Evolving wormholes and the weak energy condition

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Nonstatic Lorentzian wormholes conformally related to static wormhole geometries are found to exist for a finite (arbitrarily large) or half-infinite ($b < t \le \infty$) interval of time, with the required energy-momentum tensor satisfying the weak energy condition. Features of such spacetimes are discussed, along with several examples.

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Lorentzian wormholes have been quite popular over the last few years for a variety of reasons. Among them, we have the exciting possibility of constructing time machines with these spacetimes. Several such constructions have been suggested by various authors [1,2]. However, the basic problem with the existence of these spacetimes as solutions to the Einstein field equations of classical general relativity (GR) has been the fact that they require an energy-momentum tensor whose components violate the energy conditions [3]. In fact, the weakest of these conditions, namely, the weak energy condition (WEC) or its averaged version has been found to be violated by the matter required to support a wormhole. Although, as pointed out by Visser [4], there is no experimental evidence that clearly supports the WEC (in fact the Casimir effect [5] clearly demonstrates its violation), physicists (or, more specifically, general relativists) tend to believe in these conditions for primarily two reasons—(i) almost all known and physically realizable forms of matter satisfy them and (ii) they are among the assumptions necessary to prove the Hawking-Penrose singularity theorems [6]. A way out of this contradiction between wormhole existence and WEC violation was to take refuge in quantum theory. It has been shown (much before the WEC was proposed) [7] that there exist quantum states for which the expectation value of the energy-momentum tensor violates all energy conditions. On the other hand, some authors have also considered the possibility of using modified gravitational actions in order to obtain wormholes from "normal matter" [8,9].

However, most of the efforts in understanding Lorentzian wormholes and WEC violations have concentrated on static geometries. The only place where a nonstatic geometry is used is in the "time-machine" constructions due to Morris, Thorne, and Yurtsever [1] and Novikov [2]. The aim of this paper is to show that within classical general relativity there exist Lorentzian wormholes which are nonstatic and which do not require WEC violating matter to support them. These wormholes, as will be shown, exist for a finite- (however large) semi-infinite time interval and represent evolving geometries. During the evolution, the shape of the wormhole changes in the embedding space—the throat radius expands or contracts and the rate of change of the embedding function increases or decreases. One can draw an analogy of these geometries with the usual Friedmann-Robertson-Walker (FRW) universe ($k=1$) with S^3 spacelike sections. The difference is that we have spacelike sections which are $R\times S^2$ with a wormhole metric. Therefore, in a certain sense, one can christen these spacetimes as "wormhole universes."

We begin our analysis with the ansatz for the metric and the energy-momentum tensor. These are

$$ds^2=\Omega^2(t)\left[-dt^2+\frac{dr^2}{1-b(r)/r}+r^2d\Omega_2^2\right], \tag{1}$$

$$T_{00}=\rho(r,t),\quad T_{11}=\tau(r,t),\quad T_{22}=T_{33}=p(r,t)\ . \tag{2}$$

Here $\Omega^2(t)$ is the conformal factor, finite and positive definite throughout the domain of t. One can also rewrite the metric in (1) using "physical time" instead of "conformal time." This would mean replacing t with $\tau=\int\Omega(t)dt$ and therefore $\Omega(t)$ would become $R(\tau)$ where the latter is the functional form of the metric in the τ coordinate. However, it is convenient for us to use "conformal time." Translating all the results for t into those for τ is a trivial exercise. $\rho(r,t)$, $\tau(r,t)$, and $p(r,t)$ are the components of the energy momentum tensor in the frame given by the one-form basis $e^0=\Omega(t)dt$,

$$e^1=\Omega(t)dr/[1-b(r)/r]^{1/2}\ ,$$

$$e^2=\Omega(t)r\,d\theta\ ,$$

and

$$e^3=\Omega(t)r\sin\theta\,d\phi\,d\Omega_2^2$$

is the line element on the two-sphere. $b(r)$ is the usual "shape function" as defined by Morris and Thorne [3]. It will be assumed to satisfy all the conditions required for a spacetime to be a Lorentzian wormhole: i.e., $b(r)/r\le 1$; $b(r)/r\to 0$ as $r\to\infty$; at $r=b_0$, $b(r)=b_0$; $r>b_0$. The Einstein field equations with the ansatz (1) and (2) turn out to be (units $8\pi G=c=1$)

$$\frac{3}{\Omega^2}\left[\frac{\dot\Omega}{\Omega}\right]^2+\frac{1}{\Omega^2}\frac{b'}{r^2}=\rho(r,t)\ , \tag{3}$$

$$-\frac{2}{\Omega^2}\left[\frac{\ddot\Omega}{\Omega}\right]+\frac{1}{\Omega^2}\left[\frac{\dot\Omega}{\Omega}\right]^2-\frac{1}{\Omega^2}\frac{b}{r^3}=\tau(r,t)\ , \tag{4}$$

$$-\frac{2}{\Omega^2}\left[\frac{\ddot{\Omega}}{\Omega}\right]+\frac{1}{\Omega^2}\left[\frac{\dot{\Omega}}{\Omega}\right]^2+\frac{1}{\Omega^2}\frac{b-b'r}{2r^3}=p(r,t) . \quad (5)$$

The overdots denote derivatives with respect to t and the primes are derivatives with respect to r. The WEC ($T_{\mu\nu}u^\mu u^\nu \geq 0$ V nonspacelike u^μ) reduces to the following inequalities for the case of a diagonal energy-momentum tensor:

$$\rho \geq 0, \quad \rho+\tau \geq 0 \quad \text{and}\ \rho+p \geq 0 \quad \text{for all}\ (r,t) \quad (6)$$

From (3), (4), and (5) one can write down three inequalities which have to be satisfied if the WEC is not violated. These are

$$\frac{3}{\Omega^2}\left[\frac{\dot{\Omega}}{\Omega}\right]^2+\frac{1}{\Omega^2}\frac{b'}{r^2}\geq 0 , \quad (7)$$

$$\frac{1}{\Omega^2}\left[\frac{b'r-b}{r^3}+2\left\{2\left[\frac{\dot{\Omega}}{\Omega}\right]^2-\frac{\ddot{\Omega}}{\Omega}\right\}\right]\geq 0 , \quad (8)$$

$$\frac{1}{\Omega^2}\left\{\frac{b'r+b}{2r^3}+2\left[2\left[\frac{\dot{\Omega}}{\Omega}\right]^2-\frac{\ddot{\Omega}}{\Omega}\right]\right\}\geq 0 . \quad (9)$$

Several important facts should be noted here in comparison with the case of a static geometry. Equation (7) is trivially satisfied if $b' \geq 0$ irrespective of the geometry being static or nonstatic. However, if it is nonstatic then one can satisfy (7) even for the case when $b'<0$. In fact, one obtains the inequality

$$|b'| \leq 3\left[\frac{\dot{\Omega}}{\Omega}\right]^2 r^2 . \quad (10)$$

For every $t=$const slice (10) has to hold true, which means

$$\frac{|b'|}{r^2}\leq \min\left[3\left[\frac{\dot{\Omega}}{\Omega}\right]^2\right] , \quad (11)$$

where "min" denotes the minimum of the function for the given time interval.

For a static geometry (8) can never be satisfied, as shown by Morris and Thorne [3]. But, for a nonstatic geometry with $b' \geq 0$ one can satisfy (8). We require

$$\frac{b-b'r}{r^3}\leq \min\left\{2\left[2\left[\frac{\dot{\Omega}}{\Omega}\right]^2-\frac{\ddot{\Omega}}{\Omega}\right]\right\} . \quad (12)$$

Stated explicitly (12) implies that the value of $(b-b'r)/r^3$ for all r must be less than or equal to the minimum value of the function

$$2\left[2\left[\frac{\dot{\Omega}}{\Omega}\right]^2-\frac{\ddot{\Omega}}{\Omega}\right] ,$$

in the corresponding domain of t. However, we need

$$F(t)=2\left[2\left[\frac{\dot{\Omega}}{\Omega}\right]^2-\frac{\ddot{\Omega}}{\Omega}\right]>0 \quad \text{for all}\ t . \quad (13)$$

Equation (13) can be written in a more precise form by introducing a function $\chi(t)=\Omega/\dot{\Omega}$. We have

$$\frac{d\chi}{dt}>(-1) . \quad (14)$$

With $b' \geq 0$ and (13) one clearly sees that (9) is satisfied. Therefore, from this very simple analysis it is clear that nonstatic spherically symmetric Lorentzian wormhole geometries can exist with the required matter not violating the WEC. However, the fact that $\Omega(t)$ be finite everywhere and satisfy the condition (13) implies that these wormholes exist for finite or semiinfinite intervals of time (however large). At the end points of these intervals we have singularities.

Before we construct explicit examples it is necessary to discuss the embedding in R^3 of a $t=$const, $\theta=\pi/2$ slice briefly. Since our geometry is nonstatic each such slice will be different—more precisely the value of the function $\Omega(t)$ as $t=t_0$ (t_0 lies in the interval in which the wormhole exists) will dictate the shape and features at that instant. The metric (denoted by $\overline{ds}^2$) on such a slice takes the form

$$\overline{ds}^2=\Omega^2(t_0)\left[\frac{dr^2}{1-b(r)/r}+r^2d\phi^2\right] . \quad (15)$$

Define

$$r'=\Omega(t_0)r . \quad (16)$$

Thus (15) takes the form

$$\overline{ds}^2=\frac{dr'^2}{1-a(r')\Omega(t_0)/r'}+r'^2d\phi^2 , \quad (17)$$

where $a(r')$ is the functional form of $b(r)$ in the r' coordinate. The minimum value of r' which determines the throat radius is evaluated from

$$a(b_0')\Omega(t_0)=b_0' . \quad (18)$$

This clearly shows the dependence on $\Omega(t)$. Using the mathematics of embedding as outlined by Morris and Thorne [3], we can write

$$\frac{dz(r')}{dr'}=\left[\frac{a(r')\Omega(t_0)}{r'-a(r')\Omega(t_0)}\right]^{1/2} , \quad (19)$$

where $z(r')$ is the embedding function. Integrating (19) one can obtain the $z(r')$ for the slice at $t=t_0$.

We now discuss several examples to illustrate the facts mentioned above. The form of $b(r)$ to be chosen is $b(r)=b_0$ where b_0 is a constant. Such a form of $b(r)$ is not allowed for the case of a static geometry as it leads (from the Einstein equations) to $\rho=0$ ($\tau,p\neq 0$). However for a nonstatic geometry we can choose $b(r)=b_0$. For $\Omega(t)$ different functional forms can be assumed such as power laws, exponentials, trigonometric, and hyperbolic forms. Table I shows the various cases in a compact way. The condition (12) for the case $b(r)=b_0$ leads to an inequality of the form

$$r^3 \geq \max\{b_0[F(t)]^{-1}\} .$$

Since $r>b_0$ this leads to

$$b_0^2 \geq \max\{[F(t)]^{-1}\} .$$

TABLE I. The table shows for different choices of $\Omega(t)$ the corresponding functional forms of $F(t)$, the constraints on the minimum values of the throat radii, and the maximum ranges of t for which the wormhole can exist without collapsing into a singularity or allowing WEC violation of the matter stress energy.

$\Omega(t)$	$F(t)$	Lower bound on b_0 from Eq. (12)	Domain of t from $F(t)>0$ and finiteness of $\Omega(t)$
t^n	$2n(n+1)t^{-2}$	$b_0^2 \geq \max[t^2/2n(n+1)]$	$0<t<\infty$
$e^{\omega t}$	$2\omega^2$	$b_0^2 \geq (2\omega^2)^{-1}$	$-\infty<t\leq\infty$
$e^{-\omega t}$	$2\omega^2$	$b_0^2 \geq (2\omega^2)^{-1}$	$-\infty\leq t<\infty$
$\sin\omega t$	$2\omega^2(2\cot^2\omega t+1)$	$b_0^2 \geq \max\{[4\omega^2\cot^2\omega t+2\omega^2]^{-1}\}$	$\frac{m\pi}{\omega}<t<(m+1)\frac{\pi}{\omega}$
$\tanh\omega t$	$4\omega^2\mathrm{cosech}^2\omega t$	$b_0^2 \geq \max\{[4\omega^2\mathrm{cosech}^2\omega t]^{-1}\}$	$-\infty<t<0$ or $0<t<\infty$
$\sqrt{a^2+t^2}$	$\frac{2(2t^2-a^2)}{(a^2+t^2)^2}$	$b_0^2 \geq \max\left[\frac{(a^2+t^2)^2}{2(2t^2-a^2)}\right]$	$t^2>\frac{a^2}{2},\ t^2<\infty$
$\frac{t^2+a^2}{t^2+b^2}$	$\frac{4(b^2-a^2)(3t^2-a^2)}{(t^2+a^2)^2(t^2+b^2)}$	$b_0^2 \geq \max\left[\frac{(t^2+a^2)^2(t^2+b^2)}{4(b^2-a^2)(3t^2-a^{2})}\right]$	$-\infty<t<-\frac{a}{\sqrt{3}}$ or $\frac{a}{\sqrt{3}}<t<\infty$

We therefore obtain a lower bound on the throat radius parameter b_0. The constraint (13) together with $\Omega(t)\neq 0$ for all t implies a certain domain of validity for t. One could also have used a different form of $b(r)$ [e.g., $b(r)=\sqrt{b_0 r}$]. Certain characteristic features are mentioned below in a systematic way.

(i) The first five solutions in Table I exist for a finite amount of time which can be arbitrarily small or large. Curvature singularities signaled by the divergence of a component of the Riemann tensor appear at the boundaries of the time intervals mentioned against each solution.

(ii) Singularities occur at finite values of t only for the solutions of the form $\Omega(t)\sim t^n$ (at $t=0$), $\Omega(t)\sim \tanh\omega t$, (at $t=0$), $\Omega(t)\sim\sin\omega t$ [at $t=m\pi/\omega,(m+1)\pi/\omega$]. The latter case can be an example of a "wormhole universe" which begins with a "bang" and ends with a "crunch." One can think of π/ω as the "lifetime" of such a universe.

(iii) The exponential solutions can also have an expanding and a contracting phase if one considers time intervals of the form $-A<t<B$; $A,B>0$, and functional forms $e^{-\omega t}(t<0)$ and $e^{\omega t}(t>0)$ or $e^{\omega t}(t<0)$ and $e^{-\omega t}(t>0)$. The former case is that of an initial contraction followed by an expansion and the latter one leads to an initial expansion followed by a contraction.

(iv) If we assume a perfect fluid with an equation of state for the matter of the form $\rho/3=\tau=p$ then there are no wormhole solutions. In fact, the only solutions with a perfect fluid matter stress energy are the flat, closed, or open Friedmann-Robertson-Walker universes.

(v) If the energy-momentum tensor is assumed to be conformally invariant [i.e., $\mathrm{Tr}(T_{\mu\nu})=0$] then we have

$$\frac{\ddot{\Omega}}{\Omega}+\frac{b'}{3r^2}=0 . \tag{20}$$

Therefore, the only possible wormhole solution is the one for which $b(r)=\text{const}$ and $\Omega(t)=at+b$ (a and b being two constants).

(vi) The last entry in Table I provides an example of a solution valid for $-\infty<t<-a/\sqrt{3}$ or $a/\sqrt{3}$ or $a/\sqrt{3}<t<\infty$. Within the time domain $(-a/\sqrt{3},a/\sqrt{3})$, the WEC will be violated. By choosing a to be very small, one can construct a solution for $-\infty<t<\infty$ for which the required energy-momentum tensor will satisfy the WEC modulo, the fact that there would be a "flash" of WEC violation. Such a "flash" will exist only for an infinitesimal time interval. Also, for the solution with $\Omega(t)\sim(a^2+t^2)^{1/2}$ one can construct a solution for $-\infty<t<\infty$ with a similar "flash" of WEC violation for $-a/\sqrt{2}<t<a/\sqrt{2}$. It is important to note that the solution with $\Omega(t)\sim(t^2+a^2)/(t^2+b^2)$ for $-\infty\leq t\leq\infty$ will be nonsingular everywhere.

We conclude with a few remarks. We have shown that it is possible to have nonstatic Lorentzian wormhole spacetimes with WEC satisfying energy-momentum tensors. These "wormhole universes," however, exist for a finite–semi-infinite time interval. Allowing WEC violation for a finite (however small) time interval, we have shown that there is a solution for $-\infty<t<\infty$. Questions of traversability and possible models of time machines could be logical extensions of this paper. These will be discussed in a future article [10]. On the other hand, one knows that nonstatic spacetimes are backgrounds in which the study of quantum field theory yields processes such as "particle creation." Also, for our wormhole geometries one can do such an analysis. Finally, the fact that the WEC and the existence of Lorentzian wormholes are not entirely incompatible is perhaps quite encouraging for workers in this field.

Note added. Recently, T. Roman [Phys. Rev. D **47**, 1370 (1993)] has discussed a special Lorentzian wormhole geometry which is nonstatic. However, the corresponding energy-momentum tensor in his case violates WEC as the scale factor when substituted in the expression for $F(t)$ [Eq. (13) in this paper] gives $F(t)=0$. The analysis in this paper, however, is for those $\Omega(t)$ which satisfy the condition $F(t)>0$ strictly. As has been shown here, this condition is crucial for avoiding WEC violations at least for finite–semi-infinite time intervals.

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