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Normalizable Wave Functions for Bound States and Resonances in S-Matrix Theory

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Normalizable wave functions are constructed for bound states and resonances from the S-matrix quantities. The bound-state wave function is of the Schrödinger type. The Schrödinger equation does not give normalizable wave functions for resonances. Thus the resonance wave function is of the non-Schrödinger type. An S-matrix model is constructed to generate the desired resonance wave function. The manner in which this model departs from the conventional Schrödinger picture is discussed in detail. This S-matrix model is then applied to the P-wave two-pion system with a satisfactory numerical result for the ρ-meson radius.

I. INTRODUCTION

While there are justifiable beliefs[1] that physics comes from singularities in the complex plane, it is refreshing to note that the study of analytic properties came from the physical principles that can be best stated in terms of space and time.[2] While there are also contrary views[3] that one should stick to the quantities that are directly related to space and time, it is not wrong to assume that physics remains unchanged even if we translate the space-time property into more convenient languages such as analyticity. There are at present two distinct approaches: One is to use exclusively the S matrix and the other is to use wave functions.[4] Both approaches have their own strengths and weaknesses. Therefore, we can do better if we know both methods and know how to make up the shortcomings of one method by using the other.

The purpose of this paper is to give examples where the wave-function approach helps the S-matrix method, and vice versa. We shall specifically discuss bound states and resonances in both S-matrix and wave-function approaches. We have in mind the S matrix with the usual analytic properties. By wave function, we mean the solution of the Schrödinger equation and its possible modified form. Both bound states and resonances are represented by the S-matrix poles. There corresponds a normalizable wave function to each bound state. For resonances, however, there are only quasi-local wave functions[5] for the real physical energy near the pole and the wave function corresponding to the resonance pole is an exponentially increasing function at large distance. We would like to have normalizable wave functions for some of the resonances and have the quasi-local wave function for the physical energy coming from this completely localized base at the exact location of the resonance pole.

Let us discuss the physical significance of the above proposal. As is well known, the Schrödinger equation has two linearly independent solutions. The physical assertion that the probability of the particle being at the infinite distance be zero and that the probability be finite at the origin enables us to discard the irregular and nonlocal solutions. This condition in fact produces normalizable wave functions and the discrete energy spectrum. There are plenty of new discrete energy spectra in elementary-particle physics. Since many of them obey attractive spectral rules such as the SU(3) scheme,[6,7] we are led to suspect that these spectra may be coming from some form of boundary conditions as those of the hydrogen atom and the harmonic oscillator came from their respective boundary conditions. Now the complication arises. Some entries in these spectra are resonances while others are stable particles and stable bound states. If we subscribe to the original bootstrap idea and assert that bound states and resonances are physically equivalent, then the mixed entry in the new spectra is understandable. As we stated above, the resonance (at the exact location of the pole) and the bound state satisfy quite different boundary conditions. We are thus led to question the meaning of the "physical equivalence" between these two seemingly different states. In this paper,

we discuss a possibility of constructing normalizable wave functions for the resonances as well as for the bound states.

The complex energy plane is a very convenient place to represent the discrete spectrum of bound-state and resonance poles. But, it is not convenient to study boundary conditions with the S matrix alone.[8,9] Our aim is to construct a model that will enable us to think of every physically interesting pole in terms of a normalizable wave function. This picture is not consistent with the conventional Schrödinger formalism of quantum mechanics, but we indicate clearly where the departure occurs and discuss why that particular departure point is more reasonable than other places. In the following sections, we shall first present a concrete example where the wave function can solve the difficulties in the S-matrix calculation. We shall then construct a model that will produce normalizable wave functions for the resonances.

In Sec. II, we review and reformulate the old work of the present authors to illustrate that the S-matrix method (with the usual approximation procedures) does not always guarantee the boundary conditions required for bound states. It is shown that this trouble can be fixed by the use of the wave-function method. In Sec. III, we review the situation in the Schrödinger quantum mechanics and point out that the conventional Schrödinger picture does not give resonance wave functions satisfying the desired boundary condition in the asymptotic region. It is shown in Sec. IV that this trouble in the asymptotic region can be moved to the origin where the Schrödinger quantum mechanics is likely to break down (at least more likely than in the asymptotic region where experiments are performed). We do this by introducing an irregular solution of the Schrödinger equation that gives a localized resonance wave function and a correct phase shift up to a minus sign which cannot be detected experimentally. Removing the irregularity of this new wave function is definitely a departure from the Schrödinger quantum mechanics. In Sec. V, we discuss this possibility using analytic properties of the S matrix, and then construct a model that will give a normalizable wave function for the resonance. In Sec. VI, this model is applied to the calculation of the ρ-meson radius. Our calculation gives a satisfactory numerical result. In Sec. VII, we point out that the problems discussed in this paper can be formulated as the particle localization in S-matrix theory.

II. BOUND-STATE PERTURBATION

Perhaps the best place to illustrate the use of bound-state wave functions is the first-order perturbation theory. Let us consider the first-order energy shift in terms of wave functions and a weak perturbing potential,

$$\delta E = (\phi, \delta V \phi). \tag{1}$$

We now contemplate calculating the above quantity using only the S-matrix quantities such as the N and D functions.

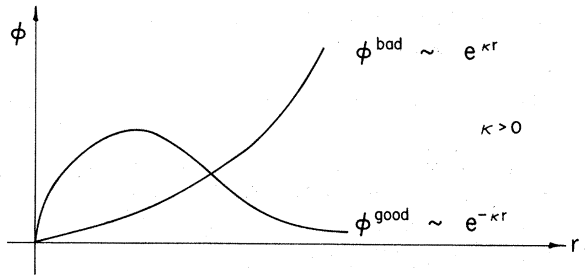

FIG. 1. "Good" and "bad" wave functions for the bound state. The bad component comes from the failure to cut off the analytic continuation of the incoming wave component in the S matrix.

The S matrix, in the physical region, contains both solutions of the Schrödinger equation satisfying the outgoing-wave [$\exp(ikr)$] and the incoming-wave [$\exp(-ikr)$] boundary conditions, respectively. Let us now consider analytic continuation of the S matrix to a bound state, in which case the momentum k becomes a purely imaginary quantity $i\kappa$ where κ is positive. During this process, the outgoing wave becomes $\exp(-\kappa r)$, while the incoming wave behaves like $\exp(\kappa r)$. For the reasons that will become clear in the following discussion, we call them "good" and "bad" wave functions, respectively. See Fig. 1. When we reach the correct binding energy, the bad wave function is to be cut off by the laws of dynamics which assure a localized probability entity for the bound state. Since this localization condition is overlooked in many S-matrix approximations, it is very easy to allow some bad wave functions to exist even at the binding energy.

Perhaps the best known work involving the above effect is the calculation of the n-p mass difference by the perturbation method developed by Dashen and Frautschi.[10] Let us review how this trouble occurs. Because they fail to cut off all bad wave functions, their approximation leads in effect to

$$\delta E = (\phi^{good}, \delta V \phi^{bad}). \tag{2}$$

Now, for the electromagnetic perturbation,

$$\delta V = \frac{1}{r} \exp(-\lambda r), \tag{3}$$

where $\lambda \to 0$, the integrand in normal circumstances

(where only good wave functions are used) would have an exponential cutoff factor

$$\exp[-(2\kappa+\lambda)r] \tag{4}$$

giving a comfortable cutoff factor even if $\lambda \to 0$. However, in the case of Eq. (2), the exponential cutoff factor is merely $\exp(-\lambda r)$ which gives a logarithmic divergence for $\lambda \to 0$. It was shown that this indeed was the source of their so-called spurious infrared divergence.[8]

It is widely believed that Dashen's calculation of the n-p mass difference is wrong and therefore does not deserve any further attention. We disagree. The Dashen-Frautschi formalism is indeed an excellent crossing point where both the wave function and the S-matrix methods can be used for the same purpose. Recognizing the seriousness of the problems of boundary conditions, the present authors devised an approximation scheme that will convert the S-matrix quantities into a "good" wave function.[8] Using this new method, we went further to calculate the size of the nucleon assuming that the nucleon is a bound state of pion and nucleon and obtained a satisfactory numerical result.[11]

The perturbation formula of Dashen and Frautschi is correct. But the approximations which are inevitable in strong-interaction calculations led to the ambiguities mentioned above, and these difficulties can be best understood in terms of wave functions. In the following sections we shall discuss a more fundamental question of assigning "good" wave functions to resonances.

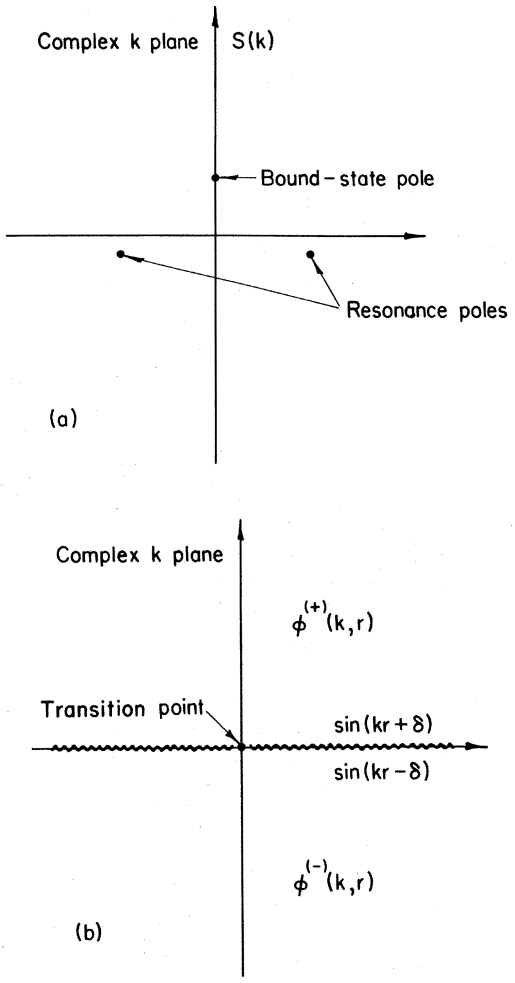

FIG. 2. Complex k planes. (a) Bound-state and resonance poles in the complex k plane. We assume that this analytic property remains unchanged in our new model. (b) Wave functions in the complex k plane. By assigning $\phi^{(-)}(k, r)$ to the lower half-plane, we obtain a localized and irregular wave function for the resonance. When a bound state becomes a resonance through a variation in the strength of potential, this transition occurs at the origin $k=0$. This transition is not analytic.

III. RESONANCE IN THE CONVENTIONAL SCHRÖDINGER PICTURE

In this section we list all the known properties of the S matrix and the wave function that will be useful in the main discussion. The S-wave Schrödinger equation is

$$\frac{d^2}{dr^2}\psi+[k^2-V(r)]\psi=0, \tag{5}$$

where the potential $V(r)$ is assumed to have the usual behavior that will give the desired analytic property of the S matrix. We can construct two linearly independent wave functions $f(-k, r)$ and $f(k, r)$ satisfying, respectively, the outgoing- and incoming-wave boundary conditions. These solutions are called the Jost solutions. We shall use throughout this paper the notation of De Alfaro and Regge.[12]

Using the Jost solutions, we can construct the regular wave function

$$\phi^{(+)}(k,r)=\frac{1}{2ik}[f(k)f(-k,r)-f(-k)f(k,r)]\,. \tag{6}$$

This solution satisfies the required regularity condition at $r=0$ and allows us to write the completeness relation for all physical solutions. This wave function furthermore has the familiar asymptotic form

$$\phi^{(+)}(k,r)\to\sin(kr+\delta), \tag{7}$$

where δ is the phase shift.

We shall assume throughout this paper that the

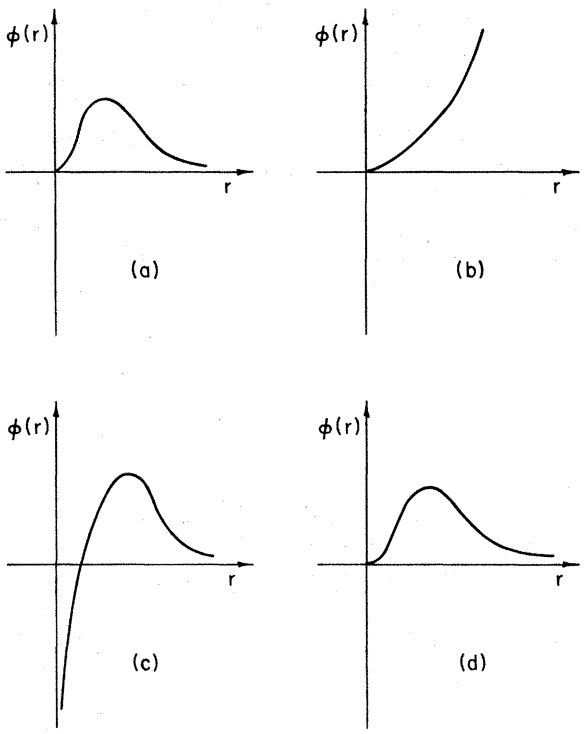

FIG. 3. Wave functions for bound states and resonances. (a) Bound-state wave function in the Schrödinger picture. It is normalizable. (b) Resonance wave function in the Schrödinger picture. It is regular at the origin but increases exponentially for large r. This wave function is not normalizable. (c) Resonance wave function in the modified picture. The wave function decreases exponentially for large r but is irregular at the origin. This wave function is not normalizable. (d) Regularized resonance wave function. The irregularity at the origin is removed. This wave function does not come from the Schrödinger equation and thus represents a departure from the Schrödinger picture. All of the resonance wave functions considered here contain an oscillatory structure within the exponential envelope. This structure has been ignored for simplicity.

S matrix defined as $S(k)=f(k)/f(-k)$ has the desired analytic properties. $f(k)$ and $f(-k)$ are called the Jost functions and are discussed extensively in the literature.[12] The entire real k axis constitutes the continuous eigenvalue spectrum. For discrete eigenvalues corresponding to bound states, the Jost function $f(-k)$ has zeros. Because the other Jost function $f(k)$ has singularities and some of them are poles, the S matrix can have poles which do not come from $f(-k)=0$. These singularities have been discussed in the literature.[13] In this paper, we restrict ourselves to only physically interesting poles which come from $f(-k)=0$.

It is well known that $f(-k)$ can have zeros only along the imaginary axis if Imk is positive. See Fig. 2. The physical states corresponding to these poles are bound states, and each bound state has a corresponding normalizable wave function. Suppose $f(-k)=0$ in Eq. (6); then

$$\phi^{(+)}(k,r)=\frac{1}{2ik}f(k)f(-k,r), \tag{8}$$

whose asymptotic form is $\exp(ikr)$ which decreases as $\exp[-(\mathrm{Im}k)r]$ for large r.

If $f(-k)$ has a zero in the lower half-plane at a point very close to the real k axis, it corresponds to a resonance. However, unlike the case of the bound state, $\phi^{(+)}(k,r)$, though regular at the origin, cannot represent a localized entity for the resonance because it increases exponentially for large r. See Fig. 3. This problem is also discussed in the literature.[12]

In Sec. II, we traced the "bad" behavior of the bound-state wave function to a certain approximation procedure. In the case of resonances, this "bad" feature comes from the analytic continuation of the Schrödinger equation and is inherent in the conventional formalism of quantum mechanics. It is a matter of changing the sign of the exponent to obtain a "good" wave function. In Sec. IV, we shall investigate the possibility of relating this sign to the quantity that is *not* measurable, and see whether we can change the sign of the exponent using this measurement aspect.

IV. RESONANCES IN THE MODIFIED PICTURE

In spite of this apparent lack of localized picture for the resonance, the most important development in particle physics during the past decade has been the formulation of the concept that particles and resonances belong to the same multiplet and that they share the common physical origin. Furthermore, there have been many successful model calculations based on the assumption that the resonances, like particles, behave like localized entities. It is thus compelling to look into the possibility of localizing the resonance wave function. We show that the Schrödinger equation has indeed a solution that satisfies the correct locality condition at least in the asymptotic region.

Let us consider the solution

$$\phi^{(-)}(k,r)=\frac{1}{2ik}[f(k)f(k,r)-f(-k)f(-k,r)]. \tag{9}$$

This wave function differs from the regular solution $\phi^{(+)}(k,r)$ by the sign of k in the Jost solutions. $\phi^{(-)}(k,r)$ has the asymptotic form

$$\phi^{(-)}(k,r)\to\sin(kr-\delta) \tag{10}$$

and becomes

$$\phi^{(-)}(k,r)=\frac{1}{2ik}f(k)f(k,r) \tag{11}$$

at the resonance energy where $f(-k)=0$ in the lower half-plane.

Both Eq. (10) and Eq. (11) can be analytically continued from the master form of Eq. (9). The asymptotic form of Eq. (10) differs from that of the regular solution by the sign of the phase shift. However, this sign of the phase shift cannot be determined *experimentally*. Therefore, both $\phi^{(+)}(k,r)$ and $\phi^{(-)}(k,r)$ give the same result as far as scattering experiments are concerned. At the resonance energy where $f(-k)=0$ in the lower half-plane, $\phi^{(-)}(k,r)$ decreases exponentially for large r and thus produces a localized picture for the resonance state.

While the analytic continuation of $\phi^{(+)}(k,r)$ to the upper half-k-plane (the so-called physical sheet) gives a good normalizable wave function for the bound state, the analytic continuation of $\phi^{(-)}(k,r)$ to the lower half-plane gives, without contradicting the scattering experiment, at least an asymptotically acceptable form for the resonance as a localized entity. Thus we are led to assign $\phi^{(+)}(k,r)$ to the upper and $\phi^{(-)}(k,r)$ to the lower half-plane. See Fig. 2.

$\phi^{(+)}(k,r)$ of the upper half-plane cannot be analytically continued to $\phi^{(-)}(k,r)$ of the lower half, nor can they be interchanged by symmetry operations such as time reversal. The newly introduced wave function $\phi^{(-)}(k,r)$, even at the resonance energy, lacks the regular behavior at the origin. See Fig. 3. For the S-wave case, the regular behavior requires $\phi \sim r$ near origin, and an r^{l+1} behavior for the lth partial wave. The new wave function behaves like r^{-l}.

This irregularity behavior cannot be eliminated within the framework of the present quantum mechanics. Earlier in this paper, we have stated that the Schrödinger quantum mechanics, if analytically continued in the conventional manner, gives a resonance wave function that cannot describe a localized entity. By introducing the above irregular solution, we have simply transferred the trouble from the asymptotic region to the origin ($r=0$), where the present formalism of quantum mechanics is likely to break down (at least more likely than in the asymptotic region). While we do not have any systematic departure from this point, we may mention that many field-theoretic calculations indeed lead to the singularities at the origin that are not familiar in nonrelativistic quantum mechanics.[14]

By considering boundary conditions and measurement aspects in the asymptotic region where experiments take place, we have been able to pack all the troubles in the Schrödinger picture into a single irregularity point at the origin. This of course creates many problems. Perhaps the immediate question to ask would be: What happens to the completeness relation if $\phi^{(-)}(k,r)$ is taken seriously as a physical wave function? We do not have a satisfactory answer. However, we point out that many field-theoretic models, such as the Bethe-Salpeter equation, lack this kind of completeness.[15]

Without discussing further the fundamental questions raised above, we shall construct a resonance model in Sec. V. This model will give a localized and regular wave function for the resonance and will be a departure from the conventional Schrödinger formalism.

V. MODEL FOR RESONANCES

We have shown in Sec. IV that the Schrödinger equation has a solution having an acceptable behavior for localized resonances in the asymptotic region. Because of the irregularity at the origin, this wave function does not have a proper meaning in quantum mechanics. However, it is not wrong to speculate that quantum mechanics breaks down in the small-distance region (within the Compton wavelength of hadrons). This is the physical basis upon which we plan to build a model that will produce a localized and normalizable resonance wave function.

The reason why we are so eager to build normalizable resonance wave functions has been given in Sec. I. The next question is naturally *how*. In order to proceed, let us look into the circumstances that led to the popular proposition that bound states and resonances are equivalent. Since Frazer's successful calculation of the nuclear electromagnetic form factors, almost all strong-interaction calculations involved a discrete representation of the continuous energy spectrum. In doing so, resonances are treated like bound states, and this technical convenience together with the more prominent discrete nature of resonance spectra eventually led to the above-mentioned proposition. Though there have been many numerical successes in this school of thought, its physical implication has not been fully explored.[9]

We are thus invited to look for a formulation of quantum mechanics in which one can obtain wave functions not by solving the Schrödinger differential equation but by summing over a complete set in a manner similar to what we do in dispersion theory.[16] As we shall see in the following discussion, the Gelfand-Levitan formalism indeed satisfies this requirement.[17] Unlike the case of solving the Schrödinger equation, the input parameters are those of the S matrix (not those of the potential). This is the point we like to exploit in this paper.

The Gelfand-Levitan (G-L) formalism is a beautiful mathematical theorem, but is not widely understood. Instead of going through rigorous mathematical steps, we shall use here first the familiar language of the completeness relation. Because there is an excellent parallelism between the G-L formalism and the completeness relation, we believe this is the easiest way to exploit the content of the Gelfand-Levitan theorem.

Let us therefore look at the situation in the completeness relation first.[16] Consider all regular solutions of the Schrödinger equation $\phi(k,r)$ normalized as

$$\lim_{r\to 0}\phi(k,r)\sim r. \tag{12}$$

Then the complete-orthonormality relation states that

$$\sum_b C_b\phi_b^*(k_b,r)\phi_b(k_b,r') + \frac{2}{\pi}\int_0^\infty k^2[f(k)f(-k)]^{-1}\phi^*(k,r)\phi(k,r')\,dk = \delta(r-r'), \tag{13}$$

where the summation b is over all bound states, and k_b is the imaginary momentum corresponding to the nth binding energy. C_b is the normalization constant:

$$\frac{1}{C_b}=\int_0^\infty |\phi_b(k_b,r)|^2\,dr. \tag{14}$$

This normalization constant can be derived from the S-matrix quantities[16]:

$$\frac{1}{C_b}=\frac{-1}{4ik^2}f(k)\frac{df(-k)}{dk}\bigg|_{k=k_b}. \tag{15}$$

In the above completeness relation, the integral measure $k^2[f(k)f(-k)]^{-1}$ is strictly an S-matrix quantity. In the sum over bound states, the coefficient C_b is also an S-matrix quantity. Thus this completeness relation formally (not uniquely) enables us to obtain the wave function $\phi(k,r)$ directly from the S-matrix quantities without using potentials.

Since we are interested in resonances, we should discuss where the resonance stands in the completeness relation of Eq. (13). The bound states are all contained in the summation. However, the resonances are included in the continuum integral. For a rapid change in phase shift such as

$$\delta(k)=\pi\,\theta(k^2-k_\rho{}^2), \tag{16}$$

where $k_\rho{}^2$ is the resonance energy, we can show that the integrand exhibits a δ-function behavior and consequently its contribution can be pulled out of the integral sign. Thus even in the Schrödinger picture, resonances form an almost discrete spectrum. But the Schrödinger picture does not give normalizable resonance wave functions.

Now we make a departure from quantum mechanics and propose the following completeness relation with normalizable resonance wave functions[18]:

$$\sum_b C_b\phi_b^*(k_b,r)\phi_b(k_b,r') + \sum_\rho C_\rho\phi_\rho^*(k_\rho,r)\phi_\rho(k_\rho,r') + \frac{2}{\pi}\int_0^\infty k^2[\tilde f(k)\tilde f(-k)]^{-1}\phi^*(k,r)\phi(k,r')\,dk = \delta(r-r'), \tag{17}$$

where all the resonance effects have been screened out from the continuum integral. We denote this effect by $k^2[\tilde f(k)\tilde f(-k)]^{-1}$. In order to accommodate the proposition of resonance–bound-state equivalence, we assert that the resonance wave function $\phi_\rho(k_\rho,r)$ has all the desirable properties of the bound-state wave function. The resonance wave function is expected to be complex. It is well known, however, that for each resonance pole there is a conjugate pole at $-k_\rho^*$. See Fig. 2. Because of this conjugate pole the over-all summation will be real. Since we are assuming that the S matrix is analytic as before, the normalization constant, which is defined as

$$\frac{1}{C_\rho}=\int_0^\infty |\phi_\rho(k_\rho,r)|^2\,dr, \tag{18}$$

is expected to be approximately equal to the real part of the S-matrix expression of Eq. (15) evaluated at the resonance energy:

$$\frac{1}{C_\rho}=\frac{1}{4ik^2}f(k)\frac{df(-k)}{dk}\bigg|_{k=k_\rho}, \tag{19}$$

where the imaginary part of this expression is expected to be small in narrow-resonance approximations.

With this preparation, let us go to the Gelfand-Levitan equation. The theorem states that one can obtain the wave function $\phi(k,r)$ *uniquely* (unlike the case of the completeness relation) from the S-matrix quantities in the following manner[16,17]:

$$\phi(k,r)=\frac{\sin kr}{k}+\int_0^r K(r,r')\frac{\sin kr'}{k}\,dr', \tag{20}$$

where the kernel $K(r,r')$ satisfies the integral equation

$$K(r,r')+g(r,r')+\int_0^r dt\,K(r,t)g(t,r')=0. \tag{21}$$

The second kernel $g(r, r')$ is to be obtained from the S-matrix:

$$g(s,t)=\int\left(\frac{\sin ks}{k}\right)\left(\frac{\sin kt}{k}\right)^{*} d\mu(E), \tag{22}$$

with

$$\begin{aligned} &d\mu(E)=C_b\delta(E-E_b)\,dE, \quad E<0 \\ &d\mu(E)=\frac{k}{\pi}\{[f(k)f(-k)]^{-1}-1\}\,dE, \quad E>0 \end{aligned} \tag{23}$$

where $E=k^2$. Here we assumed for simplicity that there is only one bound state at $E=E_b$. C_b is the normalization constant defined in Eq. (15). Since $g(s, t)$ is completely determined from the S-matrix quantities, both $K(r, r')$ and $\phi(k, r)$ come directly from the S matrix. The complex conjugation on $(\sin kt)/k$ in Eq. (22) is not needed in the original form of the G-L theorem because it is real. This will however prove convenient when we treat resonances with complex momenta.

We note here that the input-output relation in the G-L formalism is identical to that of the completeness relation. We are then led to consider a modification in the input system of Eq. (23) according to what we did in the completeness relation. Assuming for simplicity that there is only one resonance, we are led to the following set of input measures:

$$\begin{aligned} &d\mu(E)=C_b\delta(E-E_b), \quad E<0 \\ &d\mu(E)=\frac{k}{\pi}\{[\tilde{f}(k)\tilde{f}(-k)]^{-1}-1\}\,dE \\ &\qquad +C_\rho\delta(E-k_\rho{}^2)\,dE, \quad E>0. \end{aligned} \tag{24}$$

Here again, the resonance effect has been screened out from the continuum measure.

Starting from the formalism of quantum mechanics in which one obtains wave functions by summing over a complete set of states, we derived a prescription of treating resonances like bound states. Though this prescription may produce results which cannot be found in nonrelativistic quantum mechanics, this type of practice is quite common in dispersion theory and is completely in line with the present-day particle theory. We shall study consequences of this prescription in Sec. VI.

VI. APPLICATION TO THE ρ MESON

In Sec. V, we have seen how the Gelfand-Levitan measure becomes modified when resonances are treated like discrete bound states. In this section, we shall use this model to calculate a strong-interaction radius of the ρ meson.

In order to stress the resonance–bound-state equivalence, we shall briefly review what the authors did for the bound-state case.[11] If there is only one bound state with no resonances, we can consider an approximate solution of the Gelfand-Levitan equation assuming that $f(k)=1$ (which corresponds to a plane-wave approximation) throughout the continuous spectrum. Then

$$\begin{aligned} &d\mu(E)=C_b\,\delta(E-E_b)\,dE, \quad \text{for } E<0 \\ &d\mu(E)=0, \quad \text{for } E>0. \end{aligned} \tag{25}$$

If we put in this approximate measure, the integral equation for $K(r, r')$ becomes separable, and consequently the bound-state wave function becomes[11]

$$\phi_b(k,r)=\left(\frac{\sin kr}{k}\right)\left(1+C_b\int_0^r\left|\frac{\sin kt}{k}\right|^2 dt\right)^{-1}\Bigg|_{k^2=E_b}. \tag{26}$$

Since k is imaginary at the binding energy, the above wave function has an exponential cutoff factor for large r. The cutoff strength is proportional to the distance between the bound-state pole and the real k axis. This is an approximate but manifestly normalizable wave function that can be obtained directly from the S-matrix parameters. Assuming that the nucleon is a bound state of pion and nucleon, the present authors[11] went further to calculate the strong-interaction radius of the nucleon by taking

$$\begin{aligned} \langle r\rangle &=(\phi_b, r\phi_b)/(\phi_b,\phi_b) \\ &=\int_0^\infty\left(1+C_b\int_0^r\left|\frac{\sin kt}{k}\right|^2 dt\right)^{-1} dr. \end{aligned} \tag{27}$$

If there is only one resonance with no bound states, the Gelfand-Levitan measure, according to what we did in Sec. V, becomes

$$d\mu(E)=0, \quad \text{for } E<0$$

and

$$\begin{aligned} d\mu(E)=&\frac{k}{\pi}\{[\tilde{f}(k)\tilde{f}(-k)]^{-1}-1\}\,dE \\ &+C_\rho\delta(E-k_\rho{}^2)dE, \quad E>0. \end{aligned} \tag{28}$$

If we use the Breit-Wigner representation of the resonance, screening out its effect from the Jost function would take the form of factoring out its zero corresponding to the resonance pole in the following manner:

$$\tilde{f}(-k)=f(-k)/f_{\mathrm{BW}}(-k), \tag{29}$$

where $f_{\mathrm{BW}}(-k)$ is the Breit-Wigner form of the Jost function. Both $f(-k)$ and $f_{\mathrm{BW}}(-k)$ have their zeros at the resonance pole and thus $\tilde{f}(-k)$ has no zero at the pole.

Let us now consider P-wave pion-pion scattering. In this case, the Breit-Wigner representation for the ρ meson is valid for $f(-k)$ itself

throughout the entire conceivable energy region. Thus

$$\tilde{f}(-k)=f_{\mathrm{BW}}(-k)/f_{\mathrm{BW}}(-k)=1 \tag{30}$$

and

$$\begin{aligned} &d\mu(E)=0, \quad \text{for } E<0 \\ &d\mu(E)=C_\rho\delta(E-k_\rho{}^2)\,dE, \quad E>0. \end{aligned} \tag{31}$$

With this approximation, we can solve the Gelfand-Levitan equation in the same way as we did for the bound-state case. The resulting wave function is

$$\phi_\rho(k,r)=\left(\frac{\sin k_\rho r}{k_\rho}\right)\left(1+C_\rho\int_0^r\left|\frac{\sin k_\rho t}{k_\rho}\right|^2 dt\right)^{-1}. \tag{32}$$

This wave function behaves like

$$\phi_\rho(k,r)=r^{l+1} \tag{33}$$

at the origin and

$$\phi_\rho(k,r)\sim\exp(-|\mathrm{Im}k|r)$$

for large r, and thus behaves like the bound-state wave function of Eq. (26). The resonance wave function of Eq. (32), which is an approximate form, satisfies the exact normalization condition

$$\frac{1}{C_\rho}=\int_0^\infty \phi_\rho^*(k_\rho,r)\phi_\rho(k_\rho,r)\,dr. \tag{34}$$

While one cannot get this normalizable wave function by solving the Schrödinger equation, we obtained the above form by simply treating resonances like discrete bound states in the Gelfand-Levitan formalism. Using the procedure well accepted in dispersion-theory calculations, we indeed obtained an almost identical form for both resonance and bound-state wave functions. While we are not able to construct a complete theory at this time, this pair of wave functions constitutes an attractive starting point for a further study of spatial properties of the resonance–bound-state equivalence proposition.

Now the remaining task is to determine C_ρ from the resonance parameters. Again within the framework of the Breit-Wigner representation,

$$f_{\mathrm{BW}}(-k)=(k-k_\rho)(k+k_\rho^*)/k^2. \tag{35}$$

Then according to Eq. (28),

$$C_\rho=-\frac{(\mathrm{Re}k_\rho)^4}{2\,\mathrm{Im}k_\rho}. \tag{36}$$

Since $\mathrm{Im}k_\rho$ is negative, C_ρ is a positive quantity.

Using the above information, we can now calculate the strong-interaction radius of the resonance in exactly the same way as we did for the bound-state case:

$$\begin{aligned} \langle r\rangle &= (\phi_\rho, r\phi_\rho)/(\phi_\rho,\phi_\rho) \\ &= \int_0^\infty\left(1+C_\rho\int_0^r\left|\frac{\sin k_\rho t}{k_\rho}\right|^2 dt\right)^{-1} dr. \end{aligned} \tag{37}$$

This expression is strikingly similar to that of the bound state in Eq. (27).

Let us now look at the above expression as a function of the width parameter β defined as

$$\beta=-\mathrm{Im}k_\rho.$$

The integral converges for nonzero β because the integrand decreases like $\exp(-2\beta r)$ for large r. As β becomes vanishingly small, it may seem that the integral diverges like $1/\beta$. However, the C_ρ factor, together with the t integral, makes the denominator large, and this counteracts the $1/\beta$ divergence. We can in fact show analytically that the above integral is bounded by a $\beta\ln\beta$ factor. This means that $\langle r\rangle$ vanishes in the zero-width limit and that $\langle r\rangle$ increases as the resonance width increases in the narrow-resonance region. This is consistent with our physical common sense that a "loosely" bound resonance has a wider width and has a larger strong-interaction radius.

Let us now calculate the radius of the ρ meson. The P-wave generalization of Eq. (37) is[19]

$$\langle r\rangle=\int_0^\infty dr\left(1+C_\rho\int_0^r\left|\frac{j_1(k_\rho t)}{k_\rho}\right|^2 dt\right)^{-1}. \tag{38}$$

According to our computer calculations, the numerical value of the above integral becomes 0.78 F for $\Gamma=90$ MeV where $\Gamma=-4\,\mathrm{Im}k_\rho$, 0.85 F for $\Gamma=120$ MeV, and 0.88 F for $\Gamma=150$ MeV. As we predicted above, the radius increases as the width increases.

If we use the formula

$$R^2=(2\alpha+1)\frac{d\alpha}{d(k^2)},$$

given in Regge-pole theory,[20] and use the well-established value of $\alpha'=1$ GeV^{-2} for the slope of the ρ-meson trajectory, the "radius" R turns out to be 0.7 F. This is very close to our numerical value.

Starting from a model in which nonrelativistic quantum mechanics is modified in such a way that resonances and bound states are treated equally, we have obtained a very reasonable numerical result for a strong-interaction radius of the ρ meson. We have also obtained a reasonable relation between the size of the resonance and its width.

One could raise the popular question whether treatments based on nonrelativistic quantum mechanics have any validity in calculations involving resonances like the ρ meson. However, we should

note that most of the successful calculations in the quark model have been done within the framework of nonrelativistic quantum mechanics.[21,22] Though most of them are independent of the detailed behavior of radial wave functions, the quark model is basically based on a standing-wave description of the resonance, the concept which is quite similar to what we pursued in this paper. It should be noted further that radial behaviors of the quark-model wave functions are receiving increasing recent attention.[23]

VII. CONCLUDING REMARKS

We started in this paper from the premise that we should use both S-matrix and wave-function methods. We then adopted the view that the wave function comes directly from the S matrix and that the conventional quantities such as the potential and the Schrödinger equation do not have to come into our S-to-$\phi(r)$ picture.

This new approach has certain advantages. First, it gives a normalizable wave function to the resonance as is illustrated in the preceding sections. This normalizability for the resonance as well as for the bound state may have a deeper implication in the theories where bound states and resonances are treated equally.[9] Second, the S matrix is a convenient quantity to accommodate Lorentz invariance,[3] while the potential is not. Therefore, the S-and-$\phi(r)$ combination is a better set of variables in relativistic dynamics. The Bethe-Salpeter equation is a good example in this direction.[15] Third, the S-matrix quantities are directly measurable from experiments. Our ρ-meson calculation illustrates this convenience.

The central issue in this paper has been how to localize the two-body interacting system. Loosely speaking, the word "local" means that some quantity is nonzero within a specified space-time region and vanishes or almost vanishes outside this region. Indeed this locality concept is the backbone of the present-day quantum theory. Negligence or oversimplification of this problem has led to many difficulties which are often manifested by divergent integrals.[10,24] It is not clear whether our way of representing the locality property is consistent with other fundamental physical principles such as Lorentz covariance.[25] Inasmuch as the localization issue remains as one of the unsolved fundamental problems, the particle localization in S-matrix theory such as the one discussed in this paper opens up an important new line of research.

Note added in proof. In his recent paper, Chew discusses the quantum superposition principle as embodied in an S matrix.[26] In this present paper and in our previous papers,[8] we have discussed the problem of translating the S-matrix language into that of wave functions. We believe that the use of wave functions is the best tool for studying the quantum superposition principle.

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