No new symmetries of the vacuum Einstein equations

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We examine some recently proposed solutions of the linearized vacuum Einstein equations. We show that such solutions are *not* symmetries of the Einstein equations, because of a crucial integrability condition.

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I. INTRODUCTION

There is a conjecture that if a nonlinear system of partial differential equations (PDE's) possesses at least one nontrivial symmetry, then the system is integrable. A symmetry is a map from *any* solution to another solution. Nontrivial means that it should not be pure gauge. The converse is also thought to hold true. If a system admits no nontrivial symmetries, then it is conjectured it will be nonintegrable. To my knowledge, the evidence in support of these conjectures is only empirical.

A naive way to picture symmetries is to describe them as curves $q(\lambda)$ in solution space, where λ is an arbitrary parameter. A way to find $q(\lambda)$ is to note that $\dot{q}:=dq/d\lambda|_{\lambda=0}$ is a solution of the linearized system of PDE's. If one can find a nontrivial solution of the linearized PDE's system $\dot{q}$ for an *arbitrary* background solution of the PDE's system, and one can determine its integral curves, i.e., $q(\lambda)$, this yields a nontrivial symmetry. If one finds more than one symmetry, then one can explore their group structure and gain considerable insight in the structure of the system of PDE's. This method has been useful in many integrable systems, e.g., the Korteweg–de Vries (KdV) equation. Recently, this type of analysis has been applied to the vacuum Einstein equations with contradictory outcomes.

Torre and Anderson have argued that the *only* symmetries of the vacuum Einstein equations are trivial [1]. They are either constant rescalings of the metric or what Torre and Anderson call "generalized diffeomorphisms." This result suggests that the Einstein equations are *not* integrable.

On the other hand, Gürses has recently produced three sets of solutions of the linearized vacuum Einstein equations that he claims are "new symmetries" of the vacuum Einstein equations [2]. However, Gürses himself points out that the first one, type (a), is pure gauge, and corresponds to a local tetrad rotation. Hauser and Ernst have shown that the second solution of the linearized system, type (b), is an infinitesimal diffeomorphism, up to a local tetrad rotation [3]. In this work I show that the third one, type (c), is *not* a symmetry because of the existence of an integrability condition which restricts severely the background exact solution about which one is linearizing.

This work is organized as follows. In Sec. II, I briefly recall the formulation of the Einstein equations in the spinorial version of the first-order tetrad formulation. The linearized vacuum Einstein equations are given in Sec. III. In Sec. IV, I describe the form of infinitesimal variations of the tetrad that are pure gauge, either tetrad rotations, or constant rescalings, or infinitesimal diffeomorphisms. In Sec. V, I consider the ansatz that leads to Gürses type (c) solutions of the linearized vacuum Einstein equations, and to its generalization by Ernst and Hauser [4]. It is shown here the existence of an integrability condition that implies a restriction on the background solution, and thus that Gürses type (c) solutions are not symmetries of the full vacuum Einstein equations.

II. VACUUM EINSTEIN EQUATIONS

My conventions are the same as in Penrose and Rindler [5] (see also [6]). Lower case Latin letters denote space-time indices. Upper case Latin letters denote $SL(2,C)$ indices. They are raised and lowered using the antisymmetric symbol ϵ_{AB}, and its inverse, according to the rules $\lambda^A=\epsilon^{AB}\lambda_B$, $\lambda_A=\lambda^B\epsilon_{BA}$.

In a first-order formalism, the Einstein vacuum equations may be written as

$$D\theta^{AA'}:=d\theta^{AA'}+\Gamma^{AB}\wedge\theta_B{}^{A'}+\Gamma^{A'B'}\wedge\theta^A{}_{B'}=0\ , \tag{1a}$$

$$R_{AB}\wedge\theta^B{}_{A'}=0\ . \tag{1b}$$

The space-time metric is given by the (symmetrized) tensor product of two tetrad one-forms $\theta^{AA'}$:

$$g_{ab}=\theta_a^{AA'}\theta_{bAA'}\ . \tag{2}$$

The reality and the signature of the Lorentzian metric may be imposed at the level of the tetrad by requiring that $\theta_a^{AA'}=\theta_a^{AA'}=\bar{\theta}_a^{AA'}$. For the conditions that give the other signatures, see [5]. Note that complex conjugation interchanges primed and unprimed indices.

The connection one-forms $\Gamma^{AB}=\Gamma^{(AB)}$ and $\Gamma^{A'B'}=\Gamma^{(A'B')}$ are, respectively, the anti-self-dual part and the self-dual part of the spin connection compatible with $\theta^{AA'}$. Note that (1a) says that the torsion is zero. Their

*Electronic address: capo@fis.cinvestar.mx curvature two-forms R_{AB} and $R_{A'B'}$ are defined by

$$R_{AB}=d\Gamma_{AB}+\Gamma_A{}^C\wedge\Gamma_{CB} , \tag{3a}$$

$$R_{A'B'}=d\Gamma_{A'B'}+\Gamma_{A'}{}^{C'}\wedge\Gamma_{C'B'} . \tag{3b}$$

They are the anti-self-dual and self-dual parts, respectively, of the Riemannian curvature tensor constructed with g_{ab}, when (1a) is satisfied.

With the torsion-free condition (1a) satisfied, (1b) is the vacuum Einstein equation. It implies that g_{ab} given by (2) is Ricci flat. An equivalent way to write (1b) is

$$R_{AB}=\psi_{ABCD}\Sigma^{CD} , \tag{4}$$

where $\psi_{ABCD}=\psi_{(ABCD)}$ is the Weyl spinor, and the two-form $\Sigma^{AB}=\Sigma^{(AB)}$ is defined by

$$\theta^{AA'}\wedge\theta^{BB'}=\epsilon^{A'B'}\Sigma^{AB}+\epsilon^{AB}\Sigma^{A'B'} , \tag{5}$$

i.e., Σ^{AB} and $\Sigma^{A'B'}$ form a basis for the space of anti-self-dual and self-dual two-forms, respectively. The cyclic Bianchi identities are given by

$$D^2\theta^{AA'}=R_A{}^C\wedge\theta_{CA'}+R_{A'}{}^{C'}\wedge\theta_{AC'}=0 , \tag{6}$$

and the differential Bianchi identities by

$$DR_{AB}=0 , \tag{7a}$$

$$DR_{A'B'}=0 . \tag{7b}$$

III. LINEARIZED VACUUM EINSTEIN EQUATIONS

Consider now a one-parameter solution of the vacuum Einstein equations (1):

$$\theta^{AA'}(\lambda) , \quad \Gamma^{AB}(\lambda) , \quad \Gamma^{A'B'}(\lambda) .$$

Let $\dot\theta^{AA'}$ denote the derivative of $\theta^{AA'}(\lambda)$ with respect to λ evaluated at $\lambda=0$, and similarly for Γ^{AB} and $\Gamma^{A'B'}$. The linearized vacuum Einstein equations are then given by

$$D\dot\theta^{AA'}+\dot\Gamma^{AB}\wedge\theta_B{}^{A'}+\dot\Gamma^{A'B'}\wedge\theta^A{}_{B'}=0 , \tag{8a}$$

$$D\dot\Gamma_{AB}\wedge\theta^B{}_{A'}+R_{AB}\wedge\dot\theta^B{}_{A'}=0 . \tag{8b}$$

The quantities without overdots are evaluated at $\lambda=0$ on an arbitrary solution of the full vacuum Einstein equations (1). Note that (8b) may be rewritten in the equivalent way

$$D[\dot\Gamma_{AB}\wedge\theta^B{}_{A'}]+R_{AB}\wedge\dot\theta^B{}_{A'}=0 , \tag{8b'}$$

since $D\theta^{AA'}=0$.

IV. TRIVIAL DEFORMATIONS

In this section I describe the explicit form of deformations of the tetrad that are (locally) pure gauge. For concreteness, I consider solutions of the linearized vacuum Einstein equations. Note that the linearized Einstein equations enter only in the form of the deformation of the connection.

An infinitesimal SL(2,C) rotation is given by

$$\dot\theta^{AA'}=X^A{}_B\theta^{BA'}+\bar X^{A'}{}_{B'}\theta^{AB'} , \tag{9a}$$

$$\dot\Gamma_{AB}=DX_{AB} , \tag{9b}$$

$$\dot\Gamma_{A'B'}=D\bar X_{A'B'} , \tag{9c}$$

where $X^{AB}=X^{(AB)}$ is an arbitrary symmetric matrix, and $\bar X^{A'B'}=\overline{X^{AB}}$. With the help of the identity $D^2X_{AB}=2R^C{}_{(A}X_{B)C}$ and its primed version, one can verify that (9) solve (8).

The space-time metric is left invariant since

$$\begin{aligned}\dot g_{ab}&=2\dot\theta_{(a}{}^{AA'}\theta_{b)AA'}\\&=2X^A{}_C\theta_{(a}{}^{CA'}\theta_{b)AA'}+2\bar X^{A'}{}_{C'}\theta_{(a}{}^{AC'}\theta_{b)AA'}\\&=X_{AC}\epsilon^{AC}\theta_a{}^{DA'}\theta_{bDA'}+\bar X_{A'C'}\epsilon^{A'C'}\theta_a{}^{AD'}\theta_{bAD'}=0 .\end{aligned}$$

A constant rescaling of the space-time metric is generated by

$$\dot\theta^{AA'}=(c+\bar c)\theta^{AA'} , \tag{10a}$$

$$\dot\Gamma_{AB}=0 , \tag{10b}$$

$$\dot\Gamma_{A'B'}=0 , \tag{10c}$$

with c a constant.

An infinitesimal diffeomorphism may be generated by

$$\dot\theta^{AA'}=DA^{AA'}, \tag{11a}$$

$$\dot\Gamma_A{}^C\wedge\theta_{CA'}=R_{AC}A^C{}_{A'} , \tag{11b}$$

$$\dot\Gamma_{A'}{}^{C'}\wedge\theta_{AC'}=R_{A'C'}A_A{}^{C'} , \tag{11c}$$

with $A^{AA'}$ an arbitrary matrix. This solution was called type (b) by Gürses. (This solution was also considered by Pagels in a different context [7]).

To see that this solution corresponds, up to an SL(2,C) rotation, to an infinitesimal diffeomorphism, I reproduce here an argument due to Hauser and Ernst [3]. Consider the Lie derivative L_V of the tetrad along a vector field V:

$$\begin{aligned}L_V\theta^{AA'}&=i_Vd\theta^{AA'}+d(i_V\theta^{AA'})\\&=-i_V(\Gamma^{AC}\wedge\theta_C{}^{A'}+\Gamma^{A'C'}\wedge\theta^A{}_{C'})+d(i_V\theta^{AA'})\\&=-(i_V\Gamma^{AC})\theta_C{}^{A'}+\Gamma^{AC}(i_V\theta_C{}^{A'})-(i_V\Gamma^{A'C'})\theta^A{}_{C'}+\Gamma^{A'C'}(i_V\theta^A{}_{C'})+d(i_V\theta^{AA'})\\&=DX^{AA'}-Y^{AC}\theta_C{}^{A'}-Y^{A'C'}\theta^A{}_{C'} ,\end{aligned}$$

where I have set $X^{AA'}=i_V\theta^{AA'}$, and $Y^{AC}=i_V\Gamma^{AB}$. $i_V\alpha$ denotes the contracted multiplication, or interior product, of a vector field V with an arbitrary differential form α. In the third equality I have used the identity $i_V(\alpha\wedge\beta)=(i_V\alpha)\beta-\alpha(i_V\beta)$. Hence, a deformation of the type (11a) is simply the sum of an infinitesimal diffeomorphism generated by the vector field defined by $A^{AA'}=i_V\theta^{AA'}$, and a local tetrad rotation of the form (9a).

V. GÜRSES TYPE (C) SOLUTIONS OF THE LINEARIZED EINSTEIN VACUUM EQUATIONS

The ansatz proposed by Gürses for his type (c) solutions [2], and further generalized by Ernst and Hauser [4], is of the form

$$\dot{\theta}^{AA'}=h^{AA'}+\bar{h}^{AA'}\ , \tag{12a}$$

$$\dot{\Gamma}^{AC}\wedge\theta_C{}^{A'}=-Dh^{AA'}\ , \tag{12b}$$

$$\dot{\Gamma}^{A'C'}\wedge\theta^A{}_{C'}=-D\bar{h}^{AA'}\ . \tag{12c}$$

This ansatz specifies the dependence of the deformation of the spin connection on the deformation of the tetrad. It is just one way to solve the first linearized equation (8a).

There is a problem with it, however. The definition for $\dot{\Gamma}^{AB}$ is only implicit. There are 12 components in Γ^{AB}. There are 24 equations in (12b). Thus, there must be 12 additional conditions, which arise because $\dot{\Gamma}^{AC}\wedge\theta_C{}^{(A'}\wedge\theta_A{}^{B')}=0$. In terms of the right-hand side of (12b), the conditions are (see Ref. [4] for a different way to write these conditions)

$$D[h^{A(A'}\wedge\theta_A{}^{B')}]=0\ , \tag{13a}$$

similarly

$$D[\bar{h}^{(A|A'|}\wedge\theta^{B)}{}_{A'}]=0\ . \tag{13b}$$

Now, the one-forms $h^{AA'}$ and $\bar{h}^{AA'}$ may be expanded with respect to the tetrad as follows:

$$h^{AA'}=h^{ABA'B'}\theta_{BB'}\ , \tag{14a}$$

$$\bar{h}^{AA'}=\bar{h}^{ABA'B'}\theta_{BB'}\ . \tag{14b}$$

Since the antisymmetric parts give SL(2,C) rotations, or conformal scalings, as shown in the previous section, I can assume that

$$h^{ABA'B'}=h^{(AB)A'B'}=h^{AB(A'B')}\ , \tag{15a}$$

$$\bar{h}^{ABA'B'}=\bar{h}^{(AB)A'B'}=\bar{h}^{AB(A'B')}\ . \tag{15b}$$

The differential conditions (13) may then be written in the form

$$Dh^{ABA'B'}\wedge\Sigma_{AB}=0\ , \tag{16a}$$

$$D\bar{h}^{ABA'B'}\wedge\Sigma_{A'B'}=0\ . \tag{16b}$$

Note that their integrability conditions, obtained by taking the exterior covariant derivative, are satisfied automatically, using the background field equations. I will return to these conditions below.

Consider now the second linearized equation (8b′), which becomes

$$-D^2h_{AA'}+R_A{}^C\wedge h_{CA'}+R_A{}^C\wedge\bar{h}_{CA'}=0\ .$$

Developing the D^2 yields

$$R_A{}^C\wedge\bar{h}_{CA'}-R_{A'}{}^{C'}\wedge h_{AC'}=0\ .$$

Using the expansion (14), wedging with a tetrad $\theta_{BB'}$, and using the vacuum Einstein equations in the form (4), one arrives at

$$\psi_{ABCD}\bar{h}^{CD}{}_{A'B'}+\psi_{A'B'C'D'}h_{AB}{}^{C'D'}=0\ . \tag{17}$$

The second linearized equation ($8b'$) has been put in the form of an algebraic equation to be solved for $h,\bar{h}$ with respect to the Weyl spinors.

A possible solution of (17) is given by the vanishing, separately, of the two terms. This implies that the Weyl spinors and $h,\bar{h}$ are degenerate as three by three matrices, but this gives an unwanted restriction on the background.

Another possible solution is given by

$$h^{ABA'B'}=i\psi^{ABCD}B_{CD}{}^{A'B'}\ , \tag{18}$$

with $B^{ABC'D'}=\bar{B}^{ABC'D'}$. This solution was given as an ansatz by Ernst and Hauser in [4]. Gürses type (c) symmetries are of this form, with $B_{AB}{}^{C'D'}=A_A{}^{C'}\bar{A}_B{}^{D'}$, for some matrix $A_A{}^{C'}$.

At this point, I return to the differential conditions (16). For a solution of the type (18), they take the form

$$D\psi^{ABCD}B_{CD}{}^{A'B'}\wedge\Sigma_{AB}=\psi^{ABCD}DB_{CD}{}^{A'B'}\wedge\Sigma_{AB}=0\ , \tag{19}$$

together with its complex conjugate. (The differential Bianchi identities have been used in the first equality.)

For an arbitrary background solution, the Weyl spinors are arbitrary. Therefore, the vanishing of (19) implies

$$\Sigma_{(AB}\wedge DB_{CD)A'B'}=0 \tag{20}$$

with its complex conjugate.

Consider now the integrability conditions of (20), obtaining by taking its covariant exterior derivative. This gives

$$\psi_{(ABC}{}^EB_{D)E}{}^{A'B'}=0\ . \tag{21}$$

These are 15 algebraic equations on the 9 components of $B_{AB}{}^{A'B'}$. Therefore, for an arbitrary background solution, $B_{AB}{}^{A'B'}$ will have to vanish.

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