Why is spacetime Lorentzian?

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We expand on the idea that the spacetime signature should be treated as a dynamical degree of freedom in quantum field theory. It has been argued that the probability distribution for the signature, induced by massless free fields, is peaked at the Lorentzian value uniquely in $D=4$ dimensions. This argument is reviewed, and certain consistency constraints on the generalized signature (i.e., the tangent-space metric $\eta_{ab}(x)=\mathrm{diag}[e^{i\theta(x)},1,1,1]$) are derived. It is shown that only one dynamical "Wick angle" $\theta(x)$ can be introduced in the generalized signature, and the magnitude of fluctuations away from the Lorentzian signature $\delta\theta=\pi-\theta$ is estimated to be of order $(l_P/R)^3$, where l_P is the Planck length, and R is the length scale of the Universe. For massless fields, the case of $D=2$ dimensions and the case of supersymmetry are degenerate, in the sense that no signature is preferred. Mass effects lift this degeneracy, and we show that a dynamical origin of the Lorentzian signature is also possible for (broken) supersymmetry theories in $D=6$ dimensions, in addition to the more general nonsupersymmetric case in $D=4$ dimensions.

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I. INTRODUCTION

A theorem of matrix algebra states that any real symmetric matrix M can be written in the form $M=SDS^T$, where S is a real-valued matrix and D is a diagonal matrix with values ±1 and 0 along the diagonal. These diagonal entries are known as the "signature" of the matrix M, and are unique up to permutations. The metric of general relativity is normally taken to be a real symmetric matrix, and can therefore be written in the form $g_{\mu\nu}=e^a_\mu \eta_{ab} e^b_\nu$, where η_{ab} is the diagonal tangent-space metric. It has been known since the work of Minkowski that physical spacetime has a Lorentzian signature $\eta=\mathrm{diag}[-1,1,1,1]$.

The Einstein field equations $G_{\mu\nu}=-\kappa T_{\mu\nu}$ do not, however, impose any particular restriction on spacetime signature; in fact, they do not refer to signature at all. There is nothing inherent in classical general relativity which either fixes the spacetime signature to be Lorentzian, or even, given that the signature is initially Lorentzian, forces spacetime in all cases to remain Lorentzian. In this connection, several authors [1–3] have constructed solutions to the Einstein equations which evolve from Euclidean to Lorentzian signature. If signature-changing processes can occur classically, then they can presumably also occur quantum mechanically (in fact, such speculations are not uncommon in quantum cosmology, see, e.g., [1,4,5]). This then raises the question of why it is, if other signatures are dynamically accessible, that spacetime is found to be everywhere Lorentzian.

An explanation of the origin of the Lorentzian signature at the quantum level could take several forms. The simplest, and in our opinion the least satisfying, is to simply assume the existence of a constraint such as $\det(g)<0$ in the functional integration measure (this can also be done in tetrad formulation by imposing a fixed η_{ab}). Another possibility is that for some reason (perhaps the absence of certain anomalies), the Lorentzian signature is the only consistent choice at the quantum level, as may be the case in the string theory [6]. Finally, there could be dynamical reasons why the Lorentzian signature is preferred over other signatures.

In a recent paper [7] one of us suggested a dynamical origin for the Lorentzian signature; the idea is to generalize the concept of Wick rotation in path-integral quantization. Rather than viewing Wick rotation as a technicality necessary for convergence of the path integral, the Wick angle θ is treated as a dynamical degree of freedom, which is free to fluctuate. The tangent space metric then has the form

$$\eta=\mathrm{diag}[\exp(i\theta),1,\ldots,1] . \tag{1}$$

In Ref. [7] the one-loop (complex-valued) effective potential $V(\theta)$, generated by massless fields, was calculated. It was found that if the number of fermionic degrees of freedom exceeds the number of bosonic degrees of freedom, then $\mathrm{Re}[V]$ has a minimum and $\mathrm{Im}[V]$ is stationary, uniquely in $D=4$ dimensions, at $\theta=\pm\pi$, corresponding to Lorentzian signature. In this way a relation was found between the dimension of spacetime, the signature of spacetime, and the presence of the factor of i in the path amplitude $\exp(iS)$.

This paper expands further on the idea of dynamical signature. The results of Ref. [7], in a flat background space, are reviewed in Sec. II, and a quantum evolution

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equation in non-Lorentzian spacetime is proposed. Consistency conditions in curved spacetime are discussed in Sec. III. On the grounds that (i) the tangent space metric $\eta_{\mu\nu}$ is flat, (ii) the number of gravitational degrees of freedom is independent of $\eta_{\mu\nu}$, and (iii) a spin connection with appropriate properties is obtained in the Dirac action, certain strong constraints on the functional dependence of the Wick angle are deduced. These constraints turn out to be crucial in suppressing what would otherwise be unacceptably large quantum fluctuations away from Lorentzian signature. It is also shown that it is only possible to have a single dynamical Wick angle satisfying the constraints; a tangent-space metric with multiple angles,

$$\eta=\mathrm{diag}[\exp(i\theta_1),\exp(i\theta_2),\ldots,\exp(i\theta_D)]\ , \tag{2}$$

is ruled out. In Sec. IV it is shown that the cosmological constant at one-loop cannot be subtracted by a counterterm for all values of θ; in fact, if the Wick angle is dynamical, the cancellation can only be made in $D=4$ dimensions at $\theta=\pm\pi$. In Sec. V we extend the results of Ref. [7] by including mass terms for the fermionic and bosonic fields. Again requiring a minimum/stationarity condition for the one-loop effective potential $V(\theta)$ we show that, in addition to the case of $D=4$ dimensions found previously, there is also a possible solution for (broken) supersymmetric theories, at $\theta=\pm\pi$ and $D=6$. Section VI contains the conclusions.

II. THE DYNAMICAL WICK ANGLE

In the path-integral formulation of quantum field theory, it is required to evaluate Feynman path integrals of the form

$$Z_F=\int d\mu(e,\phi,\psi,\bar\psi)\exp\left[-i\int d^Dx\sqrt{-g}\,\mathcal{L}\right]\ , \tag{3}$$

where $d\mu(e,\phi,\psi,\bar\psi)$ is the integration measure for the tetrads, and other bosonic (ϕ) and fermionic ($\psi,\bar\psi$) fields. The restriction to Lorentzian spacetime is enforced by working with a fixed signature

$$\begin{aligned} g_{\mu\nu}&=e^a_\mu\eta_{ab}e^b_\nu\ ,\\ \eta_{ab}&=\mathrm{diag}[-1,1,\ldots,1]\ , \end{aligned} \tag{4}$$

and in the case of a flat background, one simply sets $g_{\mu\nu}=\eta_{\mu\nu}$. However, in order to define propagators and other correlators, it is necessary to improve the convergence properties of the Feynman amplitude e^{iS}. Note that even a zero-dimensional Gaussian integral

$$\int_{-\infty}^{\infty}dx\,x^{2n}e^{ix^2} \tag{5}$$

does not converge, when evaluated numerically, for $n\geq1$. Convergence can be improved by either adding a small imaginary mass term (the $i\epsilon$ prescription), or else by rotating the time axis into the complex plane. Rotating $t\to it$ gives the Euclidean path integral

$$Z_E=\int d\mu(e,\phi,\psi,\bar\psi)\exp\left[-\int d^Dx\sqrt{g}\,\mathcal{L}\right]\ , \tag{6}$$

where this time

$$\eta_{ab}=\mathrm{diag}[1,1,\ldots,1]\ . \tag{7}$$

Comparing the Feynman and Euclidean path integrals, it is easy to write down a path integral which interpolates between them: namely,

$$Z=\int d\mu(e,\phi,\psi,\bar\psi)\exp\left[-\int d^Dx\sqrt{g}\,\mathcal{L}\right]\ , \tag{8}$$

where

$$\eta_{ab}=\mathrm{diag}[e^{i\theta},1,\ldots,1]\ . \tag{9}$$

The Euclidean theory is obtained for $\theta=0$ and the Feynman theory for $\theta=\pi$, with the correct $i\epsilon$ prescription for propagators automatically supplied as $\theta\to\pi$.

Motivated by Lorentzian to Euclidean signature change at the classical level, we now consider the possibility that the "signature" of Eq. (9) is free to fluctuate; i.e., that θ is a dynamical degree of freedom.[1] This requires, of course, some generalization of quantum mechanics. Consider a fixed Wick angle θ anywhere in the range $-\pi<\theta<\pi$ (note that $|\theta|>\pi$ is ruled out because the kinetic term in the bosonic field action would be unbounded from below). Assuming a flat-space ($e^a_\mu=\delta^a_\mu$) background and denoting the fields collectively by ϕ, the path-integral definition of transition amplitudes is

$$G[\phi_f,t_f|\phi_i,t_i]\equiv\int_{\phi_i}^{\phi_f}d\mu(\phi)\exp\left[-\int_{t_i}^{t_f}dt\int d^{D-1}x\sqrt{g}\,\mathcal{L}\right] \tag{10}$$

and we obtain, by the usual arguments, the generalized Schrödinger equation

$$\partial_t\Psi[\phi]=-e^{i\theta/2}H\Psi[\phi]\ , \tag{11}$$

where H is the standard (and Hermitian) Hamiltonian. For any $\theta\neq\pm\pi$ the norm of Ψ can change. Therefore, to conserve probability, Ψ must be interpreted as supplying *relative* probabilities, or, equivalently,

$$\langle Q\rangle\equiv\frac{\langle\Psi|Q|\Psi\rangle}{\langle\Psi|\Psi\rangle}\ . \tag{12}$$

Equations (11) and (12) together give

$$\begin{aligned}\partial_t\langle Q\rangle&=\sin\frac{\theta}{2}\langle i[H,Q]\rangle\\ &\quad-\cos\frac{\theta}{2}(\langle HQ+QH\rangle-2\langle Q\rangle\langle H\rangle)\ .\end{aligned} \tag{13}$$

Providing Q is Hermitian, this evolution equation preserves the reality of observables, and satisfies conservation of probability.

On the other hand, for $\theta\neq\pm\pi$, conservation of energy is violated,

$$\partial_t\langle H\rangle=-2\cos\frac{\theta}{2}\langle(H-\langle H\rangle)^2\rangle \tag{14}$$

(along with Lorentz invariance), and an arbitrary initial state Ψ_{in} will eventually relax either to the ground state

[1]We will continue to refer to the (complex) entries of η as the "signature," although this is admittedly an abuse of the mathematical terminology.

Ψ_0, or else to the lowest-energy eigenstate Ψ_E for which $\langle \Psi_{in}|\Psi_E\rangle \neq 0$. There are, of course, very stringent observational limits on nonconservation of energy; see, e.g., Ref. [8]. The first problem, for a theory in which the Wick angle is allowed to fluctuate, is to show that the probability distribution is peaked at Lorentzian signature $\theta=\pi$. The next problem is to show that fluctuations away from Lorentzian signature are so strongly suppressed that observational bounds on energy conservation are not violated.

To study the first problem we need to compute the effective potential $V_{\text{eff}}(\theta)$ for the Wick angle, which is generated after integrating out all other fields. In Ref. [7] this was computed for massless fields at one-loop level. The calculation requires some assumptions about the θ dependence of the integration measure, which is otherwise just taken proportional to the (real-valued) DeWitt measure. The following assumptions were made:

(1) For free fields of mass m, the contributions to Z in Eq. (8) from each (propagating) bosonic degree of freedom are equal, and inverse to the contribution from each fermionic degree of freedom. Thus, e.g., $Z=1$ at any θ for a supersymmetric combination of free fields.

(2) The integration measure for the scalar fields is given by the real valued, invariant volume measure (DeWitt measure) in superspace $d\mu(\phi)=D\phi\sqrt{|G|}$, where G is the determinant of the scalar field supermetric $G(x,y)=\sqrt{g}\,\delta(x-y)$.

Under these assumptions, the one-loop contribution to $V_{\text{eff}}(\theta)$ due to a massless scalar field propagating in flat ($g_{\mu\nu}=\eta_{\mu\nu}$) space is

$$\exp\left[-\int d^Dx\, V_S(\theta)\right]=\det{}^{-1/2}(-\sqrt{\eta}\,\eta^{ab}\partial_a\partial_b) \qquad (15)$$

and heat-kernel regulation of the determinant gives

$$V_S(\theta)=-\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\int\frac{d^Dp}{(2\pi)^D}\times\exp[-s(e^{-i\theta/2}p_0^2+e^{i\theta/2}\mathbf{p}^2)]$$
$$=-\frac{\Lambda^D}{D(4\pi)^{D/2}}\exp[-i(D-2)\theta/4]\ , \qquad (16)$$

where Λ is a high-momentum cutoff which, given the nonrenormalizability of gravity, is taken to exist at the Planck scale. For our purposes, the choice of heat-kernel regularization is essentially unique. ζ-function and dimensional regulation methods contain implicit subtractions which remove nonlogarithmically (e.g., quadratically) divergent terms; these happen to be the terms of interest here. On the other hand, a naive momentum-space cutoff does not uniformly respect the spacetime symmetries at $\theta=0,\pm\pi$. A cutoff such as $k_0^2+\mathbf{k}^2<\Lambda^2$, which is appropriate for the Euclidean case at $\theta=0$, is clearly asymmetric at $\theta=\pm\pi$, and the reverse is true, for example, for $|k_0^2-\mathbf{k}^2|<\Lambda^2$. The same objection applies to a lattice cutoff; moreover, a regular lattice, even at $\theta=0$, does not respect the full O(D) symmetry. We are looking for a regulator which respects the symmetries at $\theta=0,\pm\pi$, and which interpolates smoothly in the range $\theta\in[-\pi,\pi]$. With these requirements, the choice of heat-kernel regularization seems almost unavoidable. In connection with the assumptions about the measure, it is worth noting that these lead, for any spin, to a contribution which can be regulated at all $\theta\in[-\pi,\pi]$ by the heat kernel technique.

For n_B massless, propagating, bosonic degrees of freedom, and n_F massless fermionic degrees of freedom, the one-loop contribution to $V_{\text{eff}}(\theta)$ becomes

$$V(\theta)=(n_F-n_B)\frac{\Lambda^D}{D(4\pi)^{D/2}}\exp[-i(D-2)\theta/4]\ . \qquad (17)$$

This potential is complex. We therefore look for a value of θ in the range $[-\pi,\pi]$ for which, simultaneously, (i) Re(V) is a minimum and (ii) Im(V) is stationary. These conditions together give us

$$\cos[(D-2)\theta/4]=0\ ,$$
$$\min\{\text{Re}[V(\theta)]\}=0\ ,\quad \theta\in[-\pi,\pi]\ . \qquad (18)$$

In searching for a solution of (18), there are five cases to consider.

(I) $n_F<n_B$. Then min Re[V]$<0\rightarrow$no solution.

(II) $n_F=n_B$ or $D=2$. Then $V(\theta)$ is independent of θ, and no θ is preferred.

(III) $n_F>n_B$ and $(D-2)\pi/4<\pi/2$. Then min Re[V]>0 $\rightarrow$no solution.

(IV) $n_F>n_B$ and $(D-2)\pi/4>\pi/2$. Then min Re[V]$<0\rightarrow$no solution.

(V) $n_F>n_B$ and $(D-2)\pi/4=\pi/2$. In this case, both conditions are satisfied at $\theta=\pm\pi$, which corresponds to Lorentzian signature. The equality $(D-2)\pi/4=\pi/2$ can, of course, only be achieved for a spacetime dimensionality $D=4$.

Since case (V), above, is the unique solution of the conditions (18), we have found an interesting connection between spacetime signature and spacetime dimension: Lorentzian signature seems to be singled out by the dynamics only in $D=4$ dimensions.

It is natural to look for generalizations. For example, just as the η of (9) interpolates between Euclidean and Lorentzian signature, one might consider metrics interpolating between a Lorentzian and a "two-time" signature, i.e.,

$$\eta_{ab}=\text{diag}[-1,e^{i\theta},1,\ldots,1]\ . \qquad (19)$$

However, it is easy to see that the kinetic term of a functional integral with such a signature is, for general $\theta\neq\pm\pi$, unbounded from below. On the other hand, one could instead consider tangent space metrics with two or more dynamical "Wick angles," e.g.,

$$\eta_{ab}=\text{diag}[e^{i\theta_1},e^{i\theta_2},\ldots,e^{i\theta_D}] \qquad (20)$$

with the $\{\theta_n\}$ suitably restricted to ensure the boundedness of the action. Finally, it is important to investigate the expected size of fluctuations away from Lorentzian signature. The magnitude of such fluctuations, which violate both Lorentz invariance and energy conservation, would have to be extremely small to be consistent with experiment. These issues will be discussed in the next section.

III. THE WICK ANGLE IN CURVED SPACETIME

$V(\theta)$ was computed above for constant θ. Before tackling the question of fluctuations away from $\theta=\pi$, we should ask whether there are any restrictions that should be imposed on the functional dependence of $\theta(x)$, apart from the condition that $|\theta|<\pi$. Since $\eta_{\mu\nu}$ is supposed to be a generalization of the flat space metric, it is reasonable to impose the condition that the Riemann tensor computed from $g_{\mu\nu}=\eta_{\mu\nu}$ vanishes. To put it another way, signature should not, by itself, generature curvature.[2] In Cartesian coordinates, with $e^a_\mu=\delta^a_\mu$, this means that θ depends only on the time coordinate $\theta=\theta(t)$. The obvious generalization of $\theta=\theta(t)$ to curved spacetime is

$$\partial_\mu\theta=fe^0_\mu\propto e^0_\mu \ . \tag{21}$$

This leads to a consistency condition

$$0=(\partial_\mu\partial_\nu-\partial_\nu\partial_\mu)\theta=\partial_\mu(fe^0_\nu)-\partial_\nu(fe^0_\mu) \ , \tag{22}$$

which imposes some extra constraints on e^0_μ. In fact, (22) is satisfied by

$$\begin{aligned}&\theta=\theta(T(x)) \ ,\\ &e^0_\mu=\partial_\mu T(x) \ .\end{aligned} \tag{23}$$

It will now be shown that these conditions on e^0_μ and θ are required by two other, quite different arguments, one of which concerns the number of degrees of freedom of the gravitational field.

In D dimensions at $\theta=0,\pm\pi$, the metric tensor $g_{\mu\nu}$ is a real symmetric matrix, and therefore has $D(D+1)/2$ degrees of freedom at each spacetime point, modulo diffeormorphisms. The metric can also be expressed in terms of vielbeins $g_{\mu\nu}=e^a_\mu\eta_{ab}e^b_\nu$, and the vielbeins have D^2 degrees of freedom (again, modulo diffeomorphisms). Naturally, the number of gravitational degrees of freedom should be the same, whether one counts metric or vielbein components. In fact, if η_{ab} is the Euclidean or Minkowski metric, one should subtract the dimension of the local Lorentz group [O(D) or O($D-1,1$)] from the number of vielbein degrees of freedom, to get the actual number of gravitational degrees of freedom. Since the number of group generators is $D(D-1)/2$, we have, for the inequivalent vierbein degrees of freedom,

$$\frac{D(D+1)}{2}=D^2-\frac{D(D-1)}{2} \ , \tag{24}$$

which is the same as the number of metric degrees of freedom.

However, for the generalized metric, the "local Lorentz" invariance is only O($D-1$). If the e^a_μ are unrestricted, then the independent vielbein degrees of freedom exceed $D(D+1)/2$ except at $\theta=0,\pm\pi$, where the number is abruptly reduced. Let us instead impose (23). Then e^0_μ contains only one degree of freedom, the e^i_μ ($i\neq0$) contain $D(D-1)$ degrees of freedom, and subtracting the dimension of the O($D-1$) group we have

$$\frac{D(D+1)}{2}=1+D(D-1)-\frac{(D-1)(D-2)}{2} \ , \tag{25}$$

which is the usual number of gravitational degrees of freedom, modulo diffeomorphisms. Thus we can impose (23) on the grounds that the dynamical Wick angle should not change the number of independent degrees of freedom of the gravitational field.

The final argument for the conditions (23) concerns fermionic fields in curved space. For the bosonic fields, the Lagrangian involves the signature only via the metric, while for Dirac fields, the signature also enters via the γ matrices, which in the tangent space should satisfy

$$\{\gamma^a,\gamma^b\}=-2\eta^{ab} \ . \tag{26}$$

The generalized Dirac action in curved space is just the usual Dirac action

$$S_D=\int d^Dx\sqrt{g}\ \bar{\psi}(-i\gamma^\mu D_\mu+m)\psi \tag{27}$$

with

$$\begin{aligned}&\gamma^\mu=e^\mu_a\gamma^a \ ,\\ &D_\mu=\partial_\mu+\tfrac{1}{2}\sigma^{ab}\omega_{\mu ab} \ ,\\ &\sigma^{ab}=\tfrac{1}{4}[\gamma^a,\gamma^b] \ ,\\ &\omega_{\mu ab}=e^\rho_a e_{b\rho;\mu} \ ,\end{aligned} \tag{28}$$

where the γ^a satisfy (26).

Equation (28) defines a spin-connection for covariant derivatives acting on spinors at arbitrary $\theta\in[-\pi,\pi]$. The question is whether those covariant derivatives have the expected properties. Of course, since even global frame invariance is broken at $\theta\neq0,\pm\pi$, we cannot demand that the spin-connection should enforce local Lorentz invariance for general θ. Certain other properties of the covariant derivative, however, are reasonable to require. Let us introduce a sort of "*ict*" notation

$$\begin{aligned}&g_{\mu\nu}=\tilde{e}^{\,a}_{\mu}\tilde{e}^{\,a}_{\nu} \ ,\\ &\tilde{e}^{\,a}_{\mu}=\begin{cases}e^{i\theta/2}e^0_\mu & (a=0) \ ,\\ e^a_\mu & (a\neq0) \ ,\end{cases}\\ &\{\gamma^a_E,\gamma^b_E\}=-2\delta^{ab} \ ,\end{aligned} \tag{29}$$

where the latin indices of $\tilde{e}$ and γ_E are raised and lowered with the Euclidean metric. In this notation, it is clear that the covariant derivative should have the property

$$\begin{aligned}0&=D_\mu g_{\alpha\beta}=D_\mu(\tilde{e}^{\,a}_{\alpha}\tilde{e}^{\,a}_{\beta})\\ &=(D_\mu\tilde{e}^{\,a}_{\alpha})\tilde{e}^{a}_{\beta}+\tilde{e}^{\,a}_{\alpha}(D_\mu\tilde{e}^{\,a}_{\beta}) \ ,\end{aligned} \tag{30}$$

which implies

$$D_\mu\tilde{e}^{\,a}_{\nu}=\tilde{e}^{\,a}_{\nu;\mu}+\tilde{\omega}^{\,a}_{\mu b}\tilde{e}^{\,b}_{\nu}=0 \tag{31}$$

and therefore

[2]Otherwise we would really be dealing with a fully complex general relativity, and we should consider complex general coordinate transformations, resulting in complex coordinates.

$$\tilde{\omega}_{\mu ab} = \tilde{e}^{\rho}_{a}\tilde{e}_{\rho b;\mu} \ . \tag{32}$$

The covariant derivative for spinor fields in "*ict*" notation would then be

$$D^{\text{"ict"}}_{\mu} = \partial_{\mu} + \tfrac{1}{2}\sigma^{ab}_{E}\tilde{\omega}_{\mu ab} \ , \qquad \sigma^{ab}_{E} \equiv \tfrac{1}{4}[\gamma^{a}_{E}, \gamma^{b}_{E}] \tag{33}$$

for arbitrary θ. It then turns out that the covariant derivative D_{μ} in Eq. (28) above, obtained simply by using generalized Dirac γ matrices in the Dirac action, and the covariant derivative $D^{\text{"ict"}}_{\mu}$ above, agree *only if conditions (23) are imposed.* In that case, it is easy to check that all derivatives of θ drop out of $\sigma^{ab}\omega_{\mu ab}$, in which case

$$\sigma^{ab}\omega_{\mu ab} = \sigma^{ab}_{E}\tilde{\omega}_{\mu ab} \tag{34}$$

and therefore $D_{\mu} = D^{\text{"ict"}}_{\mu}$. A further consequence is that D_{μ} commutes with γ^{μ}, and by Eqs. (26) and (34), and straightforward partial integration, one can verify that

$$S_D = \int d^4x \ \bar{\psi}(i\overleftarrow{D}_{\mu}\gamma^{\mu} + m)\psi\sqrt{g} \tag{35}$$

leading to the standard equation for $\bar{\psi}$. Similar considerations apply to the Weyl equation.

The conclusion is that there are three separate reasons for imposing the condition (23): namely, (1) to require that the metric $g_{\mu\nu} = \eta_{\mu\nu}$ is flat; (2) to ensure that the number of gravitational degrees of freedom (=inequivalent vielbein degrees of freedom) is independent of the Wick angle; (3) to obtain a covariant derivative for spinors with appropriate properties. These conditions, taken together, also rule out having more than one dynamical Wick angle in the tangent space metric, as in Eq. (20). The reason is that requirements (1) and (3), above, imply that $\partial_{\mu}\theta_a \propto e^{a}_{\mu}$. But then the number of inequivalent vielbein degrees of freedom would be *less* than $D(D+1)/2$, in violation of the second requirement.

However, Eq. (23) is a very severe restriction of $\theta(x)$; it means that rather than having one degree of freedom per point, which is characteristic of a field, $\theta(x) = \theta(T(x))$ has only one degree of freedom per T=const hypersurface, where the preferred time direction $\partial_{\mu}T$ is fixed by the choice of e^{0}_{μ}. Obviously, a variable which cannot vary locally is inimicable to the spirit of Lorentz invariance; but local Lorentz invariance is lost, in any case, for any[3] $\theta \neq 0, \pm\pi$. The whole argument of this paper is that Lorentz invariance can arise dynamically; it does not have to be imposed from the beginning.

We may now estimate the magnitude of fluctuations away from Lorentz signature, in flat ($e^{a}_{\mu} = \delta^{a}_{\mu}$) spacetime. It is again assumed that there is a high-frequency cutoff around the Planck scale, in which case there is roughly one degree of freedom per Planck time. Writing $\theta = \pi - \delta\theta$, the action for one Planck time (during which θ is approximately constant) is

$$\Delta S \sim \Lambda^4 V l_P \delta\theta \sim \frac{V}{l_P^3}\delta\theta \ , \tag{36}$$

where l_P is the Planck length, and V is the three-volume of the T=const hypersurface. Therefore

$$\langle \delta\theta \rangle \sim l_P^3/V \ . \tag{37}$$

Even under conservative assumptions, i.e., a closed Universe of length scale on the order of 10^{10} light years, the ratio of Planck volume to the volume of the Universe gives $\delta\theta \sim 10^{-184}$ radians. It seems safe to say that deviations from Lorentzian signature of this magnitude are undetectable. Of course, in the very early Universe, fluctuations away from Lorentzian signature could have been substantial.

Note that in this argument it was crucial that θ is constant on the preferred T=const hypersurfaces. If this were not the case, and θ could vary locally, then entropy would overwhelm the effective potential and we would instead expect $\delta\theta$ to be of the order of 1, which is surely not consistent with observation.

IV. CANCELLATION OF THE COSMOLOGICAL TERM

The effective potential $V(\theta)$ can be interpreted as a θ-dependent cosmological "constant," and the argument of this paper is based on looking for the minimum (of the real part) and stationarity (for the imaginary part) of $V(\theta)$ [the "minimum/stationary point" of $V(\theta)$]. Since $V(\pi) \neq 0$ in $D=4$ dimensions, the cosmological constant is nonzero and of order $O(\Lambda^4)$. This, of course, raises the question of how to justify expansion of the metric around flat spacetime, in computing the one-loop contribution to the determinant in Eq. (15).

It has been suggested occasionally that the cosmological term is somehow screened at large distances [9,10], and this idea, if it really works, would justify the flat-space expansion. But it is obviously important to consider other possibilities. The most conservative approach to the cosmological constant problem is simply to add a counterterm

$$S_c = \int d^Dx\sqrt{g}\ \lambda_c \tag{38}$$

to remove the induced term.[4] At first sight, it might seem that this "conservative" approach to removing the cosmological constant also removes the mechanism which singles out Lorentzian signature at $D=4$. In fact, that is not true. Writing

$$\lambda = (n_F - n_B)\frac{\Lambda^D}{D(4\pi)^{D/2}} \tag{39}$$

the total effective potential is

[3]Diffeomorphism invariance, however, is an exact symmetry at all θ.

[4]The value of λ_c, like that of all other bare masses and couplings, is assumed to be real.

$$V_T(\theta)=\lambda_c e^{i\theta/2}+\lambda e^{-i(D-2)\theta/4} \tag{40}$$

and it is clearly impossible to choose λ_c such that $V_T(\theta)=0$ for *all* θ. Instead, the object would be to choose λ_c such that $V_T=0$ at the minimum/stationary point of $V_T(\theta)$. It will now be shown that it is only possible to make such a choice in $D=4$ dimensions, where the minimum/stationary point is again Lorentzian signature.

Denoting

$$\bar{\theta}\equiv\frac{D-2}{2}\theta\ , \tag{41}$$

the condition that $V_T=0$ gives

$$\lambda_c\cos\frac{\theta}{2}+\lambda\cos\frac{\bar{\theta}}{2}=0 \tag{42}$$

for the real part,

$$\lambda_c\sin\frac{\theta}{2}-\lambda\sin\frac{\bar{\theta}}{2}=0 \tag{43}$$

for the imaginary part, while

$$\lambda_c\cos\frac{\theta}{2}-\frac{D-2}{2}\lambda\cos\frac{\bar{\theta}}{2}=0 \tag{44}$$

is the stationarity condition for $\mathrm{Im}[V_T]$. Equations (42) and (44) simply that

$$\bar{\theta}=(2n+1)\pi\ ,\quad \theta=\pi\ , \tag{45}$$

where we have used the fact that $|\theta|\le\pi$ (n integer). Then, from (43) we have that

$$|\lambda_c|=\lambda\ . \tag{46}$$

Now suppose D is large enough so that one can choose $\bar{\theta}\ge 3\pi$, consistent with (45). The remaining question is whether θ corresponding to this choice of $\bar{\theta}$ is the minimum point of $\mathrm{Re}[V_T]$. If D is such that $\bar{\theta}\ge 3\pi$ is possible, then it would also be possible to choose a value of $\theta=\theta'$ where $\bar{\theta}'=(D-2)\theta'/2=2\pi$, in which case

$$\begin{aligned}\mathrm{Re}[V_T(\theta')]&=\lambda_c\cos\frac{\theta'}{2}-\lambda\\&=-\lambda\left[1\pm\cos\frac{\theta'}{2}\right]\\&<0\end{aligned} \tag{47}$$

since $0<\theta'\le\pi$. This would mean that $V_T=0$ is not the minimum/stationary point, so the only other possibility is that $\bar{\theta}=\pi$. For $\theta=\pi$, this can only be true in $D=4$ dimensions, in which case

$$V_T(\theta)=\lambda\cos\frac{\theta}{2} \tag{48}$$

and $\theta=\pi$ is clearly the minimum of this potential.

V. THE CASE OF MASSIVE FIELDS

The analysis of the previous sections, applied to massless fields, was extremely simple; it is not as simple when our considerations are extended to massive fields. The problem is that the integral in Eq. (16), extended to massive fields, involves incomplete gamma functions, and the corresponding analysis becomes more involved. Our approach will be to make an m^2/Λ^2 expansion around $m=0$. There are three cases of interest. First of all, for $D=4$ and $n_F>n_B$, the mass corrections can be expected to separate the minimum point (of $\mathrm{Re}[V]$) and stationary point (of $\mathrm{Im}[V]$) slightly. We will show below that this slight separation does not destroy the Lorentzian behavior; it turns out that the minimum of the real part is still exactly at $\theta=\pm\pi$, while the stationary point of the imaginary part moves just outside the range $\theta\in[-\pi/\pi]$, provided that a certain inequality among the masses is satisfied. Thus Lorentzian signature is still the optimum θ value. The other two cases of interest are $D=2$ and $n_F=n_B$. For massless fields, $V(\theta)$ is independent of θ for those choices. The introduction of masses can be expected to remove this degeneracy, and the question is whether any new solutions of the minimum/stationarity criteria are obtained. We will find that only for the case $n_B=n_F$ at $D=6$ is it possible to have the minimum/stationary points (nearly) coincide.

The starting point of our analysis is the one-loop contribution $V_S(\theta)$ to the effective potential due to the integration over a scalar field ϕ of mass m in a flat background ($e^a_\mu=\delta^a_\mu$). This is given by the obvious extension of Eq. (15), i.e.,

$$\exp\left[-\int d^Dx\ V_S(\theta)\right]=\det{}^{-1/2}[-\sqrt{\eta}(\eta^{ab}\partial_a\partial_b-m^2)]\ . \tag{49}$$

Again evaluating the determinant with heat-kernel regularization one finds

$$\begin{aligned}V_S(\theta)&=-\frac{1}{2}\int_{1/\Lambda^2}^{\infty}\frac{ds}{s}\int\frac{d^Dp}{(2\pi)^D}\exp\{-s[e^{-i\theta/2}p_0^2+e^{i\theta/2}(\mathrm{p}^2+m^2)]\}\\&=-\frac{\exp[-i(D-2)\theta/4]}{2(4\pi)^{D/2}}\int_{1/\Lambda^2}^{\infty}ds\ s^{-D/2-1}\exp(-m^2e^{i\theta/2}s)\ .\end{aligned} \tag{50}$$

The convergence of the p integration still requires that $\theta\in[-\pi,\pi]$.

As in Sec. II, the one-loop contribution to $V_{\mathrm{eff}}(\theta)$ from each bosonic (fermionic) propagating degree of freedom of mass $m_B(m_F)$ turns out to be proportional to

$$\det{}^{-1/2(+1/2)}[-\sqrt{\eta}(\eta^{ab}\partial_a\partial_b-m^2_{B(F)})]$$

[neglecting factors of $\det^p(\eta)$, which one assumes to be absorbed in the functional measure]. The heat-kernel regularized value of these determinants is complicated, as compared to the massless case of Eq. (16), by the exponential factor $\exp(-m^2 e^{i\theta/2} s)$. However, in the p-integration convergence domain, $\theta \in [-\pi, \pi]$, the integral in Eq. (50) can be expressed in terms of the incomplete gamma function [11]. Again $V(\theta)$ is complex valued in general. Summing over all one-loop bosonic (B) and fermionic (F) contributions gives

$$\mathrm{Re}[V(\theta)] = \frac{\Lambda^D}{4(4\pi)^{D/2}} \left\{ \sum_F x_F^{D/2} [e^{i\theta/2}\Gamma(-D/2, x_{F+}) + e^{-i\theta/2}\Gamma(-D/2, x_{F-})] - \sum_B x_B^{D/2} [e^{i\theta/2}\Gamma(-D/2, x_{B+}) + e^{-i\theta/2}\Gamma(-D/2, x_{B-})] \right\}, \tag{51}$$

$$\mathrm{Im}[V(\theta)] = -\frac{i\Lambda^D}{4(4\pi)^{D/2}} \left\{ \sum_F x_F^{D/2} [e^{i\theta/2}\Gamma(-D/2, x_{F+}) - e^{-i\theta/2}\Gamma(-D/2, x_{F-})] - \sum_B x_B^{D/2} [e^{i\theta/2}\Gamma(-D/2, x_{B+}) - e^{-i\theta/2}\Gamma(-D/2, x_{B-})] \right\}, \tag{52}$$

where we have defined

$$x_{B(F)\pm} \equiv \frac{m^2_{B(F)}}{\Lambda^2} e^{\pm i\theta/2} \tag{53}$$

and $\Gamma(-D/2, x_{B(F)\pm})$ is the incomplete gamma function defined in the Appendix [Eqs. (A1) and (A2)].

The stationarity condition on $\mathrm{Im}[V(\theta)]$ is defined by

$$\frac{\partial}{\partial\theta} \mathrm{Im}[V] \Big|_{\bar\theta} = 0 . \tag{54}$$

Taking advantage of the useful relation [11]

$$\frac{\partial}{\partial\theta} \Gamma(-D/2, x_{B(F)\pm}) = \mp \frac{i}{2} (x_{B(F)\pm})^{-D/2} \exp(-x_{B(F)\pm}) , \tag{55}$$

Eq. (54) becomes

$$0 = \exp\left[i\frac{D-2}{4}\bar\theta \right] \left\{ \sum_F [(\bar x_{F-})^{D/2}\Gamma(-D/2, \bar x_{F-}) - e^{-\bar x_{F-}}] - \sum_B [(\bar x_{B-})^{D/2}\Gamma(-D/2, \bar x_{B-}) - e^{-\bar x_{B-}}] \right\} + \exp\left[-i\frac{D-2}{4}\bar\theta \right] \left\{ \sum_F [(\bar x_{F+})^{D/2}\Gamma(-D/2, \bar x_{F+}) - e^{-\bar x_{F+}}] - \sum_B [(\bar x_{B+})^{-D/2}\Gamma(-D/2, \bar x_{B+}) - e^{-\bar x_{B+}}] \right\}, \tag{56}$$

where an overbar over the variable x means that this has to be evaluated at $\theta = \bar\theta$.

Eliminating the incomplete gamma function dependence of Eq. (51) by means of Eq. (56), one finds that the value of $\mathrm{Re}[V(\theta)]$ at the stationary point $\bar\theta$ is

$$\mathrm{Re}[V(\bar\theta)] = \frac{\Lambda^D}{2(4\pi)^{D/2}} \left\{ \sum_F \exp(-x_F \cos\bar\theta/2) \cos[(D-2)\bar\theta/4 + x_F \sin\bar\theta/2] - \sum_B \exp(-x_B \cos\bar\theta/2) \cos[(D-2)\bar\theta/4 + x_B \sin\bar\theta/2] \right\}, \tag{57}$$

where $x_{B(F)} \equiv m^2_{B(F)}/\Lambda^2$, and we require, in the stability range $\theta \in [-\pi, \pi]$,

$$\min[\mathrm{Re}[V(\theta)]] \approx \mathrm{Re}[V(\bar\theta)] , \tag{58}$$

where the approximate equality of the minimum and stationary points is up to $O(m^2/\Lambda^2)$ corrections. Typically, if masses are on the order of the grand unified theory (GUT) scale and Λ is on the order of the Planck scale, we would expect $m^2/\Lambda^2 \approx 10^{-8}$.

We define

$$\Delta(n)_{BF} \equiv n_F - n_B ,$$

$$\Delta(x^n)_{BF} \equiv \sum_F \left[\frac{m_F^2}{\Lambda^2}\right]^n - \sum_B \left[\frac{m_B^2}{\Lambda}\right]^n , \tag{59}$$

$$\Delta(x^n \ln x)_{BF} \equiv \sum_F \left[\frac{m_F^2}{\Lambda^2}\right]^n \ln\left[\frac{m_F^2}{\Lambda^2}\right] - \sum_B \left[\frac{m_B^2}{\Lambda^2}\right]^n \ln\left[\frac{m_B^2}{\Lambda^2}\right]$$

(where, now, n_B and n_F represent the total number, massless plus massive, of bosonic and fermionic propagating degrees of freedom). Since the GUT mass, on a logarithmic scale, is not so far from the Planck mass, we will treat $\Delta(x^n)_{BF}$ and $\Delta(x^n \ln x)_{BF}$ as being of the same order of magnitude.

The derivation of the expansions of Eqs. (51), (52), (56), and (57) is quite straightforward although tedious, and the detailed m^2/Λ^2 expansions for arbitrary signature and dimension are collected in the Appendix.

The three cases of interest, namely, (i) $n_F > n_B$ at $D=4$, (ii) $D=2$, and (iii) $n_F = n_B$, will now be considered separately.

A. $D=4$ at $\Delta(n)_{BF}>0$

Except in the degenerate cases ($D=2$ or $n_F=n_B$), small mass corrections cannot affect the conclusion of Sec. II for $D\neq 4$, namely, that the minimum and stationary points are not close to one another. For $D=4$ and $n_F > n_B$, however, mass terms will spoil the exact coincidence of the two points. In this case, only the second term of Eq. (A3) of the Appendix trivially vanishes and the stationarity condition (54) becomes

$$0 \simeq \Delta(n)_{BF}\cos\frac{\bar\theta}{2} + \left[\Delta(x^2\ln x)_{BF} + \frac{2\gamma-1}{2}\Delta(x^2)_{BF}\right]\cos\frac{\bar\theta}{2} - \frac{1}{2}\Delta(x^2)_{BF}\bar\theta\sin\frac{\bar\theta}{2} - \frac{2}{3}\Delta(x^3)_{BF}\cos\bar\theta + O[\Delta(x^4\ln x)_{BF}] \tag{60}$$

whose approximate solution is

$$\frac{\bar\theta}{2} \simeq \left[\frac{2k+1}{2}\right]\pi\left[1-\frac{\Delta(x^2)_{BF}}{\Delta(n)_{BF}}\right] + O[\Delta(x^3)_{BF}] \tag{61}$$

(γ is the Euler constant and $k=0,-1$). For the real part of $V(\theta)$, from Eq. (A5) of the Appendix evaluated at $D=4$, we find

$$\mathrm{Re}[V(\theta)]|_{D=4} \simeq \frac{\Lambda^4}{2(4\pi)^2}\left[\frac{1}{2}\Delta(n)_{BF}\cos\frac{\theta}{2} - \Delta(x)_{BF} - \frac{1}{2}\left[\Delta(x^2\ln x)_{BF} + \frac{2\gamma-3}{2}\Delta(x^2)_{BF}\right]\cos\frac{\theta}{2} + \frac{1}{4}\Delta(x^2)_{BF}\theta\sin\frac{\theta}{2} + O[\Delta(x^3)_{BF}]\right] . \tag{62}$$

In the stability range $\theta\in[-\pi,\pi]$, the value $|\theta|=\pi$ is still the minimum of the real part of $V(\theta)$ for $\Delta(n)_{BF}>0$. The stationary condition point of the imaginary part is at $|\theta|=\pi+\epsilon$, where ϵ is $O[\Delta(x^2)]$. Moreover, if

$$\Delta(x^2)_{BF} < 0 \tag{63}$$

then $\epsilon > 0$, and the stationary point lies just outside the stability range. For masses at the GUT scale, and cutoff at the Planck length, this means that the stationary point is

$$|\bar\theta| = \pi + O(10^{-16}) , \tag{64}$$

which is certainly very close to the minimum point at $|\theta|=\pi$. Moreover, $\theta=\pm\pi$ is as close as it is possible to come to the stationary point in the stability range. We conclude that for $D=4$ at $n_F>n_B$, Lorentzian signature is still the optimum value of θ, as in the massless case.

B. $D=2$ at $\Delta(n)_{BF}\neq 0$

The approximate stationarity condition for $\mathrm{Im}[V(\theta)]$ in $D=2$ dimensions and arbitrary $\Delta(n)_{BF}$ turns out to be

$$0 \simeq [\Delta(x\ln x)_{BF} + \gamma\Delta(x)_{BF}]\cos\frac{\bar\theta}{2} - \frac{1}{2}\Delta(x)_{BF}\bar\theta\sin\frac{\bar\theta}{2} - \Delta(x^2)_{BF}\cos\bar\theta + \frac{1}{4}\Delta(x^3)_{BF}\cos\frac{3\bar\theta}{2} + O[\Delta(x^4\ln x)_{BF}] , \tag{65}$$

while, for the real part,

$$\mathrm{Re}[V(\theta)]|_{D=2} \simeq \frac{\Lambda^2}{8\pi}\left[\Delta(n)_{BF} + [\Delta(x\ln x)_{BF} + (\gamma-1)\Delta(x)_{BF}]\cos\frac{\theta}{2} - \frac{1}{2}\Delta(x)_{BF}\theta\sin\frac{\theta}{2} + O[\Delta(x^2)_{BF}]\right] , \tag{66}$$

which follows from Eq. (A5) of the Appendix.

Given that $\Delta(x)_{BF}$ is of the same order as $\Delta(x\ \ln x)$, and ruling out any special fine-tunings among the masses, there is no reason at all that the stationary point of Im(V) should coincide with the minimum of Re[V].

Next we consider the $n_F=n_B$ case, separately in dimensions $D=2-6$ and $D>6$.

C. $\Delta(n)_{BF}=0$ at $D=2$

The stationarity condition for $D=2$ with $\Delta(n)_{BF}=0$ is identical to the corresponding condition at $\Delta(n)_{BF}\neq 0$, while the equation for Re[V] differs only by a constant. Barring fine-tuning among the masses, the minimum and stationary points are not close together.

D. $\Delta(n)_{BF}=0$ at $D=3$

In three dimensions, the stationarity condition becomes

$$0\simeq -2\Delta(x)_{BF}\cos\frac{\bar\theta}{4}+\frac{8}{3}\sqrt{\pi}\Delta(x^{3/2})_{BF}\cos\frac{\bar\theta}{2}-3\Delta(x^2)_{BF}\cos\frac{3\bar\theta}{4}+\frac{5}{9}\Delta(x^3)_{BF}\cos\frac{5\bar\theta}{4}+O[\Delta(x^4\ln x)_{BF}] \tag{67}$$

whose approximate solution is

$$\frac{\bar\theta}{4}\simeq\left[\frac{2k+1}{2}\right]\pi+O\left[\frac{\Delta(x^{3/2})_{BF}}{\Delta(x)_{BF}}\right] . \tag{68}$$

This stationary point is well outside the convergence domain $[-\pi,\pi]$.

E. $\Delta(n)_{BF}=0$ at $D=4$

For the stationarity condition we have

$$0\simeq\left[\Delta(x^2\ln x)_{BF}+\frac{2\gamma-1}{2}\Delta(x^2)_{BF}\right]\cos\frac{\bar\theta}{2}-\frac{1}{2}\Delta(x^2)_{BF}\bar\theta\sin\frac{\bar\theta}{2}-\frac{2}{3}\Delta(x^3)_{BF}\cos\bar\theta+O[\Delta(x^4\ln x)_{BF}] , \tag{69}$$

while, for the real part,

$$\mathrm{Re}[V(\theta)]|_{D=4}\simeq\frac{\Lambda^4}{2(4\pi)^2}\left[-\Delta(x)_{BF}+\frac{1}{4}\Delta(x^2)_{BF}\theta\sin\frac{\theta}{2}-\tfrac{1}{2}\left[\Delta(x^2\ln x)_{BF}+\frac{2\gamma-3}{2}\Delta(x^2)_{BF}\right]\cos\frac{\theta}{2}+O[\Delta(x^3)_{BF}]\right] \tag{70}$$

and, in general, the minimum/stationary points do not coincide.

F. $\Delta(n)_{BF}=0$ at $D=5$

In the case $D=5$, Eq. (A3) of the Appendix with $\Delta(n)_{BF}=0$ becomes

$$0\simeq\frac{2}{3}\Delta(x)_{BF}\cos\frac{\bar\theta}{4}+\Delta(x^2)_{BF}\cos\frac{\bar\theta}{4}-\frac{16}{15}\sqrt{\pi}\Delta(x^{5/2})_{BF}\cos\frac{\bar\theta}{2}+\Delta(x^3)_{BF}\cos\frac{3\bar\theta}{4}+O[\Delta(x^4\ln x)_{BF}] \tag{71}$$

whose approximate solution is

$$\frac{\bar\theta}{4}\simeq\left[\frac{2k+1}{2}\right]\pi+O\left[\frac{\Delta(x^2)_{BF}}{\Delta(x)_{BF}}\right] . \tag{72}$$

As in the previous case with $D=3$, the stationary point is far outside the stability domain.

G. $\Delta(n)_{BF}=0$ at $D=6$

In six dimensions, Eq. (A3) of the Appendix for the stationary point becomes

$$0\simeq\Delta(x)_{BF}\cos\frac{\bar\theta}{2}+\frac{1}{3}\left[\Delta(x^3\ln x)_{BF}+\frac{6\gamma-5}{6}\Delta(x^3)_{BF}\right]\cos\frac{\bar\theta}{2}-\frac{1}{6}\Delta(x^3)_{BF}\bar\theta\sin\frac{\bar\theta}{2}+O[\Delta(x^4\ln x)_{BF}] \tag{73}$$

whose solution is

$$\frac{\bar{\theta}}{2} \simeq \left[\frac{2k+1}{2}\right]\pi\left[1-\frac{1}{3}\frac{\Delta(x^3)_{BF}}{\Delta(x)_{BF}}\right] + O\left[\frac{\Delta(x^4\ln x)_{BF}}{\Delta(x)_{BF}}\right] . \tag{74}$$

Therefore, the stationary point of $\mathrm{Im}[V(\theta)]$ can be just outside $[-\pi,\pi]$ if the following inequality holds:

$$\Delta(x^3)_{BF}/\Delta(x)_{BF} < 0 . \tag{75}$$

Moreover, Eq. (A5) of the Appendix gives

$$\mathrm{Re}[V(\theta)]|_{D=6} \simeq \frac{\Lambda^6}{2(4\pi)^3}\left[-\frac{1}{2}\Delta(x)_{BF}\cos\frac{\theta}{2} + O[\Delta(x^3\ln x)_{BF}]\right] \tag{76}$$

and this has a minimum in the stability domain exactly at $\theta=\pm\pi$, if

$$\Delta(x)_{BF} < 0 . \tag{77}$$

This means that the case of $D=6$ and $\Delta(n)_{BF}=0$ is similar to $D=4$ and $n_F > n_B$. Assuming two inequalities, namely, $\Delta(x)_{\rm BF}<0$ and $\Delta(x^3)_{BF}>0$, we find that $\min \mathrm{Re}[V]$ is at $|\theta|=\pi$, and the stationary point of $\mathrm{Im}[V]$ is at $|\theta|=\pi+\epsilon$, where ϵ is positive and $O(m^4/\Lambda^4)$. As in the $D=4$ case, Lorentzian signature is the optimum θ value in the range $[-\pi,\pi]$.

H. $\Delta(n)_{BF}=0$ at $D>6$

Finally, we consider the cases $D>6$ with $\Delta(n)_{BF}=0$. For the stationary part

$$0 \simeq 2\left[\frac{D-4}{D-2}\right]\Delta(x)_{BF}\cos\left[\frac{(D-4)\bar{\theta}}{4}\right] - \left[\frac{D-6}{D-4}\right]\Delta(x^2)_{BF}\cos\left[\frac{(D-6)\bar{\theta}}{4}\right] + \frac{1}{D-6}\left[\frac{D-8}{3}+\frac{32[1-h(D-8)]}{D(D-2)(D-4)}\cos^2\left[\frac{D\pi}{2}\right]\right]\Delta(x^3)_{BF}\cos\left[\frac{(D-8)\bar{\theta}}{4}\right] + O[\Delta(x^4\ln x)_{BF}] \tag{78}$$

so that

$$\frac{D-4}{4}\bar{\theta} \simeq \left[\frac{2k+1}{2}\right]\pi + O\left[\frac{\Delta(x^2)_{BF}}{\Delta(x)_{BF}}\right] , \tag{79}$$

while, for the real part,

$$\mathrm{Re}[V(\theta)]|_{D>6} \simeq \frac{\Lambda^D}{2(4\pi)^{D/2}}\left[-\frac{2}{D-2}\Delta(x)_{BF}\cos\left[\frac{(D-4)\theta}{4}\right] + O[\Delta(x^2)_{BF}]\right] . \tag{80}$$

It is readily seen from Eq. (80) that, in general, the minimum of $\mathrm{Re}[V(\theta)]$ is not at $\bar{\theta}$. This eliminates from consideration all dimensions $D>6$.

We have, throughout, treated $\Delta(x)_{BF}$ and $\Delta(x\ln x)_{BF}$ as being of the same order of magnitude. If the Planck scale is *not* a fundamental cutoff, so that Λ can be taken arbitrarily large, or if the mass generation scale is many orders of magnitude less than the presumed grand unification scale, then it is appropriate to treat $\Delta(x\ln x)_{BF} \gg \Delta(x)_{BF}$. In that case, in addition to Lorentzian solutions at $D=4$ ($n_F>n_B$) and $D=6$ ($n_F=n_B$) we find additional Lorentzian solutions at $D=2$ [$\Delta(n)_{BF}$ arbitrary], and $D=4$ ($n_F=n_B$).

Finally, since nonzero mass terms displace the stationary point slightly away from the minimum point at $D=4$ ($n_F>n_B$) and $D=6$ ($n_F=n_B$), the exact cancellation of the cosmological constant found in Sec. IV is no longer quite exact. Although the real part of $V_T(\theta)$ can be canceled exactly at the minimum point ($|\theta|=\pi$), one would expect a small imaginary part, of order m^4/Λ^4, left over, which in principle constitutes a contribution to the measure.

VI. CONCLUSIONS

Two fundamental facts about spacetime are its Lorentzian signature and $D=4$ dimensionality. An equally fundamental feature of quantum mechanics, which distinguishes it from any sort of classical field theory or diffusion process, is the appearance of $\sqrt{-1}$ in the Feynman amplitude and Schrödinger equation. The proposal that the spacetime signature (i.e., the tangent space metric) is dynamical provides an intriguing relation among these three facts. The i of quantum mechanics can be traced to the factor $\sqrt{g}=|e|\sqrt{\eta}$ in the path amplitude, which becomes just $\exp(iS)$ at Lorentzian signature. By allowing the tangent-space metric η_{ab} to interpolate *continuously* between different signatures (which requires that entries of η can rotate into the complex plane), we have found by a simple one-loop argument that Lorentzian signature is dynamically selected, for $n_F > n_B$, uniquely in $D=4$ dimensions. In broken supersymmetry theories, there is also a possibility for Lorentzian signature at $D=6$. With the help of curved-space consistency conditions, it has been further argued that fluctuations away from Lorentzian signature at $D=4$ are enormously suppressed (except, perhaps, in the very early Universe) and are certainly undetectable in the present epoch.

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APPENDIX

In this appendix we give a list of basic definitions and equations, which we use and to which we make reference in the main text, for the case of arbitrary signature and dimensions. In general, the incomplete gamma function $\Gamma(\alpha,x)$ is described by two different series expansions in the variable x for the case of α integer (D even) or fractional (D odd). The two expansions can be combined by writing

$$\Gamma(-D/2,x_\pm)=\frac{(-1)^{D/2}}{(D/2)!}\left[E_1(x_\pm)-e^{-x_\pm}\sum_{n=0}^{D/2-1}\frac{(-1)^n n!}{(x_\pm)^{n+1}}\right]\cos^2\left[\frac{D\pi}{2}\right] + \left[\frac{(-1)^{(D+1)/2}\sqrt{\pi}}{(D/2)!}-\sum_{n=0}^{\infty}\frac{(-1)^n(x_\pm)^{n-D/2}}{n!(n-D/2)}\right]\sin^2\left[\frac{D\pi}{2}\right], \tag{A1}$$

where $E_1(x_\pm)$ is the exponential integral function,

$$E_1(x_\pm)\equiv-\left[\gamma+\ln x_\pm+\sum_{n=1}^{\infty}\frac{(-x_\pm)^n}{n!n}\right], \tag{A2}$$

and γ is the Euler constant [11]. The effect of $\cos^2(D\pi/2)[\sin^2(D\pi/2)]$ in Eq. (A1) is just to select out one of the two (exact) expansions for the case when D is even [odd] [11].

Moreover, using the series expansion (A1) and assuming $x \ll 1$, one can easily rewrite the stationarity condition for $\mathrm{Im}[V]$, Eq. (56) of Sec. V, up to order $O[\Delta(x^4\ln x)_{BF}]$ as[5]

$$\begin{aligned}0\simeq{}&\frac{2(2-D)}{D}\Delta(n)_{BF}\cos\left[\frac{(D-2)\bar\theta}{4}\right]\\&+2\left[\frac{D-4}{D-2}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[D-2-\frac{4h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]\Delta(x)_{BF}\cos\left[\frac{(D-4)\bar\theta}{4}\right]\\&+(-1)^{D/2+1}\frac{2}{(D/2)!}\Delta(x^{D/2}\ln x)_{BF}\cos^2\left[\frac{D\pi}{2}\right]\cos\left[\frac{\bar\theta}{2}\right]h(6-D)\\&+(-1)^{D/2+1}\frac{2}{(D/2)!}\Delta(x^{D/2})_{BF}\left\{\left[\gamma\cos^2\left[\frac{D\pi}{2}\right]+(-1)^{-1/2}\sqrt{\pi}\sin^2\left[\frac{D\pi}{2}\right]\right]\cos\left[\frac{\bar\theta}{2}\right]\right.\\&\qquad\left.-\frac{\bar\theta}{2}\sin\left[\frac{\bar\theta}{2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right\}h(6-D)\\&+(-1)^{D/2}\frac{2}{(D/2)!}\Delta(x^{D/2+1})_{BF}\cos^2\left[\frac{D\pi}{2}\right]\cos(\bar\theta)h(4-D)\\&+\frac{(-1)^{D/2+1}}{2(D/2)!}\Delta(x^{D/2+2})_{BF}\cos^2\left[\frac{D\pi}{2}\right]\cos\left[\frac{3\bar\theta}{2}\right]h(2-D)\end{aligned}$$

[5]We restrict our analysis to the case of dimension $D\geq 2$, i.e., we do not consider the case of a $D=1$, single-particle quantum mechanics.

$$-\left[\frac{D-6}{(D-4)}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[D-2-\frac{16h(D-6)}{(D-2)(D-4)}-\frac{8h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]\Delta(x^2)_{BF}\cos\left[\frac{(D-6)\bar{\theta}}{4}\right]$$
$$+\left[\frac{D-8}{3(D-6)}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[\frac{D-2}{3}-\frac{32h(D-8)}{(D-2)(D-4)(D-6)}-\frac{16h(D-6)}{(D-2)(D-4)}-\frac{4h(D-4)}{(D-2)}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]$$
$$\times\Delta(x^3)_{BF}\cos\left[\frac{(D-8)\bar{\theta}}{4}\right]+O[\Delta(x^4\ln x)_{BF}]\;, \tag{A3}$$

where $h(y)$ is the Heaviside step function

$$h(y)\equiv\begin{cases}1\;, & y\geq 0\;,\\ 0\;, & y<0\;.\end{cases} \tag{A4}$$

Similarly, expansion of Eqs. (51) and (52) of Sec. V gives

$$\mathrm{Re}[V(\theta)]\simeq\frac{\Lambda^D}{2(4\pi)^{D/2}}\left[\frac{2}{D}\Delta(n)_{BF}\cos\left[\frac{(D-2)\theta}{4}\right]\right.$$
$$-2\left[\frac{1}{(D-2)}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[1+\frac{2h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]\Delta(x)_{BF}\cos\left[\frac{(D-4)\theta}{4}\right]$$
$$+\frac{(-1)^{D/2+1}}{(D/2)!}\Delta(x^{D/2}\ln x)_{BF}\cos^2\left[\frac{D\pi}{2}\right]\cos\left[\frac{\theta}{2}\right]h(6-D)$$
$$+\frac{(-1)^{D/2+1}}{(D/2)!}\Delta(x^{D/2})_{BF}\left\{\left[\gamma\cos^2\left[\frac{D\pi}{2}\right]+(-1)^{-1/2}\sqrt{\pi}\sin^2\left[\frac{D\pi}{2}\right]\right]\cos\left[\frac{\theta}{2}\right]\right.$$
$$\left.-\frac{\theta}{2}\sin\left[\frac{\theta}{2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right\}h(6-D)$$
$$+\frac{(-1)^{D/2}}{(D/2)!}\Delta(x^{D/2+1})_{BF}\cos^2\left[\frac{D\pi}{2}\right]\cos(\theta)h(4-D)$$
$$+\frac{(-1)^{D/2+1}}{4(D/2)!}\Delta(x^{D/2+2})_{BF}\cos^2\left[\frac{D\pi}{2}\right]\cos\left[\frac{3\theta}{2}\right]h(2-D)$$
$$+\left[\frac{1}{D-4}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[1+\frac{8h(D-6)}{(D-2)(D-4)}+\frac{4h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]$$
$$\times\Delta(x^2)_{BF}\cos\left[\frac{(D-6)\theta}{4}\right]$$
$$+\left[-\frac{1}{3(D-6)}\sin^2\left[\frac{D\pi}{2}\right]-\frac{1}{D}\left[\frac{1}{3}+\frac{16h(D-8)}{(D-2)(D-4)(D-6)}+\frac{8h(D-6)}{(D-2)(D-4)}\right.\right.$$
$$\left.\left.+\frac{2h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]$$
$$\left.\times\Delta(x^3)_{BF}\cos\left[\frac{(D-8)\theta}{4}\right]+O[\Delta(x^4\ln x)_{BF}]\right]\;, \tag{A5}$$

$$\mathrm{Im}[V(\theta)]\simeq\frac{\Lambda^D}{2(4\pi)^{D/2}}\left[-\frac{2}{D}\Delta(n)_{BF}\sin\left[\frac{(D-2)\theta}{4}\right]\right.$$
$$+2\left[\frac{1}{(D-2)}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[1+\frac{2h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]\Delta(x)_{BF}\sin\left[\frac{(D-4)\theta}{4}\right]$$
$$+\frac{(-1)^{D/2+1}}{(D/2)!}\Delta(x^{D/2}\ln x)_{BF}\cos^2\left[\frac{D\pi}{2}\right]\sin\left[\frac{\theta}{2}\right]h(6-D)$$

$$+\frac{(-1)^{D/2+1}}{(D/2)!}\Delta(x^{D/2})_{BF}\left\{\left[\gamma\cos^2\left[\frac{D\pi}{2}\right]+(-1)^{-1/2}\sqrt{\pi}\sin^2\left[\frac{D\pi}{2}\right]\right]\sin\left[\frac{\theta}{2}\right]\right.$$

$$\left.+\frac{\theta}{2}\cos\left[\frac{\theta}{2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right\}h(6-D)$$

$$+\frac{(-1)^{D/2}}{(D/2)!}\Delta(x^{D/2+1})_{BF}\cos^2\left[\frac{D\pi}{2}\right]\sin(\theta)h(4-D)$$

$$+\frac{(-1)^{D/2+1}}{4(D/2)!}\Delta(x^{D/2+2})_{BF}\cos^2\left[\frac{D\pi}{2}\right]\sin\left[\frac{3\theta}{2}\right]h(2-D)$$

$$-\left[\frac{1}{D-4}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[1+\frac{8h(D-6)}{(D-2)(D-4)}+\frac{4h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]$$

$$\times\Delta(x^2)_{BF}\sin\left[\frac{(D-6)\theta}{4}\right]$$

$$+\left[\frac{1}{3(D-6)}\sin^2\left[\frac{D\pi}{2}\right]+\frac{1}{D}\left[\frac{1}{3}+\frac{16h(D-8)}{(D-2)(D-4)(D-6)}+\frac{8h(D-6)}{(D-2)(D-4)}\right.\right.$$

$$\left.\left.+\frac{2h(D-4)}{D-2}\right]\cos^2\left[\frac{D\pi}{2}\right]\right]\Delta(x^3)_{BF}\sin\left[\frac{(D-8)\theta}{4}\right]$$

$$+O[\Delta(x^4\ln x)_{BF}]\Bigg], \tag{A6}$$

and, finally, Eq. (57) of Sec. V becomes

$$\mathrm{Re}[V(\bar\theta)]\simeq\frac{\Lambda^D}{2(4\pi)^{D/2}}\left[\cos\left[\frac{(D-2)\bar\theta}{4}\right]\left\{\Delta(n)_{BF}-\Delta(x)_{BF}\cos\frac{\bar\theta}{2}+\frac{1}{2}\Delta(x^2)_{BF}\cos\bar\theta-\frac{1}{6}\Delta(x^3)_{BF}\cos\frac{\bar\theta}{2}\left[4\cos^2\left[\frac{\bar\theta}{2}\right]-3\right]\right\}\right.$$

$$-\sin\left[\frac{(D-2)\bar\theta}{4}\right]\left\{\Delta(x)_{BF}\sin\frac{\bar\theta}{2}-\frac{1}{2}\Delta(x^2)_{BF}\sin\bar\theta\right.$$

$$\left.\left.-\frac{1}{6}\Delta(x^3)_{BF}\sin\frac{\bar\theta}{2}\left[4\sin^2\left[\frac{\bar\theta}{2}\right]-3\right]\right\}+O[\Delta(x^4)_{BF}]\right]. \tag{A7}$$

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