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QCD and the chiral critical point

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As an extension of QCD, consider a theory with "2 + 1" flavors, where the current quark masses are held in a fixed ratio as the overall scale of the quark masses is varied. At nonzero temperature and baryon density it is expected that in the chiral limit the chiral phase transition is of first order. Increasing the quark mass from zero, the chiral transition becomes more weakly first order, and can end in a chiral critical point. We show that the only massless field at the chiral critical point is a σ meson, with the universality class that of the Ising model. Present day lattice simulations indicate that QCD is (relatively) near to the chiral critical point.

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Understanding the collisions of heavy ions at ultrarelativistic energies requires a detailed knowledge of the equilibrium phase diagram for QCD at nonzero temperature and baryon density. We generalize QCD to a non-Abelian gauge theory with three colors and "2+1" flavors by holding the current quark masses in a fixed ratio as the overall mass scale is varied: $m \equiv m_{\rm up} = m_{\rm down} = r\, m_{\rm strange}$, with r a constant of order $\sim 1/20$. (For our purposes the difference between the up and down quark masses is inconsequential.) Currently, numerical simulations of lattice gauge theory [1] find that while there are lines of first order transitions coming up from $m = 0$ and down from $m = \infty$, these lines do *not* meet—there is a gap, with QCD somewhere in between. This is illustrated in Fig. 1, following a similar diagram from the results of Brown *et al.* [2].

Lines of first order transitions typically end in critical points, so it is natural to ask about the two critical points, labeled "$\mathcal{C}$" and "$\mathcal{D}$" in Fig. 1. As m decreases from $m = \infty$, the line of deconfining first order phase transitions [3] can end in a deconfining critical point, "$\mathcal{D}$" in Fig. 1. Correlation functions between Polyakov lines are infinite ranged at the deconfining critical point; by an analysis similar to that given below, one can show that $\mathcal{D}$ lies in the universality class of the Ising model, or a $Z(2)$ spin system, in three dimensions.

The opposite limit is to work up from zero quark mass. For three flavors the chiral phase transition is expected to be of first order at $m = 0$ [4, 5], so as m increases, the line of first order transitions can end in a chiral critical point, "$\mathcal{C}$" in Fig. 1. In this Rapid Communication we show that for $2 + 1$ flavors there is only one massless field at the chiral critical point, a σ meson ($J^P = 0^+$, predominantly isosinglet); the universality class is again that of the Ising model. Notice, however, that very different fields go critical at the two critical points $\mathcal{C}$ and $\mathcal{D}$.

We start at zero temperature by fitting the scalar and pseudoscalar mass spectrum in QCD to that found in a linear σ model [4–6]. For three quark flavors we introduce the field Φ ($\sim \bar{q}_{\rm left}\, q_{\rm right}$), as a complex valued, three by three matrix, $\Phi = \sum_{a=0}^{8} (\sigma_a + i\,\pi_a) t^a$; $t_{1,\cdots,8}$ are the generators of the SU(3) algebra in the fundamental representation, and t_0 is proportional to the unit matrix. Normalizing the generators as $\mathrm{tr}(t_a t_b) = \delta^{ab}/2$, $t_0 = \mathbb{1}/\sqrt{6}$.

The fields σ_a are components of a scalar ($J^P = 0^+$) nonet, those of π_a a pseudoscalar ($J^P = 0^-$) nonet. The latter are familiar, as $\pi_{1,2,3}$ are the three pions, denoted as π without subscript, and the $\pi_{4,5,6,7}$ are the four kaons, the K's. The π_8 and π_0 mix to form the mass eigenstates of the η and η' mesons, with mixing angle $\theta_{\eta\eta'}$ [10]. For notational ease we define the components of the scalar nonet analogously: we refer to $\sigma_{1,2,3}$ as the σ_π's, to $\sigma_{4,5,6,7}$ as the σ_K's, while σ_8 and σ_0 mix to form the σ_η and $\sigma_{\eta'}$. This multiplicity of eighteen fields is to be contrasted with the usual σ model with two flavors, which only has three π's and one σ meson.

The effective Lagrangian for the Φ field is taken to be [4–6]

$$\begin{aligned}\mathcal{L} = {} & \mathrm{tr}\,|\partial_\mu \Phi|^2 - \mathrm{tr}[H(\Phi + \Phi^\dagger)] + \mu^2\, \mathrm{tr}\,(\Phi^\dagger \Phi) \\ & -\sqrt{6}\, c[\det(\Phi) + \det(\Phi^\dagger)] \\ & +(g_1 - g_2)\left(\mathrm{tr}\Phi^\dagger\Phi\right)^2 + 3\, g_2\, \mathrm{tr}\,(\Phi^\dagger\Phi)^2 . \end{aligned} \quad (1)$$

The parameters of the linear σ model are the background

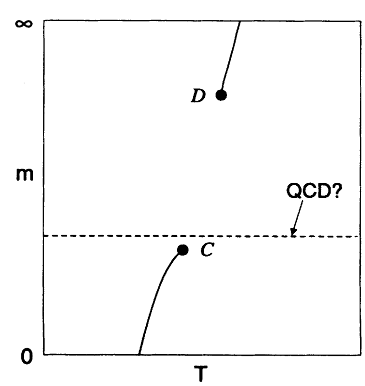

FIG. 1. Proposed phase diagram for 2+1 flavors, following Ref. [2]: $\mathcal{C}$ is the chiral critical point, $\mathcal{D}$ the deconfining critical point.

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field H, a mass parameter μ^2, an "instanton" coupling constant c, and two quartic couplings, g_1 and g_2. Neglecting effects from $m_{\rm up} \neq m_{\rm down}$, for the background field H we take $H = h_0\, t_0 - \sqrt{2}\, h_8\, t_8$. The current quark masses are then related to the background field as $m_{\rm up} = m_{\rm down} \sim h_0 - h_8$ and $m_{\rm strange} \sim h_0 + 2\, h_8$.

Throughout this paper we work exclusively at the simplest level of mean field theory. Up to differences in normalization our intermediate results agree with those of Chan and Haymaker [6]; recent analyses were also carried out by Parwani [7], Meyer-Ortmanns, Pirner, and Patkos [8] and by Metzger, Meyer-Ortmanns, and Pirner [9]. Because we attempt to fit to current experimental results [10], our fit in (2) differs from Chan and Haymaker, but is similar to that of Refs. [7–9]. Details are given elsewhere [11].

We assume that there are nonzero vacuum expectation values for σ_0 and σ_8, $\sigma_0 \to \Sigma_0 + \sigma_0$, $\sigma_8 \to -\sqrt{2}\,\Sigma_8 + \sigma_8$. Expanding the Lagrangian in powers of σ_a and π_a, expansion to linear order fixes the values of Σ_0 and Σ_8, while expansion to quadratic order gives the masses of all the fields: the mass of the pion, m_π, etc. We also need the pion decay constant $f_\pi = \sqrt{2/3}(\Sigma_0 - \Sigma_8)$ and the kaon decay constant $f_K = \sqrt{2/3}(\Sigma_0 + \Sigma_8/2)$.

There is one unexpected feature of the results [6, 7]. For the two equations of motion, the masses of the entire pseudoscalar nonet (for m_π, m_K, m_η, and $m_{\eta'}$), and the masses of half the scalar nonet (for m_{σ_π} and m_{σ_K}), the two quantities μ^2 and g_1 only enter in tandem, through the new parameter $M^2 = \mu^2 + g_1\,(\Sigma_0^2 + 2\,\Sigma_8^2)$. This means that we can fit to the pseudoscalar spectrum, and so fix M^2, and yet still be free to vary g_1: the *only* change is to alter the masses of the σ_η and the $\sigma_{\eta'}$. This technical detail plays an important role in what follows; although there must be some simple group theoretic reason for it, as of yet we do not know what it is.

There is some freedom in deciding how to fit the parameters of the linear σ model. Various kinds of fits are given by Meyer-Ortmanns, Pirner, and Patkos [8] and by Metzger, Meyer-Ortmanns, and Pirner [9]. Following the experience of Chan and Haymaker [6, 7] we do not fit to the entire pseudoscalar mass spectrum for the π, K, η, and η' mesons, since it turns out that the kaon mass is fairly insensitive to the ratio of vacuum expectation values, Σ_8/Σ_0. On the other hand, both the kaon decay constant, f_K, and the mixing angle between the η and the η', $\theta_{\eta\eta'}$, are very sensitive to this ratio. Because of this, we leave the ratio Σ_8/Σ_0 as a free parameter, and fit just to the pion decay constant f_π and to the masses for the π, η, and η' mesons. Taking the values $m_\pi = 137$ MeV, $m_\eta = 547$ MeV, $m_{\eta'} = 958$ MeV, and $f_\pi = 93$ MeV, we choose the parameters

$$\begin{aligned} &\Sigma_0 = 127\ \text{MeV}, \quad \Sigma_8 = 13\ \text{MeV}, \\ &h_0 = (290\ \text{MeV})^3, \quad h_8 = (281\ \text{MeV})^3, \\ &M^2 = +(642\ \text{MeV})^2, \quad c = 1920\ \text{MeV}, g_2 = 30, \end{aligned} \tag{2}$$

for which $\Sigma_8/\Sigma_0 \sim 0.1$. The kaon mass comes out a bit high, $m_K^{\rm fit} = 516\,\text{MeV}$ instead of the (average) experimental value of 497 MeV; the fit gives a kaon decay constant of $f_K^{\rm fit} = 109\,\text{MeV}$, which is close to the experimental value of 113 MeV; lastly, the result for the η-η' mixing angle, $\theta_{\eta\eta'}^{\rm fit} = -10.4°$, is reasonable. (The value obtained from radiative decays [12], $\theta_{\eta\eta'}^{\rm rad} = -20°$, favors even smaller values of Σ_8/Σ_0.) Because $\Sigma_8 \neq 0$, the ratio of the strange to up (= down) quark masses is $m_{\rm strange}/m_{\rm up} = (h_0 + 2h_8)/(h_0 - h_8) = 32$, and not the often quoted value of ~ 20.

The fit of (2) makes unique predictions for two masses in the scalar nonet, $m_{\sigma_\pi} = 1177$ MeV and $m_{\sigma_K} = 1322$ MeV. There are observed states [10] with these quantum numbers, the $a_0(980)$ and the $K_0^*(1430)$, respectively; the values for the σ_π and the σ_K are not too far off, although the splitting between them is too small. We note that the identification of the a_0 with the σ_π is problematic (VII.21 of Ref. [10]): the $a_0(980)$ may not be the σ_π [13], but a $K\overline{K}$ molecule [14].

There is no unique prediction for two other members of the scalar nonet, the σ_η and the $\sigma_{\eta'}$. As remarked, the masses of all other fields only depend upon the parameter M^2. In Fig. 2 we illustrate how m_{σ_η} and $m_{\sigma_{\eta'}}$ change as g_1 is varied at fixed $M^2 = +(642\ \text{MeV})^2$. We identify the $\sigma_{\eta'}$ and the σ_η with the observed states [10] with the same quantum numbers: the $f_0(975)$ and the $f_0(1400)$, respectively. With the parameters of (2), if we require that $m_{\sigma_{\eta'}} = 975$ MeV, Fig. 2 predicts that $m_{\sigma_\eta} = 1476$ MeV instead of 1400 MeV. Also, $g_1 = 40$, $\mu^2 = -(492\ \text{MeV})^2$, and the mixing angle between the σ_η and the $\sigma_{\eta'}$ is $+28°$. As before the identification of the $\sigma_{\eta'}$ with the $f_0(975)$ is open to question (VII.192 of Ref. [10]): the $f_0(975)$ may be not the $\sigma_{\eta'}$ [15], but a $K\overline{K}$ molecule [14]. For the analysis of how far QCD is from the chiral critical point, all that is important is that the $\sigma_{\eta'}$ is not light [16], so at zero temperature the quartic coupling g_1 is large.

The details of the spectrum at zero temperature are not needed to understand how a chiral critical point can arise. In mean field theory the effects of nonzero temperature or baryon density are incorporated simply by varying the mass parameter μ^2. This is valid in the limit of very high temperature, but should be qualitatively correct at all temperatures.

We begin with the SU(3) symmetric case, $h_8 = 0$. For a constant field Σ_0 the Lagrangian reduces to the potential for Σ_0:

$$\mathcal{L} = -h_0\,\Sigma_0 + \frac{1}{2}\mu^2\,\Sigma_0^2 - \frac{c}{3}\,\Sigma_0^3 + \frac{g_1}{4}\,\Sigma_0^4\,. \tag{3}$$

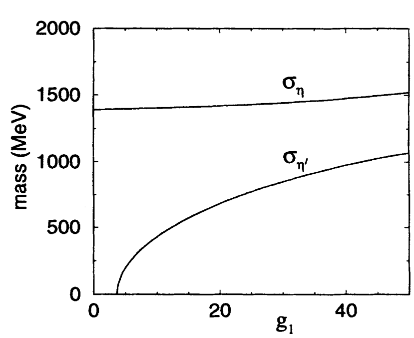

FIG. 2. Plot of the masses of the σ_η and the $\sigma_{\eta'}$ versus the coupling g_1 for the fit of 2.

This model has precisely the same phase diagram as that for the phase transition between a liquid and a gas. For zero background field, $h_0 = 0$, the instanton interaction $\det(\Phi) \sim \Sigma_0^3$ is cubic and so drives the transition first order. As h_0 increases the transition becomes more weakly first order, until at $h_0 = h_0^{\rm crit}$ the line of first order transitions ends in a critical point. For $h_0 > h_0^{\rm crit}$ there is no true phase transition, just a smooth crossover.

The critical point occurs when $h_0^{\rm crit} = c^3/(27 g_1^2)$, $\Sigma_0^{\rm crit} = c/(2 g_1)$, and $\mu^2_{\rm crit} = c^2/(3 g_1)$. At this point the potential in $\Sigma_0 - \Sigma_0^{\rm crit}$ is purely quartic, $\mathcal{L} = g_1(\Sigma_0 - \Sigma_0^{\rm crit})^4/4$, so $m^2_{\sigma_{\eta'}} = 0$. The other fields are all massive: $m_\pi^2 = m_K = m_\eta^2 = c^2/(9g_1)$, $m^2_{\eta'} = 10 m_\pi^2$, $m^2_{\sigma_\pi} = m^2_{\sigma_K} = m^2_{\sigma_\eta} = (7 + 18 g_2/g_1) m_\pi^2$. Since only the $\sigma_{\eta'}$ is massless at the chiral critical point, the similarity to the liquid gas phase transition extends to the universality class, which is that of the Ising model.

This conclusion remains true away from the case of SU(3) symmetry, $h_8 \neq 0$. Numerical analysis [11] shows that there is a single, massless field at the chiral critical point, the $\sigma_{\eta'}$, with the universality class that of the Ising model. Of course for $h_8 \neq 0$ the $\sigma_{\eta'}$ field does not remain a pure SU(3) singlet, but mixes to become part octet.

The possibility of a chiral critical point can even be seen from the calculation of the zero temperature spectrum in Fig. 2. Although we did not remark upon it before, when the coupling $g_1 \sim 3.8$, $m_{\sigma_{\eta'}} = 0$. There it appears as mere curiosity; after all, in Fig. 2 μ^2 has the value appropriate to zero temperature, while the value of μ^2 at nonzero temperature (or baryon density) must be larger. Even so, Fig. 2 does illustrate how a single field, the $\sigma_{\eta'}$, can become massless at a special point in the phase diagram.

We can further explain the nature of the entire phase diagram as a function of $m_{\rm up} = m_{\rm down}$ versus $m_{\rm strange}$, as proposed in Fig. 1 of Ref. [2]. The basic idea is along the entire line of chiral critical points, only the $\sigma_{\eta'}$ is massless, but that the singlet/octet ratio in the $\sigma_{\eta'}$ changes. At the SU(3) symmetric point, $m_{\rm up} = m_{\rm down} = m_{\rm strange}$, the $\sigma_{\eta'}$ is an SU(3) singlet. Decreasing $m_{\rm strange}$ to the critical point where $m_{\rm strange} = 0$ and $m_{\rm up} = m_{\rm down} \neq 0$, the $\sigma_{\eta'}$ becomes entirely strange, $\sigma_{\eta'} \sim \bar{s}s$.

The opposite limit of SU(2) chiral symmetry, $m_{\rm up} = m_{\rm down} = 0$, is more familiar. Assume, as in Ref. [2], that the chiral phase transition with two massless flavors is of second order, with the universality class that of an O(4) critical point. There is then a special value of $m_{\rm strange} = m^{\rm crit}_{\rm strange}$, with a line of O(4) critical points for $m_{\rm strange} > m^{\rm crit}_{\rm strange}$, and a line of first order transitions when $m_{\rm strange} < m^{\rm crit}_{\rm strange}$. Wilczek [17] observed that exactly at $m_{\rm strange} = m^{\rm crit}_{\rm strange}$, the chiral transition is in the universality class of an O(4) *tri*critical point. In our view, at the critical points along $m_{\rm strange} \geq m^{\rm crit}_{\rm strange}$, the $\sigma_{\eta'}$ is a pure SU(2) state, with $\sigma_{\eta'} \sim \bar{u}u + \bar{d}d$, while the pions are massless because $m_{\rm up} = m_{\rm down} = 0$. Whether the universality class is O(4) critical or O(4) tricritical depends upon the relevant quartic couplings.

The analysis can be extended to the case where the chiral transition is of first order for two, massless flavors, but the gap between chiral and deconfining regions remains. Then Fig. 1 of Ref. [2] would have to be modified, with a band of first order transitions about the axis $m_{\rm up} = m_{\rm down} = 0$. These first order transitions would end in chiral critical points, in the Ising universality class from the presence of massless $\sigma_{\eta'}$ fields.

Lastly we ask *How far is QCD from the chiral critical point?* In Fig. 1 we have grossly exaggerated the case, putting QCD very close to the chiral critical point. But the data of Ref. [2] do indicate that as a function of m, QCD is only about a factor of 2 from the chiral critical point.

We first *assume* that the chiral phase transition is of first order for three massless flavors because of the presence of the instanton coupling $\sim \det(\Phi)$. To find the chiral critical point we vary the current quark masses, or equivalently the background fields h_0 and h_8. We require that the ratio of strange to up quark masses equals the value found from the fit at zero temperature, $(h_0 + 2h_8)/(h_0 - h_8) = 32$. By varying h_0, h_8, μ^2, Σ_0, and Σ_8, and otherwise using the values found in (2), we find that the critical point occurs for $h_0^{\rm crit} = (62\ {\rm MeV})^3$, $h_8^{\rm crit} = (60.4\ {\rm MeV})^3$, $\mu^2_{\rm crit} = (183\ {\rm MeV})^2$, $\Sigma_0^{\rm crit} = 14.5$ MeV, and $\Sigma_8^{\rm crit} = 2.7$ MeV. We then compute the ratio between the current quark masses at the chiral critical point to those in QCD:

$$\frac{m_{\rm up}^{\rm crit}}{m_{\rm up}} = \frac{h_0^{\rm crit} - h_8^{\rm crit}}{h_0 - h_8} = 0.01 \,. \tag{4}$$

Now we readily confess that this ratio, computed in mean field theory, is at best crude. Even so, numerical simulations [2] find that the ratio in (4) is not 0.01, but ~ 0.5—mean field theory is off by almost two orders of magnitude [18].

There is an elementary explanation as to why mean field theory predicts that QCD is far from the chiral critical point. In QCD at zero temperature, we have assumed that $m_{\sigma_{\eta'}}$ is large, about 1 GeV. In contrast, at the critical temperature for $m = m_{\rm up}^{\rm crit}$, by definition $m_{\sigma_{\eta'}}$ vanishes. At the face of it, such a large change in mass is rather implausible if QCD is near the chiral critical point. Similarly, the lighter that $m_{\sigma_{\eta'}}$ is at zero temperature, the more natural it is that $m_{\rm up}$ is near $m_{\rm up}^{\rm crit}$. For example, if $m_{\sigma_{\eta'}} = 600$ MeV [16], the ratio in (4) becomes 0.06 instead of 0.01, which is still a long way from ~ 0.5.

Thus our initial assumption must be false. In the chiral limit for three massless flavors, there are two mechanisms for generating a first order transition. The first is the presence of the instanton coupling $\sim \det(\Phi)$ [4, 5]. From (4), in mean field theory this gives the wrong phase diagram, with $m_{\rm up}^{\rm crit}$ much smaller than $m_{\rm up}$. Therefore perhaps the second mechanism is operative, a type of Coleman-Weinberg transition [19].

In mean field theory quartic couplings are fixed and do not change with temperature; a Coleman-Weinberg transition is one in which the quartic couplings run from a stable into an unstable regime, and so thereby generate a finite correlation length dynamically. This phenomenon can be demonstrated rigorously in 4 and $4-\epsilon$ dimensions [19–22]; extrapolation to three dimensions, $\epsilon = 1$, is open to question.

If in the chiral limit the phase transition is of first order because it is a Coleman-Weinberg transition, it is reasonable to suggest that even for nonzero quark mass the quartic couplings change significantly with temperature. Since QCD appears to be in a region of smooth crossover, presumably the quartic couplings of the σ model do not run all the way into the unstable regime. This could explain the apparent discrepancy with mean field theory in (4). At the very least the quartic couplings do tend to run in the right direction: in the infrared limit in less than four dimensions [21, 22], the couplings run most strongly for large g_2, approximately at constant g_2 from large to small values of g_1. This type of running is *precisely* what is required at the chiral critical point: as illustrated in Fig. 2, even at zero temperature we can make $\sigma_{\eta'}$ massless by going from large to small values of g_1, keeping g_2 fixed.

Given the limitations inherent in present lattice simulations for $2+1$ flavors, we conclude with a conjecture: as suggested by our illustration in Fig. 2, perhaps QCD is *very* close to the chiral critical point. There is no good reason why it should be; but if we are lucky, then even if there is no true phase transition in QCD, the $\sigma_{\eta'}$ could still become very light. As we alluded to previously [23], a light $\sigma_{\eta'}$ may generate large domains of disoriented chiral condensates in the collisions of large nuclei at ultrarelativistic energies.

Speculation aside, we have shown how a detailed understanding of the spectrum of the scalar nonet at zero temperature, especially knowing exactly where the $\sigma_{\eta'}$ meson lies, has direct and dramatic consequences for the phase diagram of QCD at nonzero temperature and baryon density. In this way, one area of hadronic physics unexpectedly fertilizes another.

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