But, it may be argued, surely $\Psi^{(2)}$ and $\underline{W}$ will lead to different results if the above measurement of $\underline{O}$ is followed by a measurement of another observable $\underline{Q}$ characteristic of the object system. In particular, it might be thought that if the measurement of $\underline{O}$ established one of the elements of the mixture $\underline{W}$ as "the" state of the system, a second measurement of $\underline{O}$ on the same object system would surely yield the same result, while the linear combination $\Psi^{(2)}$ appears to offer no such promise. Let us see what the facts are.

Let $\underline{Q}$ have eigenstates ψ_j with eigenvalues κ_j, and let the von Neumann apparatus that measures $\underline{Q}$ have pointer observable $\underline{B}$ with eigenstates η_j and eigenvalues ν_j. Let us also assume for simplicity that the ϕ_i and the ψ_j span the same subspace of the Hilbert space of the object system so that there is an expansion,

$$\phi_i = \sum_j d_{ij}\psi_j \,, \tag{6}$$

where the d_{ij} form a unitary matrix.

We now suppose that first the observable $\underline{O}$ is measured with apparatus $\underline{A}$. Then $\underline{Q}$ is measured with apparatus $\underline{B}$. Before both measurements the joint state of object and apparatus $\underline{A}$ and $\underline{B}$ is assumed to be

$$\Psi^{(1)}=\sum_n c_n\phi_n\otimes\xi_0\otimes\eta_0. \quad (7)$$

After the interaction between the object and the apparatus $\underline{A}$ the state becomes

$$\Psi^{(2)}=\underline{U}_a\Psi^{(1)}=\sum_n c_n\phi_n\otimes\xi_n\otimes\eta_0. \quad (8)$$

Finally, the interaction with apparatus $\underline{B}$ produces

$$\Psi^{(3)}=\underline{U}_b\Psi^{(2)}=\sum_n c_n\sum_m d_{nm}\psi_m\otimes\xi_n\otimes\eta_m. \quad (9)$$

The probability that pointer $\underline{A}$ has the value μ_i is again

$$\langle\underline{P}_{1\otimes\xi_i\otimes 1}\rangle_{\Psi^{(2)}}=\langle\underline{P}_{1\otimes\xi_i\otimes 1}\rangle_{\Psi^{(3)}}=|c_i|^2. \quad (10)$$

The probability that the pointer $\underline{B}$ has the value ν_j is found to be

$$\langle\underline{P}_{1\otimes 1\otimes\eta_j}\rangle_{\Psi^{(3)}}=\sum_n|c_n|^2|d_{nj}|^2, \quad (11)$$

and the joint probability that the pointer $\underline{A}$ has the value μ_i and the pointer $\underline{B}$ has the value ν_j is

$$\langle\underline{P}_{1\otimes\xi_i\otimes\eta_j}\rangle_{\Psi^{(3)}}=|c_i|^2|d_{ij}|^2. \quad (12)$$

All of these results are the same as those obtained from the reduced-state equation (3). It follows from (12) that the conditional probability that $\underline{B}$ has the value ν_j if $\underline{A}$ has the value μ_i is $|d_{ij}|^2$, as expected.[9] In particular, if $\underline{Q}=\underline{O}$, that is, if the measurement of $\underline{O}$ is repeated, then $d_{ij}=\delta_{ij}$, and the above conditional probability tells us that according to the *unreduced* state $\Psi^{(3)}$ a second von Neumann measurement of $\underline{O}$ on the same system will yield the same result as the first measurement with certainty. This constitutes, therefore, a *derivation* of von Neumann's axiom $\underline{M}$.[5]

It is also instructive to consider the case where one measures $\underline{O}$, then $\underline{Q}$, and then measures $\underline{O}$ once again by means of an apparatus $\underline{A}'$. This leads to the final state

$$\Psi^{(4)}=\underline{U}_a'\Psi^{(3)}=\sum_{n,m,k} c_n d_{nm} d_{km}^*\,\phi_k\otimes\xi_n\otimes\eta_m\otimes\xi_k'. \quad (13)$$

Computing from this the joint probability that the pointer on apparatus $\underline{A}$ reads μ_i and the pointer on apparatus $\underline{A}'$ reads μ_j, one gets

$$\langle\underline{P}_{1\otimes\xi_i\otimes 1\otimes\xi_j'}\rangle_{\Psi^{(4)}}=\sum_m|c_i|^2|d_{im}|^2|d_{jm}|^2, \quad (14)$$

which is *not* identical to $|c_i|^2$ if $\underline{Q}$ fails to commute with $\underline{O}$. Again the result (14) agrees with the corresponding result computed from the reduced state.

This result (14) depends in an essential way upon the correlation of the coefficients in Eq. (13) with the state η_m of apparatus $\underline{B}$. Without this important correlation, Eq. (14) would become

$$|\sum_m c_i d_{im} d_{jm}^*|^2=|c_i|^2\delta_{ij},$$

that is, the original state of the object would be restored despite the intervening measurement of the noncommuting observable $\underline{Q}$. This would indeed be contrary to experience. Furthermore, since no information on the outcome of the measurement of $\underline{Q}$ is assumed, the result (14) can depend in no way upon an "act of observation" of the observable $\underline{Q}$ or the apparatus $\underline{B}$, but is merely the result of the physical interaction between the object and the apparatus $\underline{B}$. This fact should be experimentally observable, for example by the succession of Stern-Gerlach arrangements discussed by Wigner.[3]

The observer who reads the pointer on an apparatus can be included in the description. One enlarges the Hilbert space to include also the observer who during some period of time interacts with the apparatus (looks at the pointer). The subsequent correlation between the pointer position and the observer's memory is again confirmed by evaluating the expectation values of the appropriate projection operators in the entire Hilbert space. This correlation can be extended to other observers by interaction (communication) with them. Correlations of the memory contents of many observers establish the "objectivity" of an event.

OTHER KINDS OF MEASUREMENTS

von Neumann gave a constructive proof for the existence of measuring interactions which result in transformations of the type $\underline{U}_a$.[6] However, it has been pointed out, particularly by Margenau,[10] that in practice the von Neumann definition represents a too restricted class of measurement interactions. The von Neumann apparatus as defined by Eq. (2) merely correlates its state with that of the object system without affecting the state of the object. It is easy to generalize this by appending to the von Neumann apparatus a device which changes the final state of the object after the pointer has been set. The transformation (2) is then replaced by an expression

$$\underline{U}_a'\Psi^{(1)}=\sum_{n,m} c_n g_{nm}\phi_m\otimes\xi_n. \quad (2')$$

This apparatus records the state which characterized the object *before* the measurement.

Or one could construct an apparatus in which the object state is modified before the pointer is set. Then (2) becomes

$$\underline{U}_a''\Psi^{(1)}=\sum_{n,m} c_n g_{nm}\phi_m\otimes\xi_m \quad (2'')$$

and the apparatus records the state of the object *after* the measurement. In general, the state-changing function and the von Neumann function are both performed by the same piece of apparatus.

The measurement interaction (2′) permits the description of measurements in which photons are absorbed, provided the vacuum is included as a photon state. It would appear that all measurement interactions fall into one of three categories represented by $\underline{U}_a$, $\underline{U}_a'$, and $\underline{U}_a''$.

The modifications (2′) and (2″) do not affect the conclusions drawn above in any essential way. The conclusion stands that the unreduced state is in complete agreement with all observations and all "experience," including the correlation between successive measurements.

INTERPRETATION OF THE REDUCED STATE

Since the reduced state (3) predicts the abovementioned phenomena as correctly as the unreduced state does, the question arises: Wherein lies the difference between the reduced and the unreduced states? The answer is obtained by comparing the expectation values of an arbitrary Hermitian operator $\underline{G}$ with respect to the two states. We have

$$\langle \underline{G}\rangle_{\Psi^{(2)}} = \sum_{n,m} c_n^* c_m G_{nn,mm}, \qquad (15)$$

$$\langle \underline{G}\rangle_{\underline{W}} = \sum_n |c_n|^2 G_{nn,nn}, \qquad (16)$$

where

$$G_{nm,kl} = (\phi_n \otimes \xi_m, \underline{G}\phi_k \otimes \xi_l). \qquad (17)$$

If $\underline{G}$ commutes with either $\underline{O}\otimes \underline{1}$ or with $\underline{1}\otimes \underline{A}$, the right-hand side of (15) becomes identical to (16). Therefore, the only way of distinguishing between the reduced and the unreduced states is by measuring an observable that does not commute with the object-observable $\underline{O}$ and does not commute with the apparatus-pointer-observable $\underline{A}$. Let $\underline{Z}$ be such an observable.

It is clear that if the measurement of $\underline{O}$ is followed by a measurement of $\underline{Z}$ the result of the measurement of $\underline{O}$ is irretrievably lost. Because $[\underline{O}, \underline{Z}] \neq 0$ a repetition of the measurement of $\underline{O}$ will not yield the original result with certainty. Similarly, because $[\underline{A}, \underline{Z}] \neq 0$ the pointer of the apparatus $\underline{A}$ will not retain its original position with certainty. Hence, following the interaction responsible for the measurement of $\underline{Z}$, all records of the original measurement of $\underline{O}$ and all possibilities of reconstructing its result with certainty have been lost.

For the scientific purposes of verification and comparison it is essential to preserve the record of the outcome of a measurement. This means that the object and apparatus must be protected against any interaction that would enable one to determine the value of an observable of the type $\underline{Z}$.

In principle this can always be done. For example, the z component of the electron spin of a certain hydrogen atom could serve as a satisfactory pointer, so long as the atom is protected from any interaction that might flip its spin.

In practice, however, one attempts to exclude $\underline{Z}$ interactions by requiring that the algebra of observables which describes the apparatus be Abelian, for then there exists no observable $\underline{Z}$. Operationally this means that the apparatus is a "classical" object.[11] Precisely how to describe a classical system in terms of quantum mechanics does not appear to be firmly established yet, though the approach of Jauch[12] seems reasonable and promising. In any case, as observed by Fine,[1] the classical apparatus can at best be Abelian in an approximate sense. That is neither surprising nor troublesome. Every classical property has some very slight chance of being modified by a "quantum fluctuation," and every physical measurement, whether performed on a microscopic or a macroscopic system, is at best reliable only to within some arbitrarily small but finite degree of accuracy.

In the case of a classical pointer, the difficulty of measuring $\underline{Z}$ and thereby distinguishing between the reduced and the unreduced states is the same as the difficulty one has in observing quantum effect on classical properties in general. Perhaps this difficulty can be overcome by using for a pointer one of the cooperative phenomena which produce quantum effects on a macroscopic scale. However, in the absence of any actually observed distinctions between the reduced and unreduced states, the unreduced state has the clear advantage of logical simplicity and consistency.

von Neumann[5,6] applied the concept of "reduction" not to the complete physical system, as in Eq. (3), but only to the object system. More generally, he applied this concept to any subsystem defined by an arbitrary division of all interacting systems. Such a division is commonly known as a von Neumann "cut."

von Neumann also provided a completely causal explanation for the reduction of the state of a subsystem.[6] He proved that the joint state (2) *implies* that the state of the object system by itself is given by the mixture $\sum_n |c_n|^2 \underline{P}_{\phi_n}$ and that the state of the apparatus subsystem is given by the mixture $\sum_n |c_n|^2 \underline{P}_{\xi_n}$, each component of which corresponds to a definite pointer position.

This reduction of subsystems from a pure state before the interaction to a mixture after the interaction is therefore a causal consequence of the physical interaction and takes place during the finite time interval of the interaction. von Neumann's own insistence upon describing this reduction of subsystems as a separate process is therefore difficult to justify.

Finally, von Neumann showed[6] that the components of the two mixtures describing the object and the apparatus are correlated so that whenever the object is in state ϕ_i the apparatus is in state ξ_i. Using these correlations one obtains the interpretation of the reduced state (3) as the state of the separate object and apparatus systems; namely, again, that state which correctly describes all observations except those of type $\underline{Z}$.

As has been pointed out elsewhere,[13] all normal physical measurement operations involve a strict separation of object and apparatus systems, except during the measurement interaction. By imposing this restriction explicitly, a perfectly correct description of measurements can be obtained also on the basis of the reduced state.[10, 14, 15]

CONCLUSIONS

(1) The unreduced state vector correctly describes the probability of finding the apparatus pointer in any one of its possible positions. It also describes correctly the correlation between the outcomes of successive measurements on the same system.

(2) There is no noncausal state reduction associated with the measurement process.

(3) The von Neumann reduction of subsystem states is an entirely causal process which occurs during the interaction of the subsystem with other subsystems.

(4) No extra–quantum-mechanical processes are required for the complete description of a measurement. Specifically, neither the classical nature of the apparatus nor extraphysical properties of the observer's "consciousness" are needed.

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[1]Arthur Fine, Phys. Rev. D 2, 2783 (1970).

[2]B. D'Espagnat, Nuovo Cimento Suppl. 4, 828 (1966).

[3]E. P. Wigner, Am. J. Phys. 31, 6 (1963).

[4]E. P. Wigner, *Contemporary Physics* (International Atomic Energy Agency, Vienna, 1969), Vol. II, p. 431.

[5]J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, Berlin, 1932), Chaps. IV and V.

[6]Ref. 5, Chap. VI.

[7]J. Earman and A. Shimony, Nuovo Cimento 54B, 332 (1968).

[8]L. E. Ballentine, Rev. Mod. Phys. 42, 358 (1970).

[9]These facts are, of course, neither particularly startling, nor are they new. For example, Margenau (Ref.10) has stated these results and criticized the "projection postulate." The present derivation emphasizes that the calculation of correlations requires no "extension" of quantum mechanics, as suggested in Refs. 8 and 10. It is also intended to clarify the operational meaning of such correlations.

[10]H. Margenau, Phil. Sci. 30, 138 (1963); Ann. Phys. (N.Y.) 23, 469 (1963).

[11]This function of the "classical" apparatus is of course entirely distinct from the requirement that a precise measurement of an observable that fails to commute with an additive conserved observable requires a large apparatus. See E. P. Wigner, Z. Physik 131, 101 (1952); H. Araki and M. M. Yanase, Phys. Rev. 120, 622 (1960).

[12]J. M. Jauch, Helv. Phys. Acta 37, 293 (1964).

[13]P. A. Moldauer (unpublished).

[14]G. Ludwig, *Die Grundlagen der Quantenmechanik* (Springer, Berlin, 1954).

[15]*Added in proof.* Two other authors have described measurements without "reduction." Hugh Everett, III, Rev. Mod. Phys. 29, 454 (1957), rederived von Neumann's correlated mixtures of subsystem states and interpreted them as specifying different noncombining "branches" of the "real" world. However, measurements of observables of type $\underline{Z}$ do recombine Everett's branches. Philip Pearle, Am. J. Phys. 35, 742 (1967), described for special cases many of the general results of this paper. I do not follow his conclusion that quantum mechanics requires completion through the introduction of "hidden variables."