# Dispersion of Newton's constant: A transfer matrix formulation of quantum gravity

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A transfer matrix formalism applicable to certain reparametrization-invariant theories, including quantum gravity, is proposed. In this formulation it is found that every stationary state in quantum gravity satisfies a Wheeler-DeWitt equation, but each with a different value of the Planck mass; the value $M^4_{\text{Planck}}$ turns out to be proportional to the eigenvalue of the evolution operator. As a consequence, the fact that the Universe is nonstationary implies that it is not in an eigenstate of Newton's constant.

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## I. INTRODUCTION

In nonparametrized Hamiltonian mechanics, there is a set of conjugate variables $\{q^i,p_i\}$ and a time parameter $t$. Classical trajectories are fixed by specifying initial $\{q_0^i\}$ and final $\{q_f^i\}$ coordinates, and also a time lapse $\Delta t$. In parametrized mechanics there is also a set of conjugate variables $\{q^\mu,p_\mu\}$ and an evolution parameter $\tau$. The difference is that in parametrized theories, a classical trajectory is fixed by the initial $\{q_0^\mu\}$ and final $\{q_f^\mu\}$ coordinates alone; the values of $\tau$ that happen to be associated with those coordinates are irrelevant. An example is the case of a free relativistic particle, where specifying $\{x^\mu\}$ at the initial and final points determines the trajectory. A field-theory example is general relativity, where the initial and final three-manifold is sufficient, together with Einstein's equations, to determine the four-manifold between them. Since the $\{q^\mu\}$ of parametrized theories contain time information, applying standard quantization prescriptions is, in effect, "quantizing time." This poses no problem when the Hamiltonian of the parametrized theory is parabolic, as in parametrized nonrelativistic mechanics, or parametrized relativistic scalar field theory. But for parametrized theories with hyperbolic Hamiltonians, such as the free relativistic particle, or quantum gravity, standard quantization procedures can lead to serious difficulties in identifying an appropriate evolution parameter, and a conserved non-negative norm [1]. In some cases, e.g., a free relativistic particle in flat space, these problems can be easily overcome; in others, such as a free relativistic particle moving in an arbitrary curved background, they are much more problematic. In quantum gravity, these difficulties are known as the "problem of time."

In this paper I will propose a transfer matrix formalism for quantum gravity, and certain other parametrized theories of the type described above. Since a transfer matrix, by definition, is an evolution operator, this proposal is intended as a possible resolution of the time problem in quantum gravity. The method treats all dynamical degrees of freedom of parametrized theories on an equal footing: all are operators, none in particular is an evolution parameter. This method will first be illustrated for a one-dimensional "universe," whose action is that of a free relativistic particle. The formalism will then be extended to minisuperspace-type actions, typical of quantum cosmology, and finally (with caveats regarding operator ordering and regularization) to full quantum gravity.

## II. A ONE-DIMENSIONAL UNIVERSE

A free relativistic particle is the simplest example of a system with a reparametrization invariant Lagrangian; for this reason it is often used as a "warmup" exercise [1,2] for higher-dimensional reparametrization-invariant theories, such as strings and quantum gravity. To briefly recall some of the familiar analogies: The action of a relativistic particle[1]

$$S=-m_0\int d\tau\left[-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right]^{1/2} \tag{1}$$

is invariant under reparametrizations $\tau\to f(\tau)$; the Einstein-Hilbert action is invariant under general coordinate transformations $x^\mu\to x'^\mu(x)$. In phase space one can also write, for the particle theory,

$$\begin{aligned} S&=\int d\tau\left[p_\mu\frac{dx^\mu}{d\tau}-NH\right],\\ H&=\frac{1}{2m_0}(p^\mu p_\mu+m_0^2)\,, \end{aligned} \tag{2}$$

and this leads to the Hamiltonian constraint $H=0$, which is just the mass-shell constraint. The analogous steps, in the Arnowitt-Deser-Misner (ADM) decomposition for gravity, lead to the super-Hamiltonian and supermomentum constraints. Upon quantization, the Hamiltonian

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[1]Signature convention $\eta=\text{diag}[-1,1,1,1]$.

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constraint becomes a constraint on physical states $H\psi=0$, which is the Klein-Gordon equation in the case of a particle, and the Wheeler-DeWitt equation for quantum gravity.

If one slices spacetime into a series of spacelike hypersurfaces $\Sigma_T$ with a parameter $T=T(x)$ (the time coordinate) labeling each hypersurface, the following norm is preserved by the Klein-Gordon equation:

$$\langle \psi|\psi\rangle_T=\frac{i}{2m_0}\int d\Sigma_T^\mu(\psi^*\partial_\mu\psi-\psi\partial_\mu\psi^*) \ . \tag{3}$$

This norm is not positive definite, in general (although in flat spacetime one can restrict to positive states, in which case the norm *is* positive definite[2]). A similar construction can be made in quantum gravity. It is possible to slice superspace into hypersurfaces such that a norm analogous to (3) is independent of the hypersurface, providing $\Psi$ satisfies the Wheeler-DeWitt equation. It is not clear how to make such a norm non-negative. Nevertheless, the analogy to the relativistic particle case suggests (and the idea goes back to the classic paper of DeWitt [3]) that some coordinate in superspace can be interpreted as a time evolution parameter in quantum gravity. There have been many subsequent variations and extensions of this idea; a recent comprehensive review is found in [4].

On the other hand, there is one aspect of the relativistic particle example which seems quite different from the situation in four-dimensional (4D) gravity. In the case of the relativistic particle, the observer is obviously *external* to the particle. The information contained in the Klein-Gordon wave function refers to measurements that can be made by such external observers, who are free to measure the particle position on any spacelike hypersurface. In contrast, observers in 4D gravity (at least, human observers) are *internal* to the system in question (the Universe), since we live in spacetime rather than superspace. It is possible to find and quantize a one-dimensional model which reflects also *this* aspect of the 4D case?

Such a model is obtained by simply reinterpreting the action of Eq. (1). Consider an observer living in a one-dimensional universe parametrized by a single coordinate $\tau$, who is able to make observations on a four-component "field" $x^\mu(\tau)$ in that one-dimensional spacetime. The reparametrization invariant action of this field is taken to be that of Eq. (1). The Hilbert space consists of a wave functions $\phi(x^\mu)$ with the norm

$$\langle \phi|\phi\rangle=\int d^4x\,\mu(x)\phi^*(x^\mu)\phi(x^\mu) \ , \tag{4}$$

where $\mu(x)$ is a measure to be determined. Note that since all components $x^\mu$ are observable, the integral is taken over the full four-dimensional "field" space. The fact that all components $\{x^\mu\}$ are observable and all are integrated over in the norm (4) is the main (and crucial) difference between the "one-dimensional universe," in which none of the observables is to be interpreted as an evolution parameter, and the relativistic particle.

In classical relativistic mechanics, the dynamics of a particle moving in curved space, or in some external potential, can be described covariantly by a parametrized trajectory $x^\mu(\tau)$. In quantum physics the configuration space is Hilbert space, and dynamics can be presumably also be described as a parametrized trajectory $\psi(x^\mu,\tau)$ in Hilbert space. Let us postulate a corresponding Schrödinger equation

$$i\hbar\partial_\tau\psi(x,\tau)=H\psi(x,\tau) \ , \tag{5}$$

where $H$ is an operator, Hermitian in the measure $\mu(x)$ and invariant with respect to Lorentz transformations, such that

$$x^\mu_{\rm cl}(\tau)\equiv\langle\psi(x,\tau)|x^\mu|\psi(x,\tau)\rangle \tag{6}$$

is a parametrized solution of the classical equations of motion. In this way, the parameter $\tau$ running along a trajectory of Hilbert space can be identified with the variable $\tau$ parametrizing a certain classical trajectory, corresponding to the motion of the center of the wave packet. Of course reparametrizations of $\tau$ have no physical significance, and the Schrödinger equation above can be made to look covariant with respect to reparametrizations of $\tau$ by introducing an "einbein" for the trajectory

$$i\hbar\partial_\tau\psi(x,\tau)=e(\tau)H\psi(x,\tau) \ . \tag{7}$$

However, assuming that $H$ is $\tau$ independent, the trajectory through Hilbert space depends only on an initial state, and not on $e(\tau)$, which can always be set to $e=1$ by a reparametrization. From here on we set $e=1$.

I will now show how an evolution operator $H$ and measure $\mu(x)$, with the required properties, may be obtained from a transfer matrix formalism. The starting point is the Euclidean action corresponding to Eq. (1), obtained rotating the field space metric $(\eta_{\mu\nu})$ from Lorentzian to Euclidean signature, and extracting an overall factor of $i$; i.e.,

$$iS_{\rm Lorentz}(g_{\mu\nu}=\eta_{\mu\nu})\rightarrow -S_{\rm Euclid}(g_{\mu\nu}=\delta_{\mu\nu}) \ . \tag{8}$$

Then, generalizing from $D=4$ to arbitrary $D$, the transfer matrix $\mathcal{T}_\epsilon$ is defined by

$$\begin{aligned}\psi(y,\tau+\epsilon)&=\mathcal{T}_\epsilon\psi(y,\tau)=\exp(-H_\epsilon\epsilon/\hbar)\psi(y,\tau)\\&=\int d^Dx\,\mu(x)\exp(-S[y,x]/\sqrt{\epsilon\hbar})\psi(x,\tau) \ ,\end{aligned} \tag{9}$$

where $S[y,x]$ is the Euclidean action of the classical trajectory connecting (and terminating at) points $x^\mu$ and $y^\mu$. Since $S[x,y]=S[y,x]$, and $S[y,x]$ is real, it follows that $H_\epsilon$ is Hermitian in the measure $\mu$. Note that by the usual trick of integration by parts we have

[2]More generally, it is possible to identify a conserved non-negative norm in spacetimes with a timelike Killing vector field.

$$0=\int\left[\prod_n d^D x\,\mu(x_n)\right]\left[\frac{1}{\sqrt{\epsilon\hbar}}\frac{\partial S[\{x_i\}]}{\partial x_k^\mu}-\frac{\partial}{\partial x_k^\mu}\ln\mu(x_k)\right]\exp[-S[\{x_i\}]/\sqrt{\epsilon\hbar}]$$
$$=\left\langle\cdots\left[\frac{1}{\sqrt{\epsilon\hbar}}\frac{\partial S[\{x_i\}]}{\partial x_k^\mu}-\frac{\partial}{\partial x_k^\mu}\ln\mu(x_k)\right]\cdots\right\rangle, \tag{10}$$

where

$$\frac{\partial S[\{x_i\}]}{\partial x_k^\mu}=\frac{\partial}{\partial x_k^\mu}\sum_n S[x_{n+1},x_n]$$
$$\to\frac{\delta S[x(\tau)]}{\delta x^\mu(\tau')} \tag{11}$$

and where $S[x(\tau)]$ is the continuum action of the path $x^\mu(\tau)$. Formally then, in the $\epsilon\to 0$ limit, we obtain the quantum version of the Euclidean equations of motion

$$\langle\cdots\delta S/\delta x\cdots\rangle=0\ . \tag{12}$$

The measure is given by

$$\mu^{-1}(y)=(\sqrt{\epsilon\hbar})^D\lim_{\epsilon\to 0}\int\frac{d^D x}{(\sqrt{\epsilon\hbar})^D}\exp(-S[y,x]/\sqrt{\epsilon\hbar})\ . \tag{13}$$

This expression is motivated by requiring that (i) the transfer matrix is the identity operator in the $\epsilon\to 0$ limit, and (ii) symmetries of the action become symmetries of the integration measure. Finally, the evolution operator $H$ in (5) is obtained by the $\epsilon\to 0$ limit,

$$H\equiv\left[\lim_{\epsilon\to 0}H_\epsilon\right]_{\delta\to\eta}, \tag{14}$$

followed by a rotation back to the Lorentzian signature ($\delta\to\eta$).

This defines the quantization procedure. Next I will compute the operator $H$ and obtain the classical limit of (5), first with a flat space metric $g_{\mu\nu}=\eta_{\mu\nu}$, then with a curved space metric.

Let $z^\mu=x^\mu-y^\mu$. Since a rotation to Euclidean signature has been performed we can easily evaluate the integral of Eq. (13),

$$S[y,x]=m_0\sqrt{\delta_{\mu\nu}z^\mu z^\nu}=m_0|z|\ ,$$
$$\mu^{-1}(y)=\int d^D z\exp\left[-\frac{m_0|z|}{\sqrt{\epsilon\hbar}}\right] \tag{15}$$
$$=\frac{2\pi^{D/2}}{\Gamma(D/2)}(D-1)!\left[\frac{\sqrt{\epsilon\hbar}}{m_0}\right]^D,$$

and Eq. (9),

$$\psi(y,\tau+\epsilon)=\int d^D z\,\mu\exp\left[-\frac{m_0|z|}{\sqrt{\epsilon\hbar}}\right]\left[\psi(y)+\frac{\partial\psi}{\partial y^\mu}z^\mu+\frac{1}{2}\frac{\partial^2\psi}{\partial y^\mu\partial y^\nu}z^\mu z^\nu+O(z^3)\right]$$
$$=\left[1+\epsilon\hbar\frac{D+1}{2m_0^2}\delta^{\mu\nu}\partial_\mu\partial_\nu+O(\epsilon^2)\right]\psi(y,\tau)\ . \tag{16}$$

Identifying $H_\epsilon$, and taking the limit and signature rotation prescribed in (14) we obtain

$$H=-\hbar^2\frac{D+1}{2m_0^2}\eta^{\mu\nu}\partial_\mu\partial_\nu \tag{17}$$

and it is easy to see that the eigenstates of $H$,

$$\psi_{\mathcal{E}}(x^\mu,\tau)=e^{-i\mathcal{E}\tau/\hbar}\phi_{\mathcal{E}}(x^\mu)\ , \tag{18}$$

are all solutions of the Klein-Gordon equation

$$\left[\hbar^2\eta^{\mu\nu}\partial_\mu\partial_\nu+\frac{2m_0^2\mathcal{E}}{D+1}\right]\phi_{\mathcal{E}}=0\ , \tag{19}$$

where the mass-shell parameter

$$m^2=-\frac{2m_0^2\mathcal{E}}{D+1} \tag{20}$$

is proportional to the eigenvalue of the evolution operator $H$.

The important point to notice, in this very simple example, is that the transfer matrix formalism has not destroyed the constraint, since each stationary state satisfies the usual Klein-Gordon equation. What is unusual is that the dimensionful parameter in the constraint, in this case $m^2$, is different for each stationary state. In effect, $m^2$ has become a dynamical quantity, such as energy, rather than a fixed parameter.

The classical equations of motion can be obtained by WKB methods, or more simply by just making the replacement

$$H[x^\mu,-i\hbar\partial_\mu]\to H[x^\mu,p_\mu]\ ,$$
$$H_c[x,p]=\lim_{\hbar\to 0}H[x,p]\ , \tag{21}$$

and applying the classical equations

$$H_c=\mathcal{E},\quad\frac{dx^\mu}{d\tau}=\frac{\partial H_c}{\partial p_\mu},\quad\frac{dp_\mu}{d\tau}=-\frac{\partial H_c}{\partial x^\mu}\ , \tag{22}$$

where $\mathcal{E}$ is fixed from the initial conditions. This gives us

$$\eta^{\mu\nu}p_\mu p_\nu = -M^2, \quad \frac{dx^\mu}{d\tau} = \frac{D+1}{m_0^2}\eta^{\mu\nu}p_\nu \ , \tag{23}$$

where $M^2 = -2m_0^2\mathcal{E}/(D+1)$. This is equivalent, of course, to the usual classical equation of motion for a free relativistic particle moving in flat space, which is obtained by the rescaling

$$\tau = \frac{m_0^2 s}{M(D+1)} \ , \tag{24}$$

where $s$ is the proper time of the particle world line. The proportionality of $\tau$ to the proper time $s$ has no special significance since, as noted below Eq. (6), the parametrization can be modified simply by choosing a different $e(\tau) \neq 1$.

The nonrelativistic limit of this theory is equally trivial, but still instructive. Making factors of $c$ explicit, the general solution to the evolution equation (5) (with $p^0 > 0$) is

$$\psi(x^\mu,\tau) = \int dm \int d\mathbf{p}\, f(m,\mathbf{p}) \exp\left[ i \left[ \frac{D+1}{2m_0^2} m^2\tau - \sqrt{m^2c^4 + \mathbf{p}^2c^2}\, t + \mathbf{p}\cdot\mathbf{x} \right] \Big/ \hbar \right] , \tag{25}$$

where $t = x^0/c$. For simplicity, suppose that (i) $f(m,\mathbf{p})$ factorizes into $f(m,\mathbf{p}) = g(m)h(\mathbf{p})$, (ii) $g(m)$ is very sharply peaked around some value $M$, and (iii) $h(\mathbf{p})$ is negligible unless $|p| \ll Mc$. In that case, again rescaling

$$\tau = \frac{m_0^2 c^2}{M(D+1)} s \tag{26}$$

so that $s$ has units of time, the solution can be written

$$\psi(x^\mu, s) \approx \varphi(s,t)\psi_{\rm NR}(t,\mathbf{x}) \ , \tag{27}$$

where

$$\begin{aligned} \varphi(s,t) &= \int dm\, g(m) \exp\left[ i \left[ \frac{m^2c^2}{2M} s - mc^2 t \right] \Big/ \hbar \right] , \\ \psi_{\rm NR}(t,\mathbf{x}) &= \int d\mathbf{p}\, h(\mathbf{p}) \exp\left[ -i \left[ \frac{\mathbf{p}^2}{2M} t - \mathbf{p}\cdot\mathbf{x} \right] \Big/ \hbar \right] , \end{aligned} \tag{28}$$

and, of course, $\psi_{\rm NR}$ is a solution of the nonrelativistic Schrödinger equation for a free particle of mass $M$.

Now if $g(m)$ and $f(\mathbf{p})$ are both, e.g., Gaussians, then the spreading of the wave packet in both $t$ and $\mathbf{x}$ is given by

$$\begin{aligned} \Delta t(s) &= \left[ \Delta t_0^2 + \left[ \frac{\Delta m}{M} s \right]^2 \right]^{1/2} , \\ \Delta x(t) &= \left[ \Delta x_0^2 + \left[ \frac{\Delta p}{M} t \right]^2 \right]^{1/2} , \end{aligned} \tag{29}$$

where

$$\Delta m\, \Delta t_0 \sim \frac{\hbar}{c^2}, \quad \Delta p\, \Delta x_0 \sim \hbar \ . \tag{30}$$

Since $\bar{t} \equiv \langle t \rangle = s$, we can write $\Delta t(s) = \Delta t(\bar{t})$. Finally, it is consistent with the $\Delta m\, \Delta t_0$ uncertainty relation to take both $\Delta m$ and $\Delta t_0$ proportional to $1/c$. This means that the dispersion of the wave packet in the time direction

$$\begin{aligned} \Delta t(\bar{t}) &= \left[ \Delta t_0^2 + \left[ \frac{\Delta m}{M} \bar{t} \right]^2 \right]^{1/2} \\ &\sim 1/c \to 0 \end{aligned} \tag{31}$$

goes to zero in the nonrelativistic, $c \to \infty$ limit.

The lesson of this simple exercise is that, in the $c \to \infty$ limit, the one-dimensional universe has acquired a clock. By a "clock," I mean a nonstationary observable (or set of observables) whose dispersion is negligible, and whose evolution is independent of the other observables in the system. Measuring such an observable gives a value for the evolution parameter, which itself is *not* an observable. In the one-dimensional universe, the "clock" is the $t$ component of the field $x^\mu(\tau)$; this is because its dispersion can be made to vanish, and its value becomes perfectly correlated with $\tau$, in the nonrelativistic limit. Equivalently, we can just say that the observable $t$ behaves classically in the $c \to \infty$ limit.

The extension of the transfer matrix formalism to a one-dimensional universe with a curved field-space metric $g_{\mu\nu}$ is straightforward. The reparametrization invariant action in this case is

$$S = -m_0 \int d\tau \left[ -g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right]^{1/2} . \tag{32}$$

Again we rotate from Lorentzian to Euclidean signature, extract an $i$, and obtain the action for Riemannian metrics. To evaluate the integrals in Eqs. (9) and (13), it is useful to introduce Riemann normal coordinates $\xi^\mu$ around the point $y^\mu$. The classical action for a trajectory running from $\xi_1^\mu = 0$ to $\xi_2^\mu = \xi$ is

$$\Delta S[0,\xi] = m_0 \sqrt{\delta_{ab}\xi^a\xi^b + O(\xi^5)} \ . \tag{33}$$

The $O(\xi^5)$ terms will not contribute to $H$ in the $\epsilon \to 0$ limit, and can be dropped. The measure is then

$$\mu^{-1}(y)=(\sqrt{\epsilon\hbar})^D \lim_{\epsilon\to 0}\int \frac{d^D\xi}{(\sqrt{\epsilon\hbar})^D}\det\left[\frac{\partial z^\mu}{\partial\xi^\nu}\right]\exp(-m_0|\xi|/\sqrt{\epsilon\hbar})$$
$$=\frac{2\pi^{D/2}}{\Gamma(D/2)}(D-1)!\left[\frac{\sqrt{\epsilon\hbar}}{m_0}\right]^D\frac{1}{\sqrt{g(y)}}=\frac{\sigma}{\sqrt{g(y)}}\ . \tag{34}$$

The transfer matrix is computed as in the flat-space case, again with the help of Riemann normal coordinates:

$$\psi(y,\tau+\epsilon)=\int\frac{d^4z}{\sigma}\sqrt{g(y+z)}\exp(-m_0\sqrt{g_{\mu\nu}z^\mu z^\nu}/\sqrt{\epsilon\hbar})\psi(y+z,\tau)$$
$$=\int\frac{d^4\xi}{\sigma}\left[1-\frac{1}{6}R_{\alpha\beta}\xi^\alpha\xi^\beta+\cdots\right]\exp\left[-\frac{m_0|\xi|}{\sqrt{\epsilon\hbar}}\right]\left[\psi(y,\tau)+\frac{\partial\psi}{\partial\xi^\mu}\xi^\mu+\frac{1}{2}\frac{\partial^2\psi}{\partial\xi^\mu\partial\xi^\nu}\xi^\mu\xi^\nu+O(\xi^3)\right]$$
$$=\left[1+\epsilon\hbar\frac{D+1}{2m_0^2}\partial^\mu\partial_\mu-\epsilon\hbar\frac{D+1}{6m_0^2}R+O(\epsilon^2)\right]\psi(y(\xi),\tau)\ , \tag{35}$$

where $R$ is the curvature scalar. Transforming back from Riemann normal coordinates and taking the limit (14) gives

$$H=-\frac{D+1}{2m_0^2}\hbar^2\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu+\hbar^2\frac{D+1}{6m_0^2}R\ , \tag{36}$$

where $g=|\det(g_{\mu\nu})|$. The appearance of the curvature scalar term in the evolution operator $H$ is related to the choice of measure, and, in turn, the ordering of operators in Eq. (36). There are other, more complicated choices of measure that could have been made in this problem, but these would only affect the coefficient of $R$ (cf. Refs. [5,6]).

The classical equations of motion are again obtained either from the WKB approximation, or else by the prescription (21), which gives

$$H_c=\frac{D+1}{2m_0^2}g^{\mu\nu}p_\mu p_\nu\ . \tag{37}$$

Setting $H_c=\mathcal{E}$, and $M^2\equiv-2m_0^2\mathcal{E}/(D+1)$, and also rescaling $\tau$ according to (24), we recover the classical equations of a relativistic particle moving in curved space, with the identification of the evolution parameter $\tau$ as the proper time.

At the quantum level, eigenstates of the mass-shell parameter are stationary states of the evolution operator $H$. The remarks made above for the flat-space case, noting that all stationary states obey the constraint equation but with different values of the mass-shell parameter, of course apply to the curved space example also. Nonstationarity, in the one-dimensional universe, must be attributed to dispersion in the mass-shell parameter $M^2$.

As a final remark we may ask whether, since the evolution parameter is identified with proper time in the classical limit [given the choice $e(\tau)=1$ in Eq. (7)], the proper-time could have been used as an evolution parameter from the beginning. This might be done by replacing $S[y,x]$ and $\epsilon$ in Eq. (9) with a weight $S[(y,s+\Delta s),(x,s)]$, representing the minimal action between points $x$ and $y$ of paths constrained to have proper time $\Delta s$. The problem with this is that the amplitude between $(x,s_1)$ and $(y,s_2)$ would not, in general, be dominated by the classical path, unless the proper-time difference $s_2-s_1$ happened to correspond to the path length of the geodesic between $x$ and $y$. The problem can be fixed, for Green's functions, by integrating over $s_2-s_1$ (see, e.g., [7]), but then the proper time loses its function as an evolution parameter. In higher dimensions the situation is further complicated by the fact that a proper-time slicing of a simply connected Riemannian $D$-manifold into $(D-1)$-manifolds can introduce spurious singularities. Under proper-time evolution, a simply connected $(D-1)$-manifold will in general evolve into a set of disconnected $(D-1)$-manifolds [8]. It seems unlikely that such an approach would yield a Hermitian evolution operator, although it may have other applications (cf. [9]).

## III. MINISUPERSPACE MODELS

Next we consider actions of the form

$$S=\int dt\left[p_n\frac{dq^n}{dt}-N\mathcal{H}\right]\ ,$$
$$\mathcal{H}=\frac{1}{2m_0}G^{nm}p_np_m+m_0V(q)\ , \tag{38}$$

where $m_0$ is some dimensionful parameter, and the supermetric $G_{nm}$ has a Lorentzian $(-++\cdots+)$ signature. Actions of this kind typically arise in minisuperspace models of quantum cosmology. To compute $S[q+\Delta q,q]$, begin with the Hamiltonian equation

$$\dot q^{\,n}=N\frac{\partial\mathcal{H}}{\partial p_n}=\frac{N}{m_0}G^{nm}p_m\ , \tag{39}$$

insert this into the constraint equation,

$$\mathcal{H}=\frac{m_0}{2N^2}G_{nm}\dot q^{\,n}\dot q^{\,m}+m_0V=0\ , \tag{40}$$

and solve for the lapse,

$$N=\left[-\frac{1}{2V}G_{nm}\dot q^{\,n}\dot q^{\,m}\right]^{1/2}\ . \tag{41}$$

Then

$$S[q,q+\Delta]=\int_0^{\Delta t}dt\frac{m_0}{N}G_{nm}\dot{q}^n\dot{q}^m$$
$$=-m_0\int dt\sqrt{-2VG_{nm}\dot{q}^n\dot{q}^m}$$
$$=-m_0\sqrt{-2VG_{nm}\Delta q^n\Delta q^m}$$
$$=-m_0\sqrt{-\mathcal{G}_{nm}\Delta q^n\Delta q^m}\ , \tag{42}$$

where we define a modified supermetric

$$\mathcal{G}_{nm}\equiv 2VG_{nm}\ . \tag{43}$$

From here, the procedure follows exactly the same steps as in the curved metric example of the previous section. Again, rotate the signature of the supermetric $G_{nm}$ (*not* the spacetime metric $g_{\mu\nu}$) to Euclidean signature and extract a factor of $i$ to obtain the "Riemannian" action. Then, introducing Riemann normal coordinates in minisuperspace around the point $q+\Delta q$, evaluate the relevant integrals for the measure and transfer matrix. The final result for the evolution operator is

$$H=-\frac{D+1}{2m_0^2}\hbar^2\frac{1}{\sqrt{\mathcal{G}}}\frac{\partial}{\partial q^n}\sqrt{\mathcal{G}}\,\mathcal{G}^{nm}\frac{\partial}{\partial q^m}+\hbar^2\frac{D+1}{6m_0^2}\mathcal{R}\ , \tag{44}$$

where, in this case, $D$ is the dimensionality of minisuperspace, $\mathcal{R}$ is the curvature scalar formed from the modified supermetric (43), and

$$\mathcal{G}\equiv|\det(\mathcal{G}_{mn})|\ . \tag{45}$$

It should be noted that $H$ is Hermitian in the measure $\mu(q)=\sqrt{\mathcal{G}}$. This is despite the fact that, in quantum cosmology models, $V(q)$ is not positive definite, and therefore $S[q,q+\Delta q]$ can be imaginary, even after rotation of $G_{nm}$. Nevertheless, carrying out the relevant integrals and rotating back to Lorentzian signature, one still finds that $H$ is Hermitian in the appropriate measure. This can be understood from the fact that, for modified supermetrics $\mathcal{G}_{mn}$ of Euclidean signature, the operator $H_\epsilon$ is Hermitian by construction, since $S[q,q']$ is real and symmetric. But since the Hermiticity of $H_\epsilon$ does not depend on the precise functional form of $\mathcal{G}_{mn}$, the continuation to arbitrary signature can only upset Hermiticity if it introduces factors of $i$ in $H$. These factors of $i$ are avoided by making the standard Euclidean$\rightarrow$Lorentzian continuation $\sqrt{\det(g)}\rightarrow\sqrt{-\det(g)}$, in the one-dimensional universe example, and $\sqrt{\det(\mathcal{G}_{mn})}\rightarrow\sqrt{|\det(\mathcal{G}_{mn})|}$, in the minisuperspace case, for both $H$ and $\mu$.

The Schrödinger equation for stationary states is now

$$\left[-\frac{D+1}{2m_0^2}\hbar^2\frac{1}{\sqrt{\mathcal{G}}}\frac{\partial}{\partial q^n}\sqrt{\mathcal{G}}\,\mathcal{G}^{nm}\frac{\partial}{\partial q^m}+\hbar^2\frac{D+1}{6m_0^2}\mathcal{R}\right]\Psi=\mathcal{E}\Psi \tag{46}$$

or, equivalently,

$$\left[-\frac{1}{2}\hbar^2\frac{V}{\sqrt{|VG|}}\frac{\partial}{\partial q^n}\sqrt{|VG|}\frac{1}{V}G^{nm}\frac{\partial}{\partial q^m}+\frac{1}{3}\hbar^2V\mathcal{R}-\frac{2m_0^2\mathcal{E}}{D+1}V\right]\Psi=0\ . \tag{47}$$

This equation may look more recognizable if we identify

$$M^2=-\frac{2m_0^2\mathcal{E}}{D+1}\ , \tag{48}$$

then reorder the factors in the Laplacian, and drop the curvature scalar term to obtain

$$\text{“}\left[-\frac{1}{2}\hbar^2G^{nm}\frac{\partial}{\partial q^n}\frac{\partial}{\partial q^m}+M^2V\right]\Psi=0\ ,\text{”} \tag{49}$$

where the quotation marks mean that this equation is correct up to operator-ordering terms. From this it is easy to see that the equation satisfied by stationary states, Eq. (47), is simply the Wheeler-DeWitt constraint equation, with the dimensionful parameter identified as $M^2=-2m_0^2\mathcal{E}/(D+1)$, and with a particular choice of operator ordering. Once again, we see that all stationary states satisfy a Wheeler-DeWitt equation, but with different mass parameters.

As in the last section, the classical equations of motion are obtained from the classical Hamiltonian given by (21),

$$H_c=\frac{D+1}{4m_0^2V}G^{nm}p_np_m\ , \tag{50}$$

and, setting $H_c=\mathcal{E}$ and $M^2=-2m_0^2\mathcal{E}/(D+1)$, we obtain the classical constraint equation

$$\mathcal{H}_M\equiv\frac{1}{2M}G^{nm}p_np_m+MV=0\ . \tag{51}$$

Hamilton's equations give us

$$\frac{dq^n}{d\tau}=\frac{\partial H_c}{\partial p_n}=\frac{D+1}{2m_0^2V}G^{nm}p_m\ ,$$
$$\frac{dp_n}{d\tau}=-\frac{\partial H_c}{\partial q^n}$$
$$=-\frac{D+1}{4m_0^2}\left[\frac{1}{V}\frac{\partial G^{ij}}{\partial q^n}p_ip_j-\frac{1}{V^2}\frac{\partial V}{\partial q^n}G^{ij}p_ip_j\right]$$
$$=-\frac{(D+1)M}{2m_0^2V}\left[\frac{1}{2M}\frac{\partial G^{ij}}{\partial q^n}p_ip_j+M\frac{\partial V}{\partial q^n}\right]\ , \tag{52}$$

which can be rewritten as

$$\mathcal{H}_M=0\ ,$$
$$\frac{dq^n}{d\tau}=f(q)\frac{\partial\mathcal{H}_M}{\partial p_n}\ ,$$
$$\frac{dp_n}{d\tau}=-f(q)\frac{\partial\mathcal{H}_M}{\partial q^n}\ , \tag{53}$$

where

$$f(q)=\frac{(D+1)M}{2m_0^2V(q)} \tag{54}$$

and where $\mathcal{H}_M$, given in (51), is the minisuperspace Hamiltonian $\mathcal{H}$ with the parameter $m_0$ replaced by $M$.

Apart from the factor of $f(q)$, Eqs. (53) are simply the classical equations of motion of the minisuperspace action (with $m_0 \to M$). So long as $V$ is nonzero, $f(q)$, like the lapse function $N(t)$, is irrelevant in determining the classical trajectory, which depends only on the directions of the vectors $\partial_\tau q^n$ and $\partial_\tau p_n$ in phase space, and not on their magnitudes. The magnitudes of these vectors only determine the rate (compared to some analogue of proper time in minisuperspace) at which the evolution parameter runs along the classical trajectory.

This point can be made in another way. Suppose we pick an initial point in phase space $\{q,p\}_0$ and solve Eqs. (53). Denote the solutions $\bar{q}^n(\tau)$ and $\bar{p}_n(\tau)$. Then, define

$$N(\tau)_{\{q,p\}_0} \equiv f(\bar{q}(\tau)) \ . \tag{55}$$

It is easy to see that $\bar{q}^n(\tau)$ and $\bar{p}_n(\tau)$ are a solution of Hamilton's equations for the original minisuperspace action (38):

$$\begin{aligned} &\mathcal{H}_M=0 \ , \\ &\frac{dq^n}{d\tau}=N(\tau)_{\{q,p\}_0}\frac{\partial \mathcal{H}_M}{\partial p^n} \ , \\ &\frac{dp_n}{d\tau}=-N(\tau)_{\{q,p\}_0}\frac{\partial \mathcal{H}_M}{\partial q^n} \ , \end{aligned} \tag{56}$$

with $m_0$ replaced by $M$, and with a particular choice (55) for the lapse function (which will be different for each classical trajectory).

## IV. QUANTUM GRAVITY

The action of quantum gravity, in the ADM decomposition, is

$$\begin{aligned} &S=\int d^4x \left[ p^{ij}\frac{\partial g_{ij}}{\partial t}-N\mathcal{H}_x(\kappa_0^2)-N_i\mathcal{H}_x^i \right] , \\ &\mathcal{H}_x(\kappa_0^2)=\kappa_0^2 G_{ijkl}p^{ij}p^{kl}+\frac{1}{\kappa_0^2}\sqrt{g}\,(-R+2\Lambda) \ , \\ &\mathcal{H}_x^i=-2p^{ik}{}_{;k} \ , \\ &G_{ijkl}=\frac{1}{2\sqrt{g}}(g_{ik}g_{jl}+g_{il}g_{jk}-g_{ij}g_{kl}) \ , \end{aligned} \tag{57}$$

where $\sqrt{g}$ is the root determinant of the three-metric $g_{ij}$. At the classical level, $\kappa_0^2=16\pi G_N$, where $G_N$ is Newton's constant. It will be seen that this identification is modified at the quantum level, much as $m_0^2$ was replaced by $M^2$ in the examples of the preceding sections.

The presence of the shift functions $N_i$ is a serious complication, as compared to the minisuperspace models of the preceding section, where the shift functions were absent. The problem is that the classical equations of motion

$$\frac{\partial g_{ij}}{\partial t}=2\kappa_0^2 N G_{ijnm}p^{nm}+N_{i;j}+N_{j;i} \ , \tag{58}$$

which are used to solve for the momenta in terms of $\partial_t g_{ij}$, contain derivatives of the $N_i$. These shift functions can be solved for, in principle, by substituting the expression for $p^{ij}$ obtained from (58) into the supermomentum constraints $\mathcal{H}_x^i=0$, which gives the $N_i$ in terms of the lapse $N$, and then substituting these $N_i$ into the Hamiltonian constraint $\mathcal{H}_x=0$ to solve for the lapse. However, since the $N_i$ are determined, in this way, by complicated partial differential equations, this procedure leads to intractable expressions, and there is no simple form for $S[g',g]$.

A great simplification is achieved if we instead set $N_i=0$ from the beginning. In that case the supermomentum constraint is not obtained by extremizing the action, and must somehow be recovered by imposing an operator constraint on physical states:

$$Q_x[p^{ij},g_{ij}]\Psi=0 \ , \tag{59}$$

where the subscript $x$ indicates that there is a separate constraint at each point, and of course these must be mutually consistent, as well as consistent with the $\tau$-evolution operator. These constraints will be obtained below. For the moment, we just set $N_i=0$ and proceed. Denote

$$\begin{aligned} &\{a=1-6\}\leftrightarrow\{(i,j),i\le j\} \ , \\ &q^a(x)\leftrightarrow g_{ij}(x) \ , \\ &p_a(x)\leftrightarrow \begin{cases} p^{ij}(x) & (i=j), \\ 2p^{ij}(x) & (i<j) \ , \end{cases} \\ &G_{ab}(x)\leftrightarrow G^{ijnm}(x) \end{aligned} \tag{60}$$

in order to compress the number of indices somewhat. Hamilton's equations with $N_i=0$ give us

$$p_a=\frac{1}{2\kappa_0^2 N}G_{ab}\dot{q}^b \ . \tag{61}$$

Inserting this into the constraint equation

$$0=\mathcal{H}_x=\frac{1}{4\kappa_0^2N^2}G_{ab}\dot{q}^a\dot{q}^b+\frac{1}{\kappa_0^2}\sqrt{g}\,U \ , \tag{62}$$

where

$$U\equiv -R+2\Lambda \ , \tag{63}$$

and solving for the lapse gives

$$N=\left[-\frac{1}{4\sqrt{g}\,U}G_{ab}\dot{q}^a\dot{q}^b\right]^{1/2} \tag{64}$$

so we have

$$\Delta S = S[q',q] = -\frac{1}{\kappa_0^2}\int d^3x \int_0^{\Delta t} dt \sqrt{-g^{1/2} U G_{ab}\dot{q}^a \dot{q}^b}$$

$$= -\frac{1}{\kappa_0^2}\int d^3x \sqrt{-(g^{1/2} U G_{ab})_0 \Delta q^a \Delta q^b} + O(\Delta q^2)$$

$$= -\frac{1}{\kappa_0^2}\int d^3x (\sqrt{g})_0 [\sqrt{-(\mathcal{G}_{ab})_0 \Delta q^a \Delta q^b} + O(\Delta q^2)] , \tag{65}$$

where $\Delta q^a = q^a - q'^a$, and we define

$$\mathcal{G}_{ab} \equiv \frac{1}{\sqrt{g}} U G_{ab} . \tag{66}$$

The notation $(\cdots)_0$ means that the quantity in parentheses is to be evaluated at $\Delta q = 0$, i.e., at $q'$.

Once again, we implicitly rotate the signature of the superspace metric $G_{ab}$ to Euclidean signature, and extract a factor of $i$, before evaluating the transfer matrix. Note that it is the signature of superspace, not the signature of spacetime, which is rotated to the Euclidean value. At the end, of course, we rotate back to the usual signature of superspace, according to the prescription (14). It was found, in the minisuperspace example of the previous section, that the evolution operator is Hermitian in an appropriate measure $\mu(q)$, despite the fact that the potential term $V$ was not positive definite. It is expected, for similar reasons, that Hermiticity of $H$ will also be maintained in full quantum gravity, where $U$ is not positive definite.[3]

The transfer matrix is obtained from

$$\psi(q',\tau+\epsilon) = \int Dq\mu(q) e^{-\Delta S/\sqrt{\epsilon\hbar}} \left[ \psi(q') + \int d^3x \left[ \frac{\delta\psi}{\delta q^a(x)} \right] \Delta q^a(x) + \frac{1}{2}\int d^3x\, d^3y \left[ \frac{\delta^2\psi}{\delta q^a(x)\delta q^b(y)} \right] \Delta q^a(x)\Delta q^b(y) + \cdots \right]$$

$$= \psi(q',\tau) + \epsilon(T_0 + T_1 + T_2) + O(\epsilon^2) , \tag{67}$$

where the $T_n$ represents terms with $n$ derivatives of $\psi$. To find these terms we need to evaluate

$$\langle \Delta q^a(x_1)\Delta q^b(x_2)\rangle = \int D(\Delta q)(\mu)_0 \Delta q^a(x_1)\Delta q^b(x_2) \exp\left[ -\frac{1}{\kappa_0^2}\int d^3x (\sqrt{g})_0 \sqrt{(\mathcal{G}_{ab})_0 \Delta q^a \Delta q^b}/\sqrt{\epsilon\hbar} \right] . \tag{68}$$

This quantity is highly singular, and, at this point, regularization is unavoidable.

Unfortunately, the problem of regularizing a functional integral nonperturbatively, in such a way as to preserve diffeomorphism invariance, is unsolved. So I will have to resort to a naive regulator, and replace the continuum degrees of freedom, labeled by $x$, by a finite set, labeled $\{n\}$, which are taken to represent regions of equal volume. We make the correspondences

$$\begin{aligned} &\Delta q^a(x) \leftrightarrow \Delta q^a(n) , \\ &\int d^3x \sqrt{g} \leftrightarrow \frac{V}{N_p}\sum_{n=1}^{N_p} , \\ &\frac{\delta}{\delta q^a(x)} \leftrightarrow \frac{N_p\sqrt{g(n)}}{V}\frac{\partial}{\partial q^a(n)} , \\ &Dq \leftrightarrow \prod_n d^D q(n) , \end{aligned} \tag{69}$$

where $V$ is the three-volume. For pure quantum gravity, there are six independent components of $g_{ij}$, so $D=6$. Obviously, with this naive discretization we lose diffeomorphism invariance of the integration measure, and we can expect to make mistakes, even after taking the continuum limit, on certain expressions which depend crucially on the invariance of the measure. In the transfer matrix formulation, it is the operator ordering, and the presence of terms analogous to $R$ in Eq. (36) [or $\mathcal{R}$ in (47)], which are sensitive to the measure. Operator-ordering problems are notorious in quantum gravity, and I will not try to solve them here. From here on, only the principal term of the evolution operator [coming from $T_2$ in Eq. (67)], will be determined explicitly. This term is relatively insensitive to the measure, and is the only term which is important for the classical limit. But even for the $T_2$ term, the ultimate justification for the prescription (69) above will be *a posteriori*.

With the above caveats duly noted we find

[3]An actual proof of Hermiticity, however, would involve the regularization and/or operator-ordering issues discussed below.

$$\langle \Delta q^a(n)\Delta q^b(m)\rangle = \int \prod_k d^D q(k)(\mu)_0 \Delta q^a(n)\Delta q^b(m)\exp\left[-\frac{V}{N_p\kappa_0^2}\sum_k \sqrt{(\mathcal{G}_{ab})_0\Delta q^a(k)\Delta q^b(k)}/\sqrt{\epsilon\hbar}\right]$$
$$=\delta_{nm}(D+1)\hbar\epsilon\frac{\kappa_0^4 N_p^2}{V^2}\mathcal{G}^{ab} \tag{70}$$

then

$$\epsilon T_2 = \frac{1}{2}\sum_n\sum_m \frac{\partial^2\psi}{\partial q^a(n)\partial q^b(m)}\langle \Delta q^a(n)\Delta q^b(m)\rangle$$
$$=\frac{1}{2}(D+1)\epsilon\hbar\frac{\kappa_0^4 N_p^2}{V^2}\sum_n \frac{\partial^2\psi}{\partial q^a(n)\partial q^b(n)}\mathcal{G}^{ab}$$
$$\to \epsilon\hbar\frac{\beta\kappa_0^4}{V}\int d^3x\, U^{-1}G^{ab}\frac{\delta^2\psi}{\delta q^a\delta q^b}\ , \tag{71}$$

where $\beta$ is a divergent, dimensionless constant,

$$\beta \equiv \tfrac{1}{2}(D+1)N_p\ , \tag{72}$$

proportional to the number of degrees of freedom. Finally, from $T_2$ we get

$$H = -\hbar^2\frac{\beta\kappa_0^4}{V}\int d^3x\, U^{-1}G^{ab}\frac{\delta^2}{\delta q^a\delta q^b}\ , \tag{73}$$

where, again, operator-ordering and/or measure terms have been dropped.

However, this evolution operator does not yet give us general relativity, not even in the classical limit. This is easy to see from, e.g., the WKB approach, since the Hamilton-Jacobi equation corresponding to (73) is

$$\mathcal{E} = \frac{\beta\kappa_0^4}{V}\int d^3x\, U^{-1}G^{ab}\frac{\delta S}{\delta q^a}\frac{\delta S}{\delta q^b}\ , \tag{74}$$

which is not the Einstein-Hamilton-Jacobi equation of gravitation. Now, although the transfer matrix formulation as presented in Eqs. (9) and (13) was guaranteed to recover the correct classical limit, it is not hard to see what went wrong in this case. The difficulty, as already mentioned above, comes from setting the shift functions $N_i=0$ at the beginning; this means that the supermomentum constraints $\mathcal{H}_x^i=0$ were lost from the start. One could try to simply reintroduce these as physical state constraints $\mathcal{H}_x^i\Psi=0$, but this still does not lead to the Einstein-Hamilton-Jacobi equation.

Note however, that because of the factor $1/V$ in the $\tau$-evolution operator $H$ of Eq. (73), the Schrödinger equation can be written in the form

$$\int d^3x\, Q_x\Psi = 0\ , \tag{75}$$

where

$$Q_x = -\hbar^2\beta\kappa_0^4 U^{-1}G^{ab}\frac{\delta^2}{\delta q^a\delta q^b} - i\hbar\sqrt{g}\frac{\partial}{\partial\tau}\ . \tag{76}$$

The extra constraints which need to be imposed on the physical states, which then generate the usual constraint algebra of general relativity, are simply

$$Q_x\Psi = 0 \tag{77}$$

at every point $x$. It is not hard to see why. Consider an arbitrary solution of the Schrödinger equation

$$\Psi(q,\tau) = \sum_{\mathcal{E}} a_{\mathcal{E}} e^{-i\mathcal{E}\tau/\hbar}\Phi_{\mathcal{E}}(q)\ . \tag{78}$$

Since the $a_{\mathcal{E}}$ are arbitrary, the $Q$ constraint (77) requires that, for each stationary state,

$$\left[-\hbar^2 G^{ab}\frac{\delta^2}{\delta q^a\delta q^b} + \sqrt{g}\left[\frac{-\mathcal{E}}{\beta\kappa_0^4}\right]U\right]\Phi_{\mathcal{E}} = 0\ . \tag{79}$$

Identifying

$$\kappa^2 = 16\pi G_N = \left[-\frac{\beta}{\mathcal{E}}\right]^{1/2}\kappa_0^2\ ,$$
$$M_{\text{Planck}}^2 = \left[-\frac{\mathcal{E}}{\beta}\right]^{1/2}\frac{16\pi\hbar}{\kappa_0^2}\ , \tag{80}$$

as Newton's constant and the Planck mass, respectively, we see that Eq. (79) is just the Wheeler-DeWitt equation

$$\mathcal{H}_x(\kappa^2)\Phi_{\mathcal{E}} = \left[-\hbar^2\kappa^2 G^{ab}\frac{\delta^2}{\delta q^a\delta q^b} + \frac{1}{\kappa^2}\sqrt{g}\,U\right]\Phi_{\mathcal{E}} = 0 \tag{81}$$

with Newton's constant inversely proportional to $\sqrt{-\mathcal{E}}$.

Finally, we invoke the Moncrief-Teitelboim interconnection theorem [10], which says that if a state satisfies the (Wheeler-DeWitt) Hamiltonian constraint (81) at every point $x$, then that state also satisfies the supermomentum constraints

$$\mathcal{H}_x^i\Phi_{\mathcal{E}} = 0 \tag{82}$$

at every point $x$. In this way, the supermomentum constraints that were lost at the outset by fixing $N_i=0$ have been recovered. Further, given that the Hamiltonian and supermomentum constraints are consistent (commutators close on the Poincaré algebra), and that the $Q_x$ constraints (77) are implied by the Hamiltonian constraints (81), it follows that the $Q_x$ constraints are consistent not only with the evolution operator $H$, but also with each other.

The Einstein-Hamilton-Jacobi equations follow directly from a WKB treatment of the Wheeler-DeWitt constraints, and the classical limit, obtained by replacing $-i\hbar\delta/\delta q^a$ and $p_a$, follows in complete analogy to the minisuperspace case. Since the proper constraint algebra has been obtained, it is fairly obvious that the correct classical equations must fall out. But it is still nice to see this explicitly. We begin with the classical limit of the evolution operator:

$$H_c=\frac{1}{V}\int d^3x\,\beta\kappa_0^4 U^{-1}G^{ab}p_a p_b$$
$$\equiv\frac{1}{V}\int d^3x\,H_{cx}\ . \tag{83}$$

Beginning from a set of initial data $\{q,p\}_0=\{g_{ij},p^{ij}\}_0$ consistent with the supermomentum constraints

$$\mathcal{H}^i[\{q,p\}_0]=0 \tag{84}$$

we have

$$\mathcal{E}\equiv H_c[\{q,p\}_0] \tag{85}$$

and the consistency of the initial data with the $Q_x=0$ constraints gives us also

$$H_{cx}=\sqrt{g}\ \mathcal{E}\ . \tag{86}$$

Then Hamilton's equations, derived from $H_c$, are

$$\frac{dq^a(x)}{d\tau}=\frac{\beta\kappa_0^4}{UV}2G^{ab}p_b$$
$$=\int d^3x'\frac{\beta\kappa_0^4}{UV\kappa^2}\frac{\delta}{\delta p_a(x)}\mathcal{H}_{x'}(\kappa^2) \tag{87}$$

and

$$\frac{dp_a(x)}{d\tau}=-\left[-\frac{1}{V^2}\frac{\delta V}{\delta q^a(x)}\int d^3x' H_{cx'}-\frac{1}{V}\int d^3x'\frac{1}{U^2}\frac{\delta U}{\delta q^a(x)}\beta\kappa_0^4 G^{bc}p_b p_c+\frac{1}{V}\int d^3x'\frac{\beta\kappa_0^4}{U}\frac{\delta}{\delta q^a(x)}G^{bc}p_b p_c\right]\ . \tag{88}$$

Applying Eq. (86)

$$\frac{dp_a(x)}{d\tau}=-\int d^3x'\frac{1}{UV}\left[-\frac{\delta\sqrt{g}}{\delta q^a(x)}U\mathcal{E}-\frac{\delta U}{\delta q^a(x)}\sqrt{g}\ \mathcal{E}+\beta\kappa_0^4\frac{\delta}{\delta q^a(x)}G^{bc}p_b p_c\right]$$
$$=-\int d^3x'\frac{\beta\kappa_0^4}{UV\kappa^2}\frac{\delta}{\delta q^a(x)}\left[\kappa^2 G^{bc}p_b p_c+\frac{1}{\kappa^2}\sqrt{g}\ U\right]$$
$$=-\int d^3x'\frac{\beta\kappa_0^4}{UV\kappa^2}\frac{\delta}{\delta q^a(x)}\mathcal{H}_{x'}(\kappa^2)\ . \tag{89}$$

The evolution parameter $\tau$ has units of mass$\times$length. To give it conventional units, rescale $\tau\to t=\tau/M_{\text{Planck}}$, and define

$$F_x[q]\equiv\frac{M_{\text{Planck}}\beta\kappa_0^4}{U(x)V\kappa^2}\ . \tag{90}$$

Then the full set of gravitational field equations derived from $H_c$ plus the constraints is

$$\frac{dq^a(x)}{dt}=M_{\text{Planck}}\frac{\delta H_c}{\delta p_a(x)}$$
$$=\int d^3x' F_{x'}[q]\frac{\delta}{\delta p_a(x)}\mathcal{H}_{x'}(\kappa^2)\ ,$$
$$\frac{dp_a}{dt}=-M_{\text{Planck}}\frac{\delta H_c}{\delta q^a(x)} \tag{91}$$
$$=-\int d^3x' F_{x'}[q]\frac{\delta}{\delta q^a(x)}\mathcal{H}_{x'}(\kappa^2)\ ,$$
$$\mathcal{H}_x(\kappa^2)=0,\quad \mathcal{H}_x^i=0\ .$$

To see that this is classical Einstein gravity, let $\bar{q}^a(x^i,t)$ and $\bar{p}_a(x^i,t)$ be a solution of (91) for some set of initial data $\{q,p\}_0=\{g_{ij},p^{ij}\}_0$ compatible with the constraints. Then, as in the minisuperspace case, define

$$N(x,t)_{\{p,q\}_0}\equiv F_x[\bar{q}(t)] \tag{92}$$

and one sees that $\bar{q}^a(x^i,t)$ and $\bar{p}_a(x^i,t)$ is a solution of

$$\frac{dq^a(x)}{dt}=\frac{\delta H_{\text{Einstein}}}{\delta p_a(x)}\ ,$$
$$\frac{dp_a(x)}{dt}=-\frac{\delta H_{\text{Einstein}}}{\delta q^a(x)}\ ,$$
$$H_{\text{Einstein}}=\int d^3x\,N(x,t)_{\{p,q\}_0}\mathcal{H}_x(\kappa^2)\ , \tag{93}$$
$$\mathcal{H}_x(\kappa^2)=0\ ,$$
$$\mathcal{H}_x^i=0\ ,$$

which are simply the Einstein field equations in Hamiltonian form, with lapse (92), which is in general different for each classical solution, and shift $N_i=0$.

## V. DISCUSSION

To summarize, given an initial "wave function of the Universe" $\Psi[q,\tau_0]$, it has been shown how to trace its subsequent evolution along a trajectory in the Hilbert space of physical states, with the trajectory parametrized by $\tau$. The $\tau$-evolution equation for quantum gravity, up to factor-ordering terms, is

$$i\hbar\partial_\tau\Psi[q,\tau]=H\Psi[q,\tau]$$
$$=-\hbar^2\frac{\beta\kappa_0^4}{V}\int d^3x\,U^{-1}G^{ab}\frac{\delta^2}{\delta q^a\delta q^b}\Psi[q,\tau]\ , \tag{94}$$

where the space of physical states is spanned by the

eigenstates

$$H\Phi_{\mathcal{E}}[q]=\mathcal{E}\Phi_{\mathcal{E}}[q] \tag{95}$$

each of which satisfies the constraint algebra

$$\begin{aligned} &\mathcal{H}_x(\kappa^2)\Phi_{\mathcal{E}}=0 \ , \\ &\mathcal{H}_x^i\Phi_{\mathcal{E}}=0 \ , \\ &\kappa^4=-\frac{\beta}{\mathcal{E}}\kappa_0^4 \ , \end{aligned} \tag{96}$$

and where $\mathcal{H}_x(\kappa^2)$ denotes the super-Hamiltonian constraint operator, with Newton's constant $G_N=\kappa^2/16\pi$. Modulo operator-ordering issues, *every stationary physical state in the transfer matrix formulation satisfies the usual constraint algebra of general relativity, but each with a different value of Newton's constant, where $G_N^2$ is inversely proportional to the eigenvalue $\mathcal{E}$ of the evolution operator.* The number of degrees of freedom, in this formulation, is therefore $2\times\infty^3+1$; i.e., 2 degrees of freedom per point, which is the degrees of freedom of states satisfying the constraint algebra (96), and one extra degree of freedom corresponding to the Planck mass. This is only one more degree of freedom (overall, not per point) than in the standard formulations; there should be no danger of, e.g., the graviton acquiring a mass.

Thus, in the transfer matrix formulation, time evolution of states is recovered at a modest price: the Planck mass (inverse Newton's constant) becomes a dynamical quantity, analogous to energy in nonrelativistic quantum mechanics. As is the case for the time parameter in nonrelativistic quantum mechanics, or in quantum field theory on a fixed spacetime background, the evolution parameter $\tau$ is *only* an evolution parameter; it is not an observable, and there is no operator acting on the Hilbert space which corresponds to $\tau$. This avoids the problems encountered in relativistic quantum mechanics, as well in standard formulations of canonical quantum gravity, where the approach is to identify one of the operators in the theory as an evolution parameter.

We note that the spectrum of the evolution operator $H$ in (94) is unbounded from below; this is true of all the systems considered in this paper. For closed systems (such as the Universe) which do not interact with anything external, this absence of a ground state is not a problem. The eigenvalue $\mathcal{E}$ is a constant of motion, and its distribution cannot change.

In nonrelativistic quantum mechanics, any nonstationary state has a certain dispersion in its energy. For quantum gravity, the corresponding statement is that since the Universe is nonstationary, there must be a certain dispersion in $\mathcal{E}$, the eigenvalue of the evolution operator. This implies dispersion in the Planck mass $M_{\text{Planck}}$, or, equivalently, Newton's constant. Since $M_{\text{Planck}}^4 \propto -\mathcal{E}$ is conserved by the evolution equation, the fractional dispersion of Newton's constant $\Delta G_N/G_N$ should be a dimensionless constant of nature. Depending on how large this dispersion is, there could conceivably be observational consequences. I hope to return to this question at a later time.

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