Fermion back reaction and the sphaleron

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(Received 9 August 1993; revised manuscript received 18 October 1993)

Using a simple model, a new sphaleron solution which incorporates finite fermionic density effects is obtained. The main result is that the height of the potential barrier (sphaleron energy) decreases as the fermion density increases. This suggests that the rate of sphaleron-induced transitions increases when the fermionic density increases. However the rate increase is not expected to change significantly the predictions from the standard sphaleron-induced baryogenesis scenarios.

Nonconservation of baryon number in the standard model through quantum effects (anomalies) is well known [1]. While instanton-mediated baryon decays are negligible, the same cannot be said for transitions occurring because of monopole catalysis [2], high temperature [3,4], high densities [5,6], or in the presence of a heavy particle [7] (for an excellent review of these four mechanisms, see Ref. [8]).

The basis for all these transitions is the level crossing phenomenon [9] which is usually illustrated by looking at adiabatic changes in the gauge field configuration and at the accompanying variation in the energy levels for the fermions resulting in a change in the fermion number. This description neglects the effect of the fermion back reaction: a change in the fermion density can introduce a change in the gauge field configuration, just as a change of gauge field configuration can change the fermion density. This is most easily seen in the Schwinger model where this back reaction of the fermions is responsible for oscillation in the fermion number [10]. The fermion back reaction is a purely quantum mechanical effect being a direct consequence of the anomaly equation. Since the focus of fermion number violation has been in the study of solutions to the classical equation of motion (e.g., instantons and sphalerons), it is, therefore, not surprising that little attention has been paid to this back reaction. Further, it might be very difficult to properly take into account the fermion back reaction in realistic (3+1)-dimensional theories since the resolution of even seemingly straightforward related issues, such as the gauge invariance of the free energy at finite temperature and fermionic density [11], require a careful treatment of nonperturbative effects to be properly resolved [12].

Fortunately, the situation is simpler in (1+1)-dimensional models where one can, through bosonization [13], take into account the fermion back reaction at the classical level. The present work illustrates some of the nontrivial effects of this fermion back reaction using a simple model which has been extensively studied in the past, namely, the Abelian Higgs model axially coupled to fermions (see for example [6,14–18]).

The Lagrangian density describing this model is

$$\mathcal{L} = \bar{\psi} i \gamma^\mu (\partial_\mu - ie\gamma^5 A_\mu)\psi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - \lambda(|\phi|^2 - c^2)^2 , \tag{1}$$

where $D_\mu = \partial_\mu - ieA_\mu$ and the space coordinate extends from $-L$ to L. This model is to be regulated such that the gauged current $\bar{\psi}\gamma^\mu\gamma^5\psi$ is conserved while the vector current obeys the anomaly equation

$$\partial_\mu \bar{\psi}\gamma^\mu\psi \equiv \partial_\mu J^\mu = -\frac{e}{2\pi}\epsilon^{\mu\nu}F_{\mu\nu} . \tag{2}$$

Instead of working directly with the fermionic Lagrangian, it is preferable to use the Bose-equivalent form [10,13,16]

$$\mathcal{L} = \frac{1}{2}\left[\partial_\mu\chi - \frac{e}{\sqrt{\pi}} A_\mu\right]^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)(D^\mu\phi)^* - \lambda(|\phi|^2 - c^2)^2 , \tag{3}$$

where the mass term for the photon has to be included in order to give the bosonized Lagrangian the correct symmetry. The properly regularized vector current is then

$$J^\mu = \frac{1}{\sqrt{\pi}}\epsilon^{\mu\nu}\left[\partial_\nu\chi - \frac{e}{\sqrt{\pi}}A_\nu\right] \tag{4}$$

which obeys the anomaly equation (2).

The main reason for using the Bose-equivalent formulation is that the anomaly equation of the fermionic theory is present at the classical level, provided the Lagrangian is properly regularized. This ensures that the level-crossing effects connected to the anomaly are included in the classical equations of motion. This is preferable to the usual analyses (e.g., see [17–19]) where fermions are essentially ignored except for the fact that, through the anomaly equation, a change in the Chern-Simons number of a given gauge field configuration is known to be accompanied by a corresponding change in the fermion number.

The static energy density is easily obtained in the $\chi = A_0 = 0$ gauge [16]:

$$\mathcal{E} = \frac{e^2}{2\pi}A_1^2 + \left|\left[\frac{d}{dx} - ieA_1\right]\phi\right|^2 + \lambda(|\phi|^2 - c^2)^2 . \tag{5}$$

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As a result of this gauge choice, the fermion number and the Chern-Simons number are identical.

Suppose one ignores the explicit mass term for the photon, which is equivalent to ignoring the contribution from the fermions. One then finds that $\mathcal{E}$ has an infinite number of local minima such that

$$A_1 = \frac{\pi N_{\text{CS}}}{eL}, \quad \phi = c \exp(ie A_1 x) , \tag{6}$$

where N_{CS}, an integer, is the Chern-Simons number. Following [17,18], one can construct a noncontractible path interpolating between two vacuum states [23] using the parametrization

$$A_1(\tau) = \frac{\pi}{eL}(N + \tau) ,$$

$$\phi(x;\tau) = c \exp(ie A_1 x)[\cos\pi\tau + i \sin\pi\tau\Phi(z)] , \tag{7}$$

where $z = \sqrt{\lambda}\, c^2 \sin(\pi\tau)x$. As explained by Manton [23] and Carson [18], the sphaleron configuration is obtained through a minimax procedure as follows: the set of path $\{\phi(x;\tau), A_1(\tau)\}$ is the finite-energy field configurations that interpolate between two vacuum states as τ runs from 0 to 1. Therefore, as a function of τ, there must exist a point along the path where the energy reaches a maximum E_{max}. By considering the set of all such paths, one can find a function $\Phi(z)$ for which the energy E_{max} is minimal. This configuration is the desired sphaleron which is a solution to the static equations of motion that corresponds to a saddle point of the energy functional for this system. The importance of these sphaleron configurations is that they are the main contributor to baryon-number-violating processes occurring at finite temperature [3].

It is straightforward to show that the sphaleron configuration for the Abelian Higgs model without fermions is given by $\Phi(z) = \tanh(z)$ and $\tau = \frac{1}{2}$ in the limit as $L \to \infty$. The energy along the corresponding path is then

$$E(\tau) = \frac{8\sqrt{\lambda}c^3}{3}|\sin^3\pi\tau| . \tag{8}$$

This results in a periodic effective potential as function of the fermionic density (Chern-Simons number) as shown by the dashed line in Fig. 1.

In order to consider the effects of including the fermions in the system, one may reintroduce the mass term for the photon. Naively, the corresponding effective potential is then as illustrated by the solid line in Fig. 1 [20–22], where the periodic potential obtained before is simply added to a pure fermionic contribution. This results in potential barriers of varying height separating the various distinct local minima up to a maximum value for the fermion density. The computation of transition rates between a state having N fermions to a state having $N+1$ fermions is now more complicated since the transition is between two states having different energies rather than a true vacuum-to-vacuum transition.

Rather than computing the transition rate as a function of the fermionic density, suppose one simply wants to compute the critical density. Since the relevant states
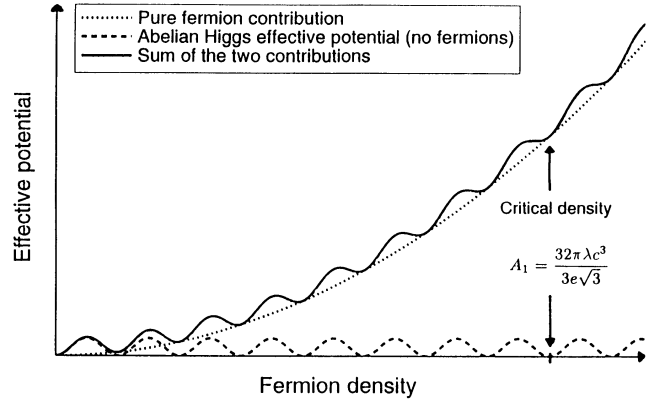

FIG. 1. Schematic representation of the naive effective potential as a function of the fermion density. The dashed line represents the Abelian Higgs contribution which has an infinite number of local minima. The dotted line is the pure fermion contribution. The solid line, obtained by adding the Abelian Higgs sphaleron contribution to the pure fermion contribution is *not* the true effective potential.

have a nonvanishing energy, it is useful, at finite density and in the $L \to \infty$ limit, to subtract an infinite constant from the energy and consider instead

$$E(\tau) = \int_{-\infty}^{\infty} dx [\mathcal{E}(N,\tau) - \mathcal{E}(N,\tau=0)] . \tag{9}$$

Because the new effective potential was obtained by assuming that one could simply add two contributions, it may seem reasonable to assume that the same function $\Phi(z) = \tanh(z)$ found before still corresponds to the sphaleron configuration. Thus, one finds

$$E(\tau) = 2\pi\tau n + \frac{8\sqrt{\lambda}c^3}{c}|\sin^3\pi\tau| , \tag{10}$$

where $n = N/L$ is the finite fermion *density* taken to be finite even in the $L \to \infty$ limit. This result shows that it is possible to find a local extremum of $E(\tau)$ only if A_1 does not exceed a critical value given by $32\pi\lambda c^3/3e\sqrt{3}$.

However, this result indicates that such a naive way of deriving an effective potential at finite density is not correct since, as has been previously found [6,15,16], no local minima of the energy exists when A_1 exceeds $2\sqrt{2}\,\pi\lambda c^3/3e\sqrt{3}$. Furthermore, this derivation does not take into account the fermion back reaction and the function $\Phi(z)$ is *not* a static solution to the complete set of equations of motion of the original Lagrangian [Eq. (3)]. Writing $\phi = \Phi \exp(i\rho)$, the static equations of motion are

$$e A_1 - 2\pi\Phi^2(\partial_x\rho - e A_1) = 0 , \tag{11}$$

$$\partial_x[\Phi^2(\partial_x\rho - e A_1)] = 0 , \tag{12}$$

$$\partial_x^2\Phi - \Phi(\partial_x\rho - e A_1)^2 - 2\lambda\Phi(\Phi^2 - c^2) = 0 . \tag{13}$$

Any solution to these equations has to be such that A_1 is spatially constant and that

$$\rho = e A_1 \int^x \left[1 + \frac{e}{2\pi\Phi^2}\right]dx' . \tag{14}$$

The periodicity requirement on ρ then implies that the

various local minima of the energy functional are such that

$$N_{CS} = \frac{e}{\pi} A_1 L = \frac{2NL}{\int_{-L}^{L}\left[1+\dfrac{e}{2\pi\Phi^2}\right]dx} \qquad (15)$$

is no longer an integer. In other words, the minima no longer coincide with pure gauge configurations. Furthermore, the difference between two adjacent minima of this quantity is also different from unity. This complicates the search for a noncontractible path joining two adjacent minima since it becomes very difficult to find a parametrization consistent with the periodic boundary condition. However, a static solution to the equations of motion resembling the Abelian Higgs sphaleron solution can still be found. If one combines Eqs. (11) and (13) one gets

$$\frac{d^2\Phi}{dx^2} = \frac{e^2 A_1^2}{\pi^2\Phi^3} + 2\lambda\Phi(\Phi^2-c^2) \qquad (16)$$

which, in the $L \to \infty$ limit, has a single nonhomogeneous solution [24] as well as two homogeneous solutions. To see this, it is convenient to parametrize A_1 in terms of a new variable γ such that

$$A_1 = \left[1-\frac{\gamma}{3}\right]\left[\frac{2\gamma\lambda}{3}\right]^{1/2}\frac{\pi c^3}{e}, \quad 0\le\gamma\le 1 . \qquad (17)$$

This parametrization is one to one on the defined interval and such that $A_1(\gamma=0)=0$ while $A_1(\gamma=1)$ is the critical density above which no local minima of the energy exists. With this parametrization, the inhomogeneous solution (sphaleron) is given by

$$\Phi_{sph} = c\left[\frac{2\gamma}{3} + (1-\gamma)\tanh^2[\sqrt{\lambda(1-\gamma)}cx]\right]^{1/2} \qquad (18)$$

and the two homogeneous solutions are

$$\Phi_1 = c\left[1-\frac{\gamma}{3}\right]^{1/2},$$
$$\Phi_2 = \frac{c}{\sqrt{6}}[\gamma+\sqrt{3\gamma(4-\gamma)}]^{1/2} . \qquad (19)$$

The difference in energy between the sphaleron solution and the lowest energy homogeneous solution (Φ_1) is $8c^3\sqrt{\lambda(1-\gamma)}/3$ which correctly reproduces the Abelian Higgs result when the fermion density is zero $(\gamma=0)$, vanishes at the critical density $(\gamma=1)$, and remains finite for all other values of γ. The result for the effective potential is given in Fig. 2. There are three important characteristics of this effective potential; firstly, its overall positive curvature due to the finite fermionic density; secondly, the existence of a critical density above which no stable solution can be found; and lastly, and perhaps most importantly, the decrease of the sphaleron energy (the barrier height) with increasing fermion number. Of course, the important question is the extrapolation of these re-
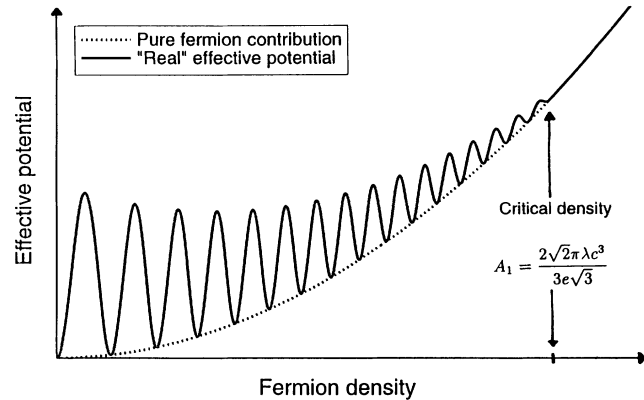

FIG. 2. Schematic representation of the real effective potential as a function of the fermion density. The main thing to note is that the height of the potential barrier decreases with increasing fermion density until it vanishes at the critical density.

sults for $(3+1)$-dimensional theories. Firstly, in order to compute transition rates at finite temperature, the relevant quantity is the free energy which also has a generally positive curvature as a function of the fermion density [4,20–22,25]. Secondly, in $(3+1)$-dimensional models there also exists a critical density [6,8], just as in the $(1+1)$-dimensional model discussed above. Hence, it is probably safe to assume that the height of the barrier between local minima (the energy of the sphaleron) decreases as the fermionic density increases so that it vanishes at and above the critical density. From this it follows that, as the fermionic density increases, sphaleron-induced transitions would occur even more rapidly than the rate suggested by conventional calculations, with the finite density effects becoming the dominant factor when the fermionic density becomes comparable to or greater than the critical density. The critical density computed in the standard model [6,8] is about 12 orders of magnitude greater than the nuclear density which is comparable to the total energy density close to the electroweak phase transition in the standard big bang model [26]. However, the critical density is the *net* fermionic density (i.e., fermion minus antifermion) which is likely to be much smaller than the overall density given the small baryon to photon ratio obtained from primordial nucleosynthesis. Thus it appears that the rate increase of sphaleron transitions due to finite density effects would be negligible. Nonetheless, it might still be desirable to compute this rate increase precisely and to do so it will be necessary to know in what way the standard sphaleron [27] is modified at finite density. It is likely that nonperturbative effects would play a crucial role thereby leading to an impossibly complicated analytical solution. However, a numerical solution should be attainable.

RAPID COMMUNICATIONS

[1] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).

[2] C. G. Callan, Phys. Rev. D **26**, 2058 (1982); V. A. Rubakov, Nucl. Phys. **B203**, 311 (1982).

[3] See, for example, V. Kuzmin, V. Rubakov, and M. Shaposhnikov, Phys. Lett. **155B**, 36 (1985); Phys. Lett. B **191**, 171 (1987); P. Arnold and L. McLerran, Phys. Rev. D **36**, 581 (1987); **37**, 1020 (1988); M. Shaposhnikov, Nucl. Phys. **B287**, 757 (1987); **B299**, 797 (1988); K. Enqvist, D. V. Nanopoulos, and S. Rao, Phys. Rev. D **38**, 1797 (1988).

[4] S. Yu. Khlebnikov and M. E. Shaposhnikov, Nucl. Phys. **B308**, 885 (1988).

[5] V. A. Rubakov and A. N. Tavkhelidze, Teor. Mat. Fiz. **65**, 250 (1985); V. A. Rubakov, A. N. Tavkhelidze, and V. F. Tokarev, *ibid.* **68**, 3 (1986); V. A. Matveev *et al.*, Nucl. Phys. **B282**, 700 (1987).

[6] V. A. Rubakov, Prog. Theor. Phys. **75**, 366 (1986).

[7] V. A. Rubakov, Nucl. Phys. **B256**, 509 (1985).

[8] V. A. Matveev *et al.*, Usp. Fiz. Nauk **156**, 253 (1988) [Sov. Phys. Usp. **31**, 916 (1988)].

[9] N. H. Christ, Phys. Rev. D **21**, 1591 (1980); J. Ambjørn, J. Greensite, and C. Peterson, Nucl. Phys. **B221**, 381 (1983).

[10] D. Boyanovsky, Nucl. Phys. **B294**, 223 (1987).

[11] A. N. Redlich and L. C. Wijewardhana, Phys. Rev. Lett. **54**, 970 (1985); K. Tsokos, Phys. Lett. **157B**, 413 (1985); A Niemi and G. Semenoff, Phys. Rev. Lett. **54**, 2166 (1985).

[12] A. R. Rutherford, Phys. Lett. B **182**, 187 (1987).

[13] S. Coleman, Phys. Rev. D **11**, 2088 (1975); S. Mandelstam, *ibid.* **11**, 3026 (1975); M. B. Halpern, *ibid.* **12**, 1684 (1975); D. Boyanovsky, I. Schmidt, and M. F. L. Golterman, Ann.

Phys. (N.Y.) **185**, 111 (1988).

[14] N. V. Krasnikov, V. A. Rubakov, and V. F. Tokarev, Yad. Fiz. **29**, 1127 (1979) [Sov. J. Nucl. Phys. **29**, 581 (1979)].

[15] V. A. Matveev *et al.*, Theor. Math. Phys. **68**, 635 (1987).

[16] A. Roberge, Phys. Rev. D **41**, 2605 (1990).

[17] A. I. Bochkarev and M. E. Shaposhnikov, Mod. Phys. Lett. A **2**, 991 (1987).

[18] L. Carson, Phys. Rev. D **42**, 2853 (1990).

[19] K. Rajagopal and N. Turok, Nucl. Phys. **B375**, 299 (1992).

[20] M. Dine *et al.*, Nucl. Phys. **B342**, 381 (1990).

[21] E. Mottola, S. Raby, and G. Starkman, in *Baryon Number Violation at the SSC*, edited by M. Mattis and E. Mottola (World Scientific, Singapore, 1990).

[22] L. McLerran, E. Mottola, and M. Shaposhnikov, Phys. Rev. D **43**, 2027 (1991).

[23] N. Manton, Phys. Rev. D **28**, 2019 (1983).

[24] Strictly speaking there are many other such solutions but, apart from possible translations, the solution given is the one with the lowest energy of all inhomogeneous solutions. The method used to derive this solution as well as a thorough analysis of this model on a circle of finite diameter are given in A. Roberge (unpublished).

[25] Note that in the $(1+1)$-dimensional model we introduced, the static energy is really just the same as the zero temperature free energy.

[26] E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Reading, Massachusetts, 1990).

[27] R. F. Klinkhamer and N. S. Manton, Phys. Rev. D **30**, 2212 (1984).