

## Entropy in black hole pair production

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Pair production of Reissner-Nordström black holes in a magnetic field can be described by a Euclidean instanton. It is shown that the instanton amplitude contains an explicit factor of  $e^{A/4}$ , where  $A$  is the area of the event horizon. This is consistent with the hypothesis that  $e^{A/4}$  measures the number of black hole states.

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### I. INTRODUCTION

The elegant laws of black hole thermodynamics [1,2] have yet to find a microscopic explanation in an underlying statistical mechanics of black hole states. Particularly interesting is the interpretation of the Bekenstein-Hawking entropy. In most cases this entropy is given by the simple formula

$$S_{\text{BH}} = A/4, \quad (1.1)$$

where  $A$  is the area of the black hole horizon in Planck units. Up to an additive constant, this formula can be derived by insertion of the semiclassical black hole mass-temperature relation [3] into the thermodynamic formula  $1/T = \partial S / \partial E$  followed by integration. Additional assumptions are required to fix the constant part of the entropy. A widely utilized, but mysterious, procedure is to fix the constant by relating it to the black hole instanton in the Euclidean path integral [4].

The relation (1.1) acquires additional meaning in light of Bekenstein's conjectured generalized second law [1], which states that the sum of the usual entropy plus  $S_{\text{BH}}$  always increases. Although there is no complete proof of this conjecture, evidence is provided by the many ingenious gedanken attempts [2] to violate the generalized second law which have been foiled by the subtle dynamics of quantum-mechanical black holes.

If the traditional connection between thermodynamics and statistical mechanics were to extend to black holes, then the number of quantum states of the black hole would be finite and given by

$$N = e^{S_{\text{BH}}}. \quad (1.2)$$

These microstates might be either "internal states" inside

the black hole or "horizon states" somehow associated with degrees of freedom of (or near) the black hole horizon, or both.

The issue of whether (1.2) can be taken literally has bearing on the vexing question of what happens to information cast into a black hole.<sup>1</sup> If one assumes that (1.2) counts *all* the black hole states, and that information is preserved, then one is forced to conclude that information escapes from a black hole at a rapid rate (proportional to the rate of area decrease) during the Hawking process. We do not think this is likely because it seems to require a breakdown of semiclassical methods for arbitrarily large black holes and at arbitrarily weak curvatures, although this point is certainly the subject of heated debates. On the other hand, one might try to account for the decrease in (1.2) during black hole evaporation by assuming that information is truly lost in the black hole interior, perhaps being absorbed by the singularity. A problem with this is that, for large neutral black holes, the spacelike slice on which the quantum Hilbert space is defined can be extended through the interior of the black hole in a manner which avoids the singularity and all strong-curvature regions. Dynamics on such a slice is weakly coupled, and it is therefore hard to see how information could be lost.<sup>2</sup>

An alternate interpretation of (1.2) is that it counts only the horizon states. One is then not pushed into the conclusion that information either rapidly escapes or is absorbed at weak coupling. Indeed if one assumes that there are  $e^{1/4}$  states per Planck area of the horizon, one

<sup>1</sup>For recent reviews see Refs. [5–8].

<sup>2</sup>Although perhaps (1.2) makes sense only with respect to a specific slicing of spacetime, which differs from the one described here.

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precisely recovers (1.2). Certainly a derivation of this strange factor would be of great interest. Of course even if such a derivation were found, it would still remain to understand why, if (1.2) counts only horizon states, the generalized second law appears to be valid.

Yet a third microphysical explanation of (1.1) is suggested by recent work [9], in which the entropy of the free scalar field vacuum outside a ball of surface area  $A$  was computed by tracing over states inside the ball. The result was found to leading order in  $A$  to be

$$S_{\text{BH}} = A \Lambda^2, \quad (1.3)$$

where  $\Lambda$  is an ultraviolet cutoff. Equation (1.3) has a microphysical explanation by construction, but it is not in terms of states at or inside the surface of the ball. Rather the entropy arises from correlations between the quantum state inside and outside the ball. It is tempting to try to relate this observation to (1.1), but this would require explaining why  $\Lambda^2$  is precisely  $\frac{1}{4}$  in Planck units. Furthermore, such an interpretation (1.1) would not appear to readily explain the validity of the generalized second law. Certainly no such law is valid in the free field example of Ref. [9].

For these reasons it is clearly of interest to seek a deeper understanding of the meaning of the black hole entropy. One promising avenue of exploration is the phenomenon of pair production of charged black holes. In Schwinger production of charged particles in a background field, the total production rate grows as the number of particle species produced. If this is extrapolated to black hole production in a background field [10,11] then one would likewise expect the rate to be proportional to the number of independent black hole states produced.<sup>3</sup> In this paper we show that the factor (1.2) indeed multiplies the pair production amplitude, consistent with its interpretation as somehow counting black hole microstates. While the nature of these supposed states is still very mysterious, we do hope that our result will constrain future interpretations.

The desired factor (1.2) is isolated from the rest of the pair-production amplitude by consideration of the family of stable solutions discussed in Ref. [13] corresponding to gravitationally corrected 't Hooft–Polyakov monopoles of charge  $q$ . For  $qM_{\text{GUT}} \ll M_{\text{Planck}}$ , these closely resemble the 't Hooft–Polyakov solutions. For  $qM_{\text{GUT}} > M_{\text{Planck}}$ , the monopole drops inside an event horizon and the solutions are identical to extremal Reissner–Nordström monopole black holes. Pair production of these monopoles can be analyzed using instanton methods. For fixed magnetic field  $B$ , consider a one-parameter family of instantons labeled by  $M_{\text{GUT}}$ . For  $qM_{\text{GUT}} \ll M_{\text{Planck}}$ , the instanton resembles the one described by Affleck, Alvarez, and Manton [14,15] as a 't Hooft–Polyakov monopole in a circular orbit in Euclidean space. For  $qM_{\text{GUT}} < M_{\text{Planck}}$ , the instanton is precisely the one found in Ref. [11] describing Reissner–

Nordström monopole pair production. At the critical value of  $M_{\text{GUT}}$  near  $M_{\text{Planck}}/q$ , where the monopole drops inside a horizon, one finds that the action discontinuously changes by precisely  $-S_{\text{BH}}$ .

Of course, even our well-funded gedanken experimentalist cannot observe this threshold because coupling constants such as  $M_{\text{GUT}}$  cannot be varied in the laboratory. Fortunately, it will be seen from a precise description of the production process that the same threshold can be observed by varying the magnetic field  $B$  while keeping  $M_{\text{GUT}}$  fixed. Our gedanken experimentalist who discovers that the production rate suddenly jumps up at precisely the critical  $B$  field which produces monopoles with horizons, will likely conclude that he has crossed a threshold for production of  $e^{S_{\text{BH}}}$  new states. This assigns a new, physical significance to the relation (1.2).

In Sec. II we briefly review Ref. [11] and present an exact formula for the pair production rate. It is, however, somewhat difficult to extract from this the contribution of the entropy because structure-dependent Coulomb terms give contributions of similar magnitude. To circumvent this difficulty, Sec. III compares this amplitude to the pair-production of a GUT monopole (with parameters tuned so that its surface is barely outside the would-be horizon) and thereby extracts the entropy factor. Finally, in Sec. IV we perform the same comparison in the two-dimensional reduced theory that arises in the weak-field limit. Although this yields exactly the same result, it provides a simplified description of the process. Section V closes with discussion. The Appendix contains a derivation of the exact action of the black-hole pair-production instanton, which is valid even for black holes of size (or charge) comparable to  $1/B$ . This extends the leading-order-in- $B$  expression given in Ref. [11].

## II. REISSNER-NORDSTRÖM PAIR PRODUCTION

The amplitude for production of magnetically charged black holes in a magnetic field can be calculated in the semiclassical approximation by finding an analogue of the Schwinger instanton in gravity coupled to electromagnetism, with the Euclidean action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{g} (-R + F_{\mu\nu} F^{\mu\nu}) - \frac{1}{8\pi} \int d^3x \sqrt{h} K, \quad (2.1)$$

where we have included the surface term written in terms of the extrinsic curvature  $K$  and boundary metric  $h$ . First, consider the solution corresponding to the background field. Because of the magnetic energy this solution is not flat, but rather for a magnetic field in the  $z$  direction is given by the Euclidean Melvin universe [16]:

$$ds^2 = (1 + \frac{1}{4} B^2 \rho^2)^2 (dt^2 + dz^2 + d\rho^2) + \frac{\rho^2}{(1 + \frac{1}{4} B^2 \rho^2)^2} d\phi^2, \quad (2.2)$$

$$F = \frac{B \rho d\rho \wedge d\phi}{(1 + \frac{1}{4} B^2 \rho^2)^2},$$

where  $-\infty < t, z < \infty$ ,  $0 < \rho < \infty$ , and  $0 < \phi < 2\pi$ . This solution corresponds to a flux tube with total flux

<sup>3</sup>See also Ref. [12].

$$\Phi = \int F = \frac{4\pi}{B} \tag{2.3}$$

through a transverse hypersurface.

The instanton describes circular motion of an extremal Reissner-Nordström black hole in the Euclidean continuation of the Melvin universe. It is given by the Ernst solution [17]

$$ds^2 = \frac{\Lambda^2}{A^2(x-y)^2} [-G(y)dt^2 - G^{-1}(y)dy^2 + G^{-1}(x)dx^2] + \frac{G(x)}{\Lambda^2 A^2(x-y)^2} dz^2, \tag{2.4}$$

$$F = dz \wedge dE.$$

Here

$$G(x) = 1 - x^2(1 + \hat{q}Ax)^2, \tag{2.5}$$

$$E = \frac{2}{\Lambda \hat{B}} \left[ 1 + \frac{1}{2} \hat{q} \hat{B} x \right], \tag{2.6}$$

$$\Lambda = \left[ 1 + \frac{1}{2} \hat{q} \hat{B} x \right]^2 + \frac{\hat{B}^2 G(x)}{4A^2(x-y)^2}, \tag{2.7}$$

and  $A$ ,  $\hat{q}$ , and  $\hat{B}$  are parameters obeying

$$\hat{q} \hat{B} < \frac{1}{4}. \tag{2.8}$$

The function of  $G$  then has four zeros,  $\zeta_1, \dots, \zeta_4$ ;  $y$  is taken to run between  $\zeta_2$  and  $\zeta_3$  and  $x$  runs between  $\zeta_3$  and  $\zeta_4$ . As these zeros are approached the metric becomes singular unless periodic identifications are made on  $t$  and  $z$ . This also forces a relation between  $A$ ,  $\hat{q}$ , and  $\hat{B}$ :

$$1 = \left[ \frac{1 + 4\hat{q}A}{1 - 4\hat{q}A} \right] \left[ \frac{1 + \frac{1}{2} \hat{q} \hat{B} \zeta_3}{1 + \frac{1}{2} \hat{q} \hat{B} \zeta_4} \right]^8. \tag{2.9}$$

This solution has topology  $S^2 \times S^2 - \{p\}$  where  $p$  corresponds to  $x = \zeta_3 = y$ . The Melvin metric is recovered asymptotically as  $x, y \rightarrow \zeta_3$ . A schematic picture of the solution is shown in Fig 1.

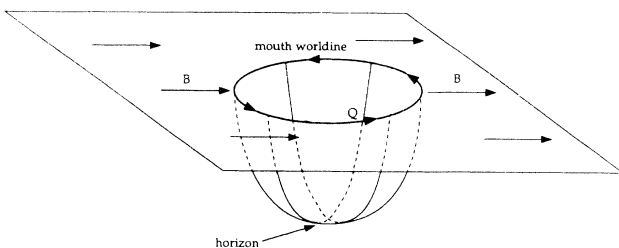


FIG. 1. Shown is a schematic representation of the Ernst solution. The topology of the solution is that of  $R^4$  asymptotically, but internally has a circulating wormhole mouth. Any given point in the “cup” region corresponds to a two-sphere.

By comparing the asymptotic form of the metric to the Melvin metric, the value of the asymptotic magnetic field can be identified:

$$B = \hat{B} (1 + \frac{1}{2} \hat{q} \hat{B} \zeta_3)^{-3} \sqrt{1 - 4\hat{q}A}. \tag{2.10}$$

The value of the charge is found by computing the integral of  $F$  over the two-sphere parametrized by  $x$  and  $z$  that encircles the horizon of the black hole; this gives

$$q = \frac{1}{4\pi} \int_{S^2} F = \frac{1}{B} \left[ 1 - \frac{1 + \frac{1}{2} \hat{q} \hat{B} \zeta_3}{1 + \frac{1}{2} \hat{q} \hat{B} \zeta_4} \right]. \tag{2.11}$$

For production of point particles of charge  $q$  and mass  $m$  the radius of the circle on which the particles travel is given by

$$l \sim m / qB. \tag{2.12}$$

In the case of extremal black holes this radius should be bigger than that of the black hole horizon, which is enforced by (2.8). The analogue of the point particle limit is given by  $qB \ll 1$ . We also want  $q > 1$  so that the black holes are larger than the Planck radius.

To interpret the instanton’s contribution to the production process it must be cut in half along the moment of time symmetry surface given by  $t = \text{const}$ . The three-geometry of this surface is that of a wormhole, with a trapped magnetic flux. The opposite ends of the wormhole correspond to the pair of black holes. Subsequent evolution arises from continuation to Lorentzian signature; the pair of mouths run away to opposite ends of the magnetic fields.

To evaluate the semiclassical production rate we need the instanton action. The action is computed by calculating its change

$$\frac{\delta S}{\delta q} \tag{2.13}$$

under an infinitesimal variation of the charge of the black hole. This can then be integrated from zero to  $q$  to give the total action, as described in the Appendix. The result is

$$S = 4\pi q^2 \frac{(1 - Bq)^2}{1 - (1 - Bq)^4}. \tag{2.14}$$

Expanding in the small parameter  $qB$  we find

$$S = \frac{\pi q}{B} - \frac{\pi}{2} q^2 + O(q^3 B). \tag{2.15}$$

The semiclassical production rate is thus

$$e^{-S} \sim \exp \left[ -\frac{\pi q}{B} + \frac{\pi}{2} q^2 + O(q^3 B) \right] \tag{2.16}$$

up to factors arising from loops. The leading term corresponds precisely to the Schwinger rate,  $\exp(-\pi m^2 / qE)$  for pair production in an electric field.

The subleading term in (2.16) is of the correct order of magnitude to correspond to the black hole entropy,

<sup>4</sup>For more details, see the Appendix.

$S_{\text{BH}} = A/4 = \pi q^2$ . However, things are not so simple as various loop corrections enter at the same order. In particular, there are Coulomb corrections that arise from the exchange of a photon line from one point on the trajectory to another. In the case of pair production of magnetic monopoles, considered in Refs. [14,15], these give a contribution  $\exp(\pi q^2)$ . In the present case one likewise expects a gravitational Coulomb correction of the same size. However, in contrast to the monopole, in the present case there is not a well-defined four-dimensional effective field theory in which excitations around the black hole can be ignored.<sup>5</sup> Therefore, there can be structure-dependent corrections to these results. This makes it difficult to compare the expected production rate and compare it to the rate (2.16) to extract factors corresponding to the number of states.

A second problem is that the approach of this section requires integrating the action up from  $q=0$ . One might worry that there could be a surface term, perhaps even infinite, arising from the end point of the integration. In the following section we will perform a different calculation that addresses both of these problems.

### III. COMPARISON TO MONOPOLE PRODUCTION

In view of the various contributions to the semiclassical rate one needs a better standard of comparison to extract the state-counting factor. Such a standard can be had by reconsidering the pair production of magnetic monopoles in a grand unified theory with gravity included. The mass and size of such a monopole are of order  $q^2 M_{\text{GUT}}$  and  $1/M_{\text{GUT}}$ , respectively, where  $M_{\text{GUT}}$  is the grand unified theory (GUT) mass scale. If we consider tuning the parameters so the  $M_{\text{GUT}}$  approaches  $M_{\text{Planck}}/q$ , then the surface of the monopole barely hovers outside the would-be horizon. Thus the solution is essentially Reissner-Nordström until very close to the horizon. Pair production of these objects is described by an instanton such as that of the preceding section, except the geometry near the bottom of the “cup” of Fig. 1 is cut off and replaced by the monopole circulating around the loop, as shown in Fig. 2. All structure-dependent corrections are therefore the same in two solutions and we can compute the structure-independent difference between black hole production and monopole production rates.

As one increases the magnetic field, the radius of the Euclidean orbit decreases. Both the radius of the cup, and of the hole at the bottom of the cup in Fig. 2 will decrease. At a critical value of the  $B$  field, the hole closes

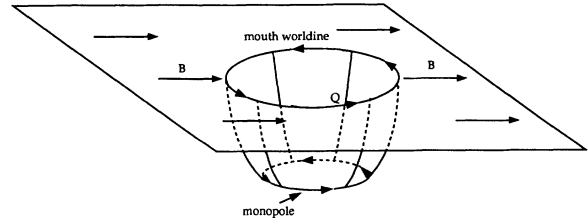


FIG. 2. Shown in the instanton of Fig. 1, but with the lower portion of the cup truncated. The resulting boundary corresponds to the  $S^2$  surface of the monopole, moving on a circular trajectory. This instanton thus describes pair production of gravitationally corrected 't Hooft–Polyakov monopoles.

up, because sufficient acceleration of a monopole with  $M_{\text{GUT}}$  just below  $M_{\text{Planck}}$  results in a horizon.<sup>6</sup> Thus the difference of the actions above and below the critical value can be interpreted as the threshold factor measured by a gedanken experimentalist who varies  $B$ .

To proceed we must calculate the difference between the classical actions for the respective processes. The difference solely from the difference between the actions of the monopole core and of the section of the bottom of the cup that it replaces. The latter tends to zero as the core is tuned to the horizon, so all we need is the monopole action.

The latter is given by

$$S_m = \int d^4x \sqrt{g} \left[ -\frac{R}{16\pi} + \mathcal{L}_m \right], \quad (3.1)$$

where the integral is over the monopole core and  $\mathcal{L}_m$  is the Lagrangian for gauge fields minimally coupled to Higgs fields that break the GUT group to  $U(1)$  (we do not need the explicit form of this action). The full solution has Killing vector  $k^\mu$  that generates  $t$  translations, i.e., rotations about the cup's axis. Using  $k^\mu$  we will rewrite (3.1) as a surface term.

Double contraction of Einstein's equations with this vector gives

$$\frac{1}{8\pi} k^\mu k^\nu R_{\mu\nu} = k^\mu k^\nu T_{\mu\nu} + \frac{1}{16\pi} k^2 R, \quad (3.2)$$

where  $k^2 = k^\mu k_\mu$ . The term involving the stress tensor is

$$k^\mu k^\nu T_{\mu\nu} = 2 \left[ \frac{\partial \mathcal{L}_m}{\partial g^{\mu\nu}} k^\mu k^\nu - \frac{1}{2} k^2 \mathcal{L}_m \right]. \quad (3.3)$$

In this expression the first term vanishes. To see this, note that for the static monopole configuration all time derivatives vanish, and furthermore a gauge may be chosen so that only spatial components of the gauge field are nonvanishing. This means that the Lagrangian cannot depend on  $g''$  as there are no indices for this to con-

<sup>5</sup>There is presumably an effective field theory which contains two-dimensional regions corresponding to the long throat outside the black hole. Such effective field theories were derived for dilatonic black holes in Refs. [18,19] and used to analyze pair production in Refs. [20,21]. It will be partially described for the Reissner-Nordström case in Sec. V.

<sup>6</sup>Note that this suggests an argument for the inclusion of topology change in quantum gravity, so that one matches to a pair-production instanton above the critical value of  $B$ .

tract with. Combining (3.2) and (3.3) therefore gives the Lagrangian

$$-\frac{R}{16\pi} + \mathcal{L}_m = -\frac{1}{8\pi k^2} k^\mu k^\nu R_{\mu\nu}. \quad (3.4)$$

The latter expression can be written as a total derivatives: the identity

$$\nabla_\nu \nabla^\nu k_\mu = -k^\nu R_{\mu\nu} \quad (3.5)$$

for Killing vectors gives

$$-k^\mu k^\nu R_{\mu\nu} = k^\mu \nabla_\nu \nabla^\nu k_\mu. \quad (3.6)$$

Then the hypersurface orthogonality of  $k^\mu$  implies

$$\nabla_\mu k_\nu = -k_{[\mu} \nabla_{\nu]} \ln k^2, \quad (3.7)$$

which can be used to show

$$\frac{1}{k^2} k^\mu \nabla_\nu \nabla^\nu k_\mu = \frac{1}{2} \square \ln k^2. \quad (3.8)$$

The action (3.1) now becomes

$$S_m = \frac{1}{16\pi} \int d\Sigma n^\mu \nabla_\mu \ln k^2, \quad (3.9)$$

where the integral is over a surface just outside the monopole core. This integral is readily evaluated using the approximate form

$$ds^2 = r^2 d\tau^2 + dr^2 + h_{ij} dx^i dx^j \quad (3.10)$$

of the metric near the horizon; here  $x^i$  parametrize the horizon two-sphere. In these coordinates  $k^\mu = (1, 0, 0, 0)$  and we find

$$S_m = \frac{1}{4} \int d^2x \sqrt{h} = \frac{1}{4} A, \quad (3.11)$$

where  $A$  is again the horizon area.

Equation (3.11) is precisely the usual black hole entropy. Monopole production is therefore suppressed relative to black hole production, and the ratio of the black hole production rate to that for monopoles is given by

$$\frac{\Gamma_{\text{BH}}}{\Gamma_m} = e^{S_{\text{BH}}}. \quad (3.12)$$

This is exactly as one would expect if in creating black hole pairs one is allowed to create an extra number of states given by (1.2).

The surface term (3.11) could equally well be understood by eliminating the monopole core and including the Gibbons-Hawking [4] surface term in the action.<sup>7</sup> We have given the above derivation in order to make the origin of this term more apparent. In the next section we will reproduce the same result from the latter approach, but will instead work in the two-dimensional theory describing the geometry of the cup portion of the instanton at small  $B$ .

#### IV. PRODUCTION IN THE TWO-DIMENSIONAL EFFECTIVE THEORY

For weak magnetic field the above instanton and the contribution of the black hole entropy can be approximately described in a two-dimensional reduced theory. To see this, recall that the spatial geometry of the extremal Reissner-Nordström solution near the horizon is that of an infinitely long throat. The latter statement follows from making the redefinition

$$r - q = qe^w \quad (4.1)$$

on the Euclidean Reissner-Nordström metric

$$ds^2 = \left[ 1 - \frac{q}{r} \right]^2 dt^2 + \frac{1}{(1 - q/r)^2} dr^2 + r^2 d\Omega_2^2. \quad (4.2)$$

The limit  $w \rightarrow -\infty$  corresponds to the vicinity of the horizon, and in this limit the metric becomes

$$ds^2 = e^{2w} dt^2 + q^2 dw^2 + q^2 d\Omega_2^2. \quad (4.3)$$

We will work in the approximation where all but the  $s$ -wave excitations are dropped, and thus the angular directions will be ignored. The corresponding two-dimensional Euclidean action is found by dimensional reduction:

$$S_2 = -\frac{1}{4} \int d^2\sigma \sqrt{g} [e^{-2\phi} R + 2e^{-2\phi} (\nabla\phi)^2 + 2 - 2q^2 e^{2\phi}], \quad (4.4)$$

where

$$ds^2 = g_{\alpha\beta} d\sigma^\alpha d\sigma^\beta + e^{-2\phi} d\Omega_2^2. \quad (4.5)$$

A two-dimensional limit also occurs in the instanton (2.4). In the limit  $B \rightarrow 0$  the radius of the black hole trajectory grows as  $1/B$ , and the size of the cup in Fig. 1 goes to infinity while the radius of the horizon is fixed. Therefore, over the cup region the low-momentum theory is effectively two dimensional. The vicinity of the horizon is given by values of  $y$  close to  $\xi_2$ , and in this vicinity the metric (2.4) takes the form

$$ds^2 = q^2 \sinh^2 w dt^2 + q^2 dw^2 + q^2 d\Omega_2^2, \quad (4.6)$$

where we have defined  $y - \xi_2 = \cosh w - 1$ . This is readily seen to give a solution to (4.4).

The effects of monopoles are incorporated in the two-dimensional theory by adding a boundary corresponding to the monopole world line. The monopole dynamics are encoded both in the boundary conditions and by the addition of operators integrated along the boundary. All operators consistent with the symmetries are expected to be present.

Monopole pair production as in Fig. 2 corresponds to terminating the surface at a circle just outside the horizon. By tuning the parameters this boundary can be taken to be at any value of  $w$  near the horizon at  $w=0$ . This can, for example, be explicitly seen from the boundary term

$$C \oint dl \quad (4.7)$$

corresponding to the monopole energy. Allowing varia-

<sup>7</sup>A related observation has been made by F. Wilczek (private communication).

tions of the fields at the boundary will yield an equation fixing the boundary radius in terms of  $C$ . In general,  $C$  and the coefficients of the boundary operators are hard to compute. Fortunately, there is only one term which has a nonvanishing contribution as the boundary circle shrinks to a point: namely [4,22],

$$-\frac{1}{2} \oint dl e^{-2\phi} k, \quad (4.8)$$

where  $k$  is the extrinsic curvature of the surface. Its coefficient must be precisely as in (4.8), otherwise the variational principle is not well defined due to surface terms involving the variation of the derivative of the metric at the boundary.

In comparing the action for monopole production to that for black hole production, the surface term (4.8) contributes only in the former case. Evaluating the surface term in that case gives  $\pi q^2$ , in agreement with (3.11).

## V. DISCUSSION

The above results show that pair production of Reissner-Nordström black holes is enhanced by an extra factor of  $e^{S_{\text{BH}}}$  over that of pair production of magnetic monopoles. This is consistent with the black hole entropy serving as a measure of the number of internal states. However, for several reasons it is far from clear why the black hole entropy is playing this role. Indeed, our calculations only include the classical action, and not the functional determinant from the functional integral. The latter is expected to count states corresponding to fluctuations about the semiclassical geometry (for further discussion see Ref. [21]). We do not understand why contributions from the *classical* action should provide a factor appearing to count states, or how these states might be described. These matters deserve further exploration.

## ACKNOWLEDGMENTS

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## APPENDIX: CALCULATION OF EXACT EUCLIDEAN ACTION

In this appendix we will derive the result (2.14) for the exact action of the instanton. As outlined in the text, this is computed by first finding its variation under a small change in the charge of the black hole, then integrating. As stated in Ref. [11], the variation of the gravitational part of the action vanishes, and we have

$$\begin{aligned} \delta S &= \frac{1}{8\pi} \int d^4x \sqrt{g} F^{\mu\nu} \delta F_{\mu\nu} \\ &= \frac{1}{4\pi} \int d^4x \sqrt{g} F^{\mu\nu} \nabla_\mu \delta A_\nu \\ &= \frac{1}{4\pi} \int d^3x \sqrt{h} n_\mu F^{\mu\nu} \delta A_\nu, \end{aligned} \quad (A1)$$

where the latter is a boundary integral with induced metric  $h$ . This boundary integral receives contributions only from the failure of  $\delta A$  to match on the equator of the horizon two-sphere and gives

$$\delta S = I \delta q, \quad (A2)$$

where

$$I = \int_{S^2} *F \quad (A3)$$

and  $S^2$  is here the “orbital” two-sphere.

First we calculate  $I$ . From (2.4) we find

$$\int_{S^2} *F = - \int_{t_-}^{t_+} dt \int_{\xi_2}^{\xi_3} dy \Lambda^2 \frac{\partial E}{\partial x}. \quad (A4)$$

Note that this integral is evaluated on the equator of the horizon two-sphere, that is, at  $x=0$ . Note also that  $G(0)=1$  and  $G'(0)=0$ . It then follows from the formulas (2.6) and (2.7) for  $E$  and  $\Lambda$  that at  $x=0$  we have

$$\Lambda^2 \frac{\partial E}{\partial x} = -\hat{q} + \frac{\hat{q} \hat{B}^2}{4A^2 y^2} - \frac{\hat{B}}{A^2 y^3}. \quad (A5)$$

It then follows that

$$\begin{aligned} I &= \Delta t \int_{\xi_2}^{\xi_3} dy \left[ \hat{q} - \frac{\hat{q} \hat{B}^2}{4A^2 y^3} + \frac{\hat{B}}{A^2 y^3} \right] \\ &= \Delta t \Delta y \left[ \hat{q} - \frac{\hat{q} \hat{B}^2}{4A^2 \xi_3 \xi_2} + \frac{\hat{B}(\xi_3 + \xi_2)}{2A^2 (\xi_3 \xi_2)^2} \right]. \end{aligned} \quad (A6)$$

The zeros  $\xi_2$  and  $\xi_3$  are explicitly found to be

$$\xi_{2,3} = \frac{1}{2\hat{q}A} (-1 \mp \sqrt{1-4\hat{q}A}). \quad (A7)$$

It then follows that

$$I = \Delta t \sqrt{1-4\hat{q}A} \left[ \frac{1}{A} - \frac{\hat{B}}{2A^2} - \frac{\hat{q} \hat{B}^2}{4A^2} \right]. \quad (A8)$$

The quantity  $\Delta t$  must have the value that makes the metric well behaved at the poles of the orbital two-sphere. This gives

$$\Delta t = \frac{2\pi}{\sqrt{1-4\hat{q}A}}. \quad (A9)$$

So we find

$$I = 2\pi \left[ \frac{1}{A} - \frac{\hat{B}}{2A^2} - \frac{\hat{q} \hat{B}^2}{4A^2} \right]. \quad (A10)$$

Unfortunately, our expression for  $I$  is in terms of the “bare” parameters  $\hat{q}$ ,  $A$ , and  $\hat{B}$ . We need an expression for  $I$  in terms of the physical parameters  $q$  (the magnetic charge) and  $B$  (the magnetic field of the Melvin metric that the Ernst metric is asymptotic to). We start by evaluating  $B$ . Define the scalar  $J$  to be the value of  $F^2$  on the axis (that is, at  $x=\xi_3$ ). In the Melvin universe  $J=2\hat{B}^2$  so in the Ernst metric  $J$  far from the black holes will approach  $2B^2$ .

Calculating  $J$  we find

$$\begin{aligned} J &= \lim_{x \rightarrow \xi_3} [2g^{zz}g^{xx}(F_{zx})^2 + 2g^{zz}g^{yy}(F_{zy})^2] \\ &= 2A^4(\xi_3 - y)^4 \left[ \lim_{x \rightarrow \xi_3} \frac{\partial E}{\partial x} \right]^2. \end{aligned} \quad (\text{A11})$$

It then follows using the formula for  $E$  (2.6) (and some straightforward but tedious algebra) that

$$\begin{aligned} J &= 2[\hat{q}(1 + \frac{1}{2}\hat{q}\hat{B}\xi_3)^{-2}A^2(\xi_3 - y)^2 \\ &\quad + \hat{B}(1 + \frac{1}{2}\hat{q}\hat{B}\xi_3)^{-3}\sqrt{1 - 4\hat{q}A}]^2. \end{aligned} \quad (\text{A12})$$

We then find

$$\begin{aligned} B &= \lim_{y \rightarrow \xi_3} \sqrt{J/2} \\ &= \hat{B}(1 + \frac{1}{2}\hat{q}\hat{B}\xi_3)^{-3}\sqrt{1 - 4\hat{q}A}. \end{aligned} \quad (\text{A13})$$

Next we need to find  $q$  in terms of the ‘‘bare’’ parameters  $A$ ,  $\hat{B}$ , and  $\hat{q}$ . We have

$$q = \frac{1}{4\pi} \int_{S^2} F, \quad (\text{A14})$$

where the integral is over the horizon two-sphere. We then find

$$\begin{aligned} q &= \frac{1}{4\pi} \int_{\xi_3}^{\xi_4} dx \int_{z_-}^{z_+} dz F_{xz} \\ &= \frac{1}{4\pi} \int_{z_-}^{z_+} dz \int_{\xi_3}^{\xi_4} dx \left[ -\frac{\partial E}{\partial x} \right] \\ &= \frac{\Delta z}{4\pi} [E(\xi_3) - E(\xi_4)] \\ &= \frac{\Delta z}{4\pi} \frac{2}{\hat{B}} \left[ \left( 1 + \frac{1}{2}\hat{q}\hat{B}\xi_3 \right)^{-1} - \left( 1 + \frac{1}{2}\hat{q}\hat{B}\xi_4 \right)^{-1} \right]. \end{aligned} \quad (\text{A15})$$

However, one can show that smoothness of the horizon two-sphere metric requires

$$\Delta z = \frac{2\pi}{\sqrt{1 - 4\hat{q}A}} \left( 1 + \frac{1}{2}\hat{q}\hat{B}\xi_3 \right)^4. \quad (\text{A16})$$

So we find

$$\begin{aligned} q &= \frac{1}{\hat{B}\sqrt{1 - 4\hat{q}A}} \left( 1 + \frac{1}{2}\hat{q}\hat{B}\xi_3 \right)^3 \left[ 1 - \frac{(1 + \frac{1}{2}\hat{q}\hat{B}\xi_3)}{(1 + \frac{1}{2}\hat{q}\hat{B}\xi_4)} \right] \\ &= \frac{1}{B} \left[ 1 - \frac{1 + \frac{1}{2}\hat{q}\hat{B}\xi_3}{1 + \frac{1}{2}\hat{q}\hat{B}\xi_4} \right]. \end{aligned} \quad (\text{A17})$$

We now find the constraint on the three bare parameters. Smoothness of the horizon two-sphere metric requires

$$\left| \frac{G'(\xi_4)}{\Lambda^2(\xi_4)} \right| = \left| \frac{G'(\xi_3)}{\Lambda^2(\xi_3)} \right|. \quad (\text{A18})$$

However, we have

$$\xi_{3,4} = \frac{1}{2\hat{q}A} (-1 + \sqrt{1 \mp 4\hat{q}A}). \quad (\text{A19})$$

The constraint then becomes

$$1 = \left| \frac{1 + 4\hat{q}A}{1 - 4\hat{q}A} \right| \left| \frac{1 + \frac{1}{2}\hat{q}\hat{B}\xi_3}{1 + \frac{1}{2}\hat{q}\hat{B}\xi_4} \right|^8. \quad (\text{A20})$$

We next find an expression for  $\hat{q}A$  in terms of the physical parameters. First define the parameter  $u$  by

$$u \equiv 1 - Bq. \quad (\text{A21})$$

Then the expression for  $q$  becomes

$$u = \frac{1 + \frac{1}{2}\hat{q}\hat{B}\xi_3}{1 + \frac{1}{2}\hat{q}\hat{B}\xi_4}. \quad (\text{A22})$$

The constraint then becomes

$$u^8 = \frac{1 - 4\hat{q}A}{1 + 4\hat{q}A}, \quad (\text{A23})$$

which yields

$$4\hat{q}A = \frac{1 - u^8}{1 + u^8}. \quad (\text{A24})$$

We now find an expression for  $\hat{B}/A$  in terms of the physical parameters. We have

$$1 + \frac{1}{2}\hat{q}\hat{B}\xi_3 = u \left( 1 + \frac{1}{2}\hat{q}\hat{B}\xi_4 \right). \quad (\text{A25})$$

Substituting the expressions for  $\xi_3$  and  $\xi_4$  we find

$$\begin{aligned} 1 + \frac{\hat{B}}{4A} (-1 + \sqrt{1 - 4\hat{q}A}) \\ = u + u \frac{\hat{B}}{4A} (-1 + \sqrt{1 + 4\hat{q}A}). \end{aligned} \quad (\text{A26})$$

Rearranging terms we have

$$1 = \frac{\hat{B}}{4A} \left[ 1 + \frac{u\sqrt{1 + 4\hat{q}A} - \sqrt{1 - 4\hat{q}A}}{1 - u} \right]. \quad (\text{A27})$$

Now substituting the expression for  $\hat{q}A$  we find

$$1 = \frac{\hat{B}}{4A} \left[ 1 + \left( \frac{2}{1 + u^8} \right)^{1/2} u(u^2 + u + 1) \right]. \quad (\text{A28})$$

Define the quantity  $f$  by

$$f \equiv 1 + \left( \frac{2}{1 + u^8} \right)^{1/2} u(u^2 + u + 1). \quad (\text{A29})$$

Then we have

$$\frac{\hat{B}}{A} = \frac{4}{f}. \quad (\text{A30})$$

We now find an expression for  $1/A$  in terms of the physical parameters. First we have

$$\begin{aligned} 1 + \frac{1}{2} \hat{q} \hat{B} \zeta_3 &= 1 + \frac{\hat{B}}{4A} (-1 + \sqrt{1 - 4\hat{q}A}) \\ &= 1 + f^{-1} \left[ -1 + \left( \frac{2}{1+u^8} \right)^{1/2} u^4 \right] \\ &= f^{-1} \left[ f - 1 + \left( \frac{2}{1+u^8} \right)^{1/2} u^4 \right] \\ &= f^{-1} \left( \frac{2}{1+u^8} \right)^{1/2} u \frac{1-u^4}{1-u}. \end{aligned} \quad (\text{A31})$$

It then follows that

$$\begin{aligned} \frac{1}{A} &= \frac{1}{B} \frac{\hat{B}}{\hat{B} A} \\ &= \frac{1}{B} \left[ 1 + \frac{1}{2} \hat{q} \hat{B} \zeta_3 \right]^{-3} \sqrt{1 - 4\hat{q}A} \frac{4}{f} \\ &= \frac{1}{B} 2f^2 u (1+u^8) \left( \frac{1-u}{1-u^4} \right)^3. \end{aligned} \quad (\text{A32})$$

We are now ready to evaluate the quantity  $I$ :

$$\begin{aligned} I &= 2\pi \left[ \frac{1}{A} - \frac{\hat{B}}{2A^2} - \frac{\hat{q}\hat{B}^2}{4A^2} \right] \\ &= \frac{2\pi}{A} \left[ 1 - \frac{\hat{B}}{2A} - \frac{1}{4} \hat{q} A \left( \frac{\hat{B}}{A} \right)^2 \right] \\ &= \frac{2\pi}{B} 2f^2 u (1+u^8) \left( \frac{1-u}{1-u^4} \right)^3 \left[ 1 - \frac{2}{f} - \frac{1-u^8}{1+u^8} \frac{1}{f^2} \right] \\ &= \frac{4\pi}{B} u (1+u^8) \left( \frac{1-u}{1-u^4} \right)^3 \left[ [f-1]^2 - 1 - \frac{1-u^8}{1+u^8} \right] \\ &= 8\pi \frac{u(u^3+u^2+u-1)}{(u^3+u^2+u+1)^2}. \end{aligned} \quad (\text{A33})$$

Now we are (finally) ready to evaluate the action:

$$\begin{aligned} S &= \int_0^q I dq \\ &= \frac{1}{B} \int_{1-Bq}^1 I du. \end{aligned} \quad (\text{A34})$$

The integral is

$$\begin{aligned} S &= \frac{4\pi}{B^2} \left[ \frac{2u^2+u+1}{u^3+u^2+u+1} \right]_1^{1-Bq} \\ &= \frac{4\pi}{B^2} \left[ 1 + \frac{u^2-u^3}{u^3+u^2+u+1} \right]_1^{1-Bq} \\ &= \frac{4\pi}{B^2} \left[ 1 + \frac{u^2(1-u)^2}{1-u^4} \right]_1^{1-Bq} \\ &= 4\pi q^2 \frac{(1-Bq)^2}{1-(1-Bq)^4}, \end{aligned} \quad (\text{A35})$$

as quoted in (2.14).

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