There is in our model sufficient freedom to ensure correct threshold behavior and perhaps some unitarity effects without affecting the crossing symmetry and asymptotic behavior of the amplitude. As argued in Ref. 1, such arbitrariness is expected since unitarity is not yet fully imposed. However, since amplitude (1) has second-sheet resonance poles, correct Regge asymptotic behavior for essentially arbitrary α, nonvanishing double-spectral functions, and a factorizable N-point generalization, it may be expected to serve as a starting point in a search for an exactly unitary amplitude.

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[1]G. Cohen-Tannoudji, F. Henyey, G. L. Kane, and W. J. Zakrzewski, Phys. Rev. Letters 26, 112 (1971).

[2]See E. T. Copson, *Asymptotic Expansions* (Cambridge Univ. Press, London, 1965).

[3]M. Suzuki, Phys. Rev. Letters 23, 205 (1969).

[4]There exist many functions which satisfy the conditions in Eq. (2). An example of such a function is

$$\frac{1}{c}\int_x^1 \exp\left(-\frac{1}{u}-\frac{1}{1-u}\right)du, \quad c=\int_0^1 \exp\left(-\frac{1}{u}-\frac{1}{1-u}\right)du\,.$$

However, the main features of our amplitude do not depend on the precise form of $N(x)$. The asymptotic behavior in the second sheet (which we have not considered in this paper) will restrict this choice.

[5]Obviously, duality in the sense that the sum over t-channel resonances equals the sum over s-channel resonances cannot be maintained in an amplitude that allows for a finite number of resonances. However it may be noted that the sequence of resonances and the Regge behavior are embodied in an inseparable manner, which is a distinctive feature of duality.

[6]We use the notation of Chan Hong-Mo, Proc. Roy. Soc. (London) A318, 379 (1970).

[7]It is clear that in our approach the Adler condition is a dynamical requirement imposed on $f(x,y)$ such that the integral (15) vanishes. Other similar approaches that produce an Adler zero do so either by an explicit kinematical restriction [see A. I. Bugrij, L. L. Jenkovsky, and N. A. Kobylinsky, Kiev Report No. ITF-71-28E, 1971 (unpublished)], or in the context of an amplitude for scalar particles (see Ref. 1).

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Special Bootstrapping Property of SU(n) Lie Algebras*

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It is shown that certain simple bootstrap conditions, applied to a hypothetical universe of mutually interacting mesons, require SU(n) symmetry of the interactions. The relevant distinguishing feature of SU(n) is that the matrices of the fundamental representation of the Lie algebra, together with the identity matrix, are a complete set.

Several years ago it was shown that the application of a certain type of bootstrap condition requires the existence of a Lie-algebra interaction symmetry.[1] Recently, several arguments have been given that some bootstrap conditions require further that the Lie group is one of the special unitary groups SU(n).[2-4] However, the arguments in these references are rather complicated. The purpose of the present note is to show that the SU(n) requirement follows simply from simple assumptions, and that a basic special property of the SU(n) algebras is involved. The basic algebraic structure of the argument used here is the same as that in Sec. III of Ref. 4, however.

We assume a set of mesons interacting with trilinear interactions, and consider a self-conjugate basis of this set (i.e., each state is identical with its antiparticle state). We write the interaction constant G as a sum of parts symmetric and antisymmetric in the first two indices, i.e., $G_{ijk}=D_{ijk}+F_{ijk}$ where $D_{ijk}=D_{jik}$ and $F_{ijk}=-F_{jik}$. Whether or not the subscripts refer to helicities as well as to internal quantum numbers depends on the model; we are not interested in the details of the model here.[5] We assume that the D and F can be defined so as to be Hermitian in the last two subscripts, i.e.,[6]

$$D_{ijk}=D^*_{ikj}, \quad F_{ijk}=F^*_{ikj}. \tag{1}$$

This condition, together with the definition of D and F, implies that D_{ijk} is real and completely symmetric, and F_{ijk} is imaginary and completely antisymmetric.

The basic bootstrap condition is taken to be either of the following two equations:

$$\sum_r G_{dcr}G^*_{abr}=\lambda\sum_r G_{adr}G^*_{bcr}, \tag{2a}$$

$$\sum_r (G_{cdr}G^*_{abr}-G_{adr}G^*_{cbr})=\gamma\sum_r (G_{acr}-G_{car})G^*_{rbd}, \tag{2b}$$

where the constant λ or γ is real, and the set of intermediate states r must be identical with the range of the external indices a, b, c, and d. The only other assumption made is that of nontriviality; we require that any solution must have at least two different interacting states.

Models leading to one or the other of these conditions are discussed in the literature, and will be mentioned only briefly here. If the process $a+b\to c+d$ is the s-channel scattering amplitude, and the initial states in the t and u Mandelstam channels are $a+c$ and $b+c$, respectively, then Eq. (2a) results from the application of a duality assumption to backward s-channel scattering.[7] The alternate condition, Eq. (2b), may be obtained from a finite-energy sum rule, with a pole-dominance assumption, applied to forward scattering.

For convenience, we define a constant H_{ijk} by the equation $H_{ijk}=G_{jik}$, so that $H_{ijk}=D_{ijk}-F_{ijk}$. The G and H are each invariant to cyclic permutations of the indices. For each of the constants G, H, F, and D, we define matrices in the space of the last two indices, i.e., G_{ijk} is the jk element of the matrix G_i.

We consider first the bootstrap condition of Eq. (2a). If one permutes cyclically the indices a, b, c, and d in this equation, and compares with the original equation (using the Hermiticity and cyclic permutation invariance of the G's), it is seen that λ must be either 1 or -1. It can be shown that there are no nontrivial solutions if $\lambda=-1$, so we take $\lambda=1$. The following three conditions are then obtained from this bootstrap equation in Ref. 4: (1) There is no nonzero solution in which all D's vanish. (2) There is no nontrivial solution in which all F's vanish, so D's and F's must both exist. (3) The F's must be proportional to the structure constants of a compact, semisimple Lie group.

If one makes use of the Hermiticity condition of Eq. (1), then (for $\lambda=1$) the condition of Eq. (2a) may be written as the db element of the matrix commutator equation,

$$[H_c, G_a]=0. \tag{3}$$

Two further equations may be obtained by using Eq. (1) to remove the asterisks in Eq. (2a) and then applying the permutation operators $(1\mp\Pi_{ac})\Pi_{cd}$, where Π_{ij} interchanges the labels i and j. These are the db elements of the matrix equations,

$$[G_c, G_a]=-2\sum_r F_{car}G_r, \tag{4}$$

$$\{G_c, G_a\}=2\sum_r D_{car}G_r, \tag{5}$$

where $\{\ \}$ denotes an anticommutator. These latter two equations are equivalent to bd elements of the following equations involving the H matrices:

$$[H_c, H_a]=2\sum_r F_{car}H_r, \tag{6}$$

$$\{H_c, H_a\}=2\sum_r D_{car}H_r. \tag{7}$$

The derivation of the SU(n) requirement from these equations is simple. Since the F_{ijk} are proportional to the structure constants of a Lie algebra, Eqs. (4) and (6) state that the G_i and H_i are representations of the algebra. If, for any state a, all F_{aij} vanish, then it follows from Eqs. (3) and (4) that G_a (and consequently D_a) commutes with all G_i and H_i. By Schur's lemma, such a D_a must be a multiple of the identity matrix in any irreducible subspace of the G or H. We consider now an irreducible subspace of the G_i representation. If the states i and j belong to the algebra (i.e., F_i and F_j are nonzero), Eqs. (4) and (5) imply that all products G_iG_j are linear combinations of the G_k and the identity matrix. Thus, the G_k and the identity matrix are a complete set for the corresponding irreducible representation of the Lie group. If this representation is of degree n, Burnside's theorem states that it contains n^2 linearly independent matrices.[8] Thus, there are n^2-1 matrices in the representation of the Lie algebra, linearly independent of each other and the identity matrix; this defines simultaneously SU(n) and the fundamental representation.

Since the H equations, Eqs. (6) and (7), differ from the corresponding G equations only in the sign of the commutator equation, the H's must be the fundamental representation conjugate to that of the G's. The meson states are direct products of these two representations, and hence correspond to singlet $\oplus$ regular representation.

Next we suppose that the starting equation is Eq. (2b); we assume at the outset that $\gamma \neq -1$. Taking the symmetric and antisymmetric parts of this equation with respect to the interchange $b \rightleftharpoons d$ separates it into an even part (involving DD and FF terms) and an odd part (involving FD terms). If the permutation operator $\Pi_{bc}(1-\Pi_{ab})$ is applied to the odd part, and the result compared with the original odd part, it can be shown that γ must be unity. By using other permutation operations of the same general nature, one can derive Eq. (2a) (with $\lambda=1$) from Eq. (2b). Hence, the conclusions are the same as those given above.

It has been suggested in the literature that the special property of SU(n) that permits bootstrap equations to be satisfied, is the existence of a symmetric, trilinear interaction involving only the regular representation.[2,3] Such an interaction is present only in the case of SU(n), with $n \geq 3$.[3] However, this is not the property that is crucial for bootstrapping. SU(2) lacks this property, but leads to solutions of the bootstrap equations of this paper and of Refs. 2 and 3, solutions in which the symmetric interactions are singlet-singlet-singlet and singlet-triplet-triplet interactions. The group property that is crucial for bootstrapping distinguishes both the SU(n) groups, and the fundamental representation of these groups. It is the property that any linear combination of the matrices representing the algebra is a linear combination of these matrices and the identity matrix.

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[1]R. E. Cutkosky, Phys. Rev. 131, 1888 (1963).

[2]R. H. Capps, Phys. Rev. 171, 1591 (1968). In this reference, the uniqueness of SU(3) among second-rank groups was shown, and the uniqueness of SU(n) suggested.

[3]C. M. Andersen and Joel Yellin, Phys. Rev. D 3, 846 (1971).

[4]R. H. Capps, Phys. Rev. D 3, 3059 (1971).

[5]If the indices refer to helicities as well as internal quantum numbers, as in Ref. 4, then for each set of states ijk, both of the interactions F and D cannot be nonzero. The argument given here is still valid, though.

[6]In many models, these conditions follow directly from vertex crossing symmetry. For example, if the indices refer to spin components as well as internal quantum numbers, so that D_{ijk} and F_{ijk} cannot both be nonzero for a specific set of indices, Eq. (1) follows from the general Hermiticity condition $G_{ijk}=G^*_{ikj}$. The fact that this Hermiticity condition is equivalent to vertex crossing in such a model is shown by R. H. Capps, Phys. Rev. 168, 1731 (1968), Sec. II. (In this reference the antisymmetric interactions are defined to be i times the F of the present paper.)

[7]This condition, used in Ref. 4, is derived by R. H. Capps, Phys. Rev. D 2, 780 (1970), Sec. II C; and 2, 2640 (1970), Sec. II A.

[8]Hermann Boerner, *Representations of Groups* (North-Holland, Amsterdam, 1963), p. 69. The theorem is stated in this reference for finite groups, but also holds for compact, infinite groups. This theorem was called to the author's attention by Dr. T. K. Kuo. The theorem was called to Dr. Kuo's attention by Dr. S. P. Wang.