Quasilocal gravitational energy

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A dynamically preferred quasilocal definition of gravitational energy is given in terms of the Hamiltonian of a 2+2 formulation of general relativity. The energy is well defined for any compact orientable spatial two-surface, and depends on the fundamental forms only. The energy is zero for any surface in flat spacetime, reduces to the Hawking mass in the absence of shear and twist, and reduces to the standard gravitational energy in spherical symmetry. For asymptotically flat spacetimes, the energy tends to the Bondi mass at null infinity and the ADM mass at spatial infinity, taking the limit along a foliation parametrized by the area radius. The energy is calculated for the Schwarzschild, Reissner-Nordström, and Robertson-Walker solutions, and for plane waves and colliding plane waves. Energy inequalities are discussed, and for static black holes the irreducible mass is obtained on the horizon. Criteria for an adequate definition of quasilocal energy are discussed.

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I. INTRODUCTION

It comes as a surprise to many that there is no agreed definition of gravitational energy (or mass) in general relativity. In Newtonian gravity, there is a material density which may be integrated over a spatial three-surface to give the mass, and the Poisson equation and Gauss theorem then yield an expression for the mass inside a two-surface, measured on the two-surface—a so-called quasilocal mass. Finding a similar quasilocal expression for the Newtonian gravitational energy is impossible, since the energy density, unlike the material density, is not a total divergence. In general relativity, material mass is merely one aspect of the energy-momentum-stress tensor, and gravitational energy, if well defined at all, is nonlocal, as follows from the equivalence principle. The gravitational field can be measured by the geodesic deviation of two observers, but a single observer cannot distinguish it from kinematical effects. Equivalently, curvature cannot be measured on a point or line, but requires a two-surface at least.

In the case of an asymptotically flat spacetime, the Bondi [1,2] and Arnowitt-Deser-Misner (ADM) [3] masses are well-defined asymptotic quantities which are generally accepted as the total mass of the spacetime as measured on spheres at null infinity and spatial infinity, respectively. The existence of these asymptotic definitions, and in particular the Bondi mass-loss result, indicate that gravitational energy is not meaningless, and that there is an actual physical process involved in radiating gravitational energy to infinity in such a spacetime. Since an appropriate definition of this energy cannot be found locally, a quasilocal definition is sought, where "quasilocal" is used in the sense originally introduced by Penrose [4], referring to a compact orientable spatial two-surface, usually spherical. This is the natural finite analogue of the asymptotic definitions.

In the case of spherical symmetry, there is a standard definition of gravitational energy due originally to Misner and Sharp [5]. This reduces to the Schwarzschild mass in vacuum, and is the unique such functional which depends only on the metric and its first derivatives. It also has the correct Newtonian limit, yielding the Newtonian mass to leading order and the Newtonian energy to the next order [6]. This definition can be thought of as the prototype of quasilocal gravitational energy, defined for metric spheres.

A rough way to understand the concept is as follows. A massive body produces a gravitational field which itself has (negative) energy, which in turn modifies the field and contributes to a total effective energy in a nonlinear, nonlocal way. Quasilocal energy is intended to measure this effective energy. For instance, outside a source in spherical symmetry, the gravitational energy is just the Schwarzschild mass, which is all that can be determined about the source from measurements of its external gravitational field. More generally, the idea is to determine the effective energy of a source by measurements on a two-surface enclosing the source.

It is important to emphasize that quasilocal energy is supposed to be a functional of two-surfaces rather than of three-surfaces with boundary, since there is a widespread tendency to regard gravitational energy as referring to three-surfaces, presumably by analogy with energy in Newtonian theory. This is not consistent with the nature of the Bondi and ADM masses, nor with that of the spherically symmetric energy. It seems that general relativity differs from Newtonian theory in that the evidence suggests that gravitational energy is associated with two-surfaces, rather than with either points or three-surfaces. This also has the practical advantage that one can "weigh" a source by taking measurements at a distance, rather than having to enter the source, with all the attendant hazards.

Hawking [7] defined a quasilocal mass which has several desirable properties: it reduces to the standard definition in spherical symmetry, it tends to the Bondi mass in an asymptotically flat spacetime, taking the limit along an affinely parametrized foliation, and it tends to zero as a sphere in any spacetime is shrunk to a point [8,9]. Unfortunately, it is nonzero for generic two-surfaces in flat spacetime. This drawback is easily corrected by adding a certain term, as is shown subsequently.

Penrose [4] emphasized the conceptual importance of quasilocal mass, and suggested a twistorial construction. This can be properly defined if the two-surface can be transplanted into a conformally flat spacetime [10,11], but generically the transplant cannot be made without damaging the structure of the two-surface [12]. Bergqvist [13] reviews various similar spinorial attempts, of which that of Dougan and Mason [14] comes closest to being a well-defined functional of two-surfaces, being well defined generically, but breaking down for the important case of marginally trapped two-surfaces. Bergqvist also shows that seven different definitions all give different results in two simple examples: namely, the Reissner-Nordström solution and the Kerr horizon. This is an unfortunate situation which indicates the need for a more critical approach to what constitutes an adequate definition. Currently, there seems to be no demand for a quasilocal energy to be even well defined, let alone geometrically or dynamically natural.

The main purpose of this article is to present, in Sec. III, a dynamically preferred quasilocal energy which is essentially the 2+2 Hamiltonian of the Einstein gravitational field. This is a well-defined functional of compact orientable spatial two-surfaces, and is geometrically natural in the sense of depending only on the fundamental forms of the two-surface, introduced in Sec. II. The energy is shown to vanish for any surface in flat spacetime in Sec. IV, to tend to the Bondi mass at null infinity in Sec. V, and to tend to the ADM mass at spatial infinity in Sec. VI, if the spacetime is asymptotically flat. Uniqueness on purely geometrical grounds is considered in Sec. VII, examples in Sec. VIII, energy inequalities in Sec. IX, and criteria for an adequate definition in Sec. X.

II. GEOMETRY OF TWO-SURFACES

It is appropriate to begin with a review of the geometry of a compact orientable spatial two-surface S embedded in spacetime, according to the 2+2 formalism developed by the author [15], which describes null foliations of such surfaces. In this formalism, a basis (e_1^a, e_2^a) for S is completed to a spacetime basis (u^a, v^a, e_1^a, e_2^a) by Lie propagation: $(\mathcal{L}_u, \mathcal{L}_v)(u^a, v^a, e_1^a, e_2^a)=0$, where $\mathcal{L}$ denotes the Lie derivative. The commuting vectors (u^a, v^a) are referred to as the evolution vectors, since Lie propagation in these directions enables the spacetime in a neighborhood of S to be developed. (This is analogous to the Cauchy problem, where the single evolution direction may be decomposed into a lapse function and a shift vector.) The choice of evolution vectors is partially fixed by demanding that the three-surfaces $(\mathcal{L}_u S, \mathcal{L}_v S)$ are null, $u^a l_a = v^a n_a = 0$, where the null normals (l_a, n_a) satisfy

$$\nabla_{[a} l_{b]} = \nabla_{[a} n_{b]} = 0 \ , \quad l_a l^a = n_a n^a = 0 \ ,$$

$$l_a n^a = -e^m \ , \quad h_{ab} l^b = h_{ab} n^b = 0 \ ,$$

$$g_{ab} = h_{ab} - e^{-m}(l_a n_b + n_a l_b) \ ,$$

and g_{ab} is the spacetime metric, h_{ab} the induced two-metric of S, and m is called the scaling function. The remaining freedom in the evolution vectors is given by the shift 2-vectors $r^a = h^a_b u^b$ and $s^a = h^a_b v^b$. The metric then decomposes into

$$g_{ab} = \begin{bmatrix} r_c r^c & r_c s^c - e^{-m} & r_b \\ r_c s^c - e^{-m} & s_c s^c & s_b \\ r_a & s_a & h_{ab} \end{bmatrix}$$

in the above basis. Note that the null normals are uniquely defined only up to interchange $(l_a, n_a) \mapsto (n_a, l_a)$ and boosts $(l_a, n_a) \mapsto (\lambda l_a, \lambda^{-1} n_a)$. This freedom is left open.

The dynamically independent first derivatives of the metric consist of the expansions

$$\theta = \tfrac{1}{2} h^{cd} \mathcal{L}_{u-r} h_{cd} \ , \quad \tilde{\theta} = \tfrac{1}{2} h^{cd} \mathcal{L}_{v-s} h_{cd} \ ,$$

the (traceless) shears

$$\sigma_{ab} = h^c_a h^d_b \mathcal{L}_{u-r} h_{cd} - \tfrac{1}{2} h_{ab} h^{cd} \mathcal{L}_{u-r} h_{cd} \ ,$$

$$\tilde{\sigma}_{ab} = h^c_a h^d_b \mathcal{L}_{v-s} h_{cd} - \tfrac{1}{2} h_{ab} h^{cd} \mathcal{L}_{v-s} h_{cd} \ ,$$

the "inaffinities"

$$\nu = \mathcal{L}_{u-r} m \ , \quad \tilde{\nu} = \mathcal{L}_{v-s} m \ ,$$

and the "twist," or "anholonomicity," or commutator of the null normals,

$$\omega_a = \tfrac{1}{2} e^m h_{ab} (\mathcal{L}_u s^b - \mathcal{L}_v r^b - \mathcal{L}_r s^b) \ .$$

These fields encode the extrinsic curvature of S, or equivalently the momenta conjugate to the configuration fields (h_{ab}, r^a, s^a, m). In [15], the vacuum Einstein equations are written in a first-order from in terms of $(h_{ab}, r^a, s^a, m, \theta, \tilde{\theta}, \sigma_{ab}, \tilde{\sigma}_{ab}, \nu, \tilde{\nu}, \omega_a)$, which constitutes a 2+2 analogue of the ADM 3+1 formalism. The initial data for the associated characteristic initial value problem can be taken as $(h_{ab}, m, \theta, \tilde{\theta}, \omega_a)$ on S, (σ_{ab}, ν) on $\mathcal{L}_u S$, and $(\tilde{\sigma}_{ab}, \tilde{\nu}, s^a)$ on $\mathcal{L}_v S$, with r^a being prescribed over the whole four-dimensional patch. Of this data, r^a, s^a, m, ν, and $\tilde{\nu}$ represent coordinate freedom on the respective surfaces, and will be set to zero henceforth.

The remaining dynamical variables are, in traditional language, the first fundamental form h_{ab}, the (null) second fundamental forms $\sigma_{ab} + \theta h_{ab}$ and $\tilde{\sigma}_{ab} + \tilde{\theta} h_{ab}$, and the normal fundamental form ω_a [16], with the null normals fixing a preferred basis for the fibers of the normal bundle, up to boosts and interchange. These fundamental forms are the only geometrical invariants of a two-surface in spacetime, and are also the dynamically independent parts of the gravitational field referred to a two-surface. It is therefore most natural to seek a definition of quasilocal energy in terms of these forms. In fact, of the previous definitions, this is the case only for the Hawking mass:

$$M_H=\frac{1}{8\pi}\left[\frac{A}{16\pi}\right]^{1/2}\int_S\mu(\mathcal{R}+\theta\tilde{\theta})\ ,\quad A=\int_S\mu\ ,$$

where μ is the area two-form and $\mathcal{R}$ the Ricci scalar of h_{ab}.

III. DEFINITION OF THE ENERGY

To measure the energy of a field on a spatial three-surface, the Hamiltonian of a 3+1 formulation is usually taken. On a compact orientable spatial two-surface S, the analogous dynamical object for the Einstein field is the Hamiltonian two-form, which may be quoted from [15] as

$$8\pi\mathcal{H}=-\mu(\mathcal{R}+\theta\tilde{\theta}-\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab}-2\omega_a\omega^a)\ ,$$

where the notation and choice of evolution are as in Sec. II, and the 8π has been inserted to agree with the units $G=1$ used in most papers on quasilocal energy. This is a preferred dynamical quantity in the sense that variation of the (full) Hamiltonian yields the Einstein equations in a form adapted to null foliations of spatial two-surfaces. The Hamiltonian is obtained from a Lagrangian which is the Einstein-Hilbert Lagrangian in 2+2 form, up to a total divergence. In a 3+1 context, the energy is defined as the integral of the Hamiltonian over the three-surface, but in a 2+2 context, $\int_S\mathcal{H}$ needs to be multiplied by a length to give energy units, and the only natural length scale is given by the area

$$A=\int_S\mu\ .$$

The proposed definition of the gravitational energy of a two-surface, or its quasilocal energy, is then

$$E=-\left[\frac{A}{16\pi}\right]^{1/2}\int_S\mathcal{H}\ ,$$

where the factor has been chosen for agreement with the Schwarzschild mass. Thus

$$E=\frac{1}{8\pi}\left[\frac{A}{16\pi}\right]^{1/2}\int_S\mu(\mathcal{R}+\theta\tilde{\theta}-\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab}-2\omega_a\omega^a)\ .$$

It should be noted that the expression is valid for any one two-surface, but that when considering the variation of energy between surfaces, the quasilocal coordinate freedom cannot be fixed as in Sec. II and the full Hamiltonian should be used.

Various immediate observations concerning E can be made. First, the total energy of two surfaces $S=S_1\cup S_2$ is greater than the sum of the individual energies, assumed positive, since $A=A_1+A_2$ and $E/\sqrt{A}=E_1/\sqrt{A_1}+E_2/\sqrt{A_2}$. More generally,

$$E\left[\bigcup_i S_i\right]\geq\sum_i E(S_i)$$

if $E(S_i)\geq 0$. Second, the term in the Ricci scalar may be integrated by the Gauss-Bonnet theorem [16]:

$$\int_S\mu\mathcal{R}=8\pi(1-g)\ ,$$

where the genus g is a topological invariant representing the number of handles of S: $g=0$ for a sphere, $g=1$ for a torus, etc. Third, E reduces to the Hawking mass M_H in the shear-free, twist-free case, and hence reduces to the standard definition in spherical symmetry. Finally, the shear term in E is precisely the addition to the Hawking mass required to yield zero in flat spacetime, as follows.

IV. FLAT SPACETIME

Recall first the contracted Gauss equation [16]:

$$\mathcal{R}+\theta\tilde{\theta}-\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab}=h^{ac}h^{bd}R_{abcd}\ ,$$

which is a purely geometrical equation describing the embedding of S, with R_{abcd} being the Riemann tensor of g_{ab}. In flat spacetime, commutativity of the null normals means that the twist vanishes, $\omega_a=0$, and the Gauss equation becomes

$$\mathcal{R}+\theta\tilde{\theta}-\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab}=0\ ,$$

so that $\mathcal{H}$ and E vanish.

Alternatively, note that the expansions and shears can be taken as equal and opposite, $\theta=-\tilde{\theta}$, $\sigma_{ab}=-\tilde{\sigma}_{ab}$, by fixing the boost freedom of the normals. For a two-surface which lies entirely in one Euclidean three-plane, Euclidean surface theory [16] can be applied. Here, the Ricci scalar is twice the Gaussian curvature, and so is defined by $\mathcal{R}=2\kappa_+\kappa_-$, where the principal curvatures $\kappa_\pm$ are the roots of the eigenvalue equation

$$\det\left[\frac{1}{\sqrt{2}}(\sigma_{ab}+\theta h_{ab})-\kappa h_{ab}\right]=0\ ,$$

since the null second fundamental form is $\sqrt{2}$ times the Euclidean second fundamental form. This yields

$$\sqrt{2}\kappa_\pm=\theta\pm\sqrt{\tfrac{1}{2}\sigma_{ab}\sigma^{ab}}\ ,$$

and so

$$\mathcal{R}-\theta^2+\tfrac{1}{2}\sigma_{ab}\sigma^{ab}=0\ ,$$

so that $\mathcal{H}$ and E vanish.

Since $\sigma_{ab}\tilde{\sigma}^{ab}\leq 0$, it also follows that the Hawking mass is nonpositive in flat spacetime, $M_H\leq 0$, vanishing only for shear-free surfaces. A similar argument shows that the Geroch mass [17] is also nonpositive in flat spacetime, since the Geroch mass is never greater than the Hawking mass [9]. This problem with the Hawking mass was well known, and its solution is simple: add the appropriate term in the shear, as determined by the Gauss equation. In this sense, E can be regarded as a modification of the Hawking mass which provides a more realistic measure of gravitational energy.

Note that the vanishing of E in flat spacetime also guarantees that E tends to zero as a sphere in any spacetime is shrunk to a point. Indeed, by following the small-sphere approximation of Horowitz and Schmidt [9], it can be seen that the leading-order asymptotic behavior is the same as for the Hawking mass, essentially because the shears and twist contribute at a lower order. Namely, $E=O(r^3)$ in terms of an area radius r, with the coefficient determined by the energy tensor of the matter. In vacuum, $E=O(r^5)$, with the coefficient determined by the Bel-Robinson tensor.

V. NULL INFINITY

Asymptotically flat spacetimes were first studied in the axisymmetric case by Bondi *et al.* [1], and in general by Sachs [2]. They defined an asymptotic mass M_B on slices of null infinity, and showed that it decreased to the future, which is interpreted as a loss of mass due to gravitational radiation. Briefly sketching the approach: on a foliation of null surfaces labeled by u, spherical polar coordinates (r,ϑ,φ) were introduced, such that each surface S of constant (u,r) had area form $\mu=r^2\sin\vartheta\, d\vartheta\wedge d\varphi$, and hence area $A=4\pi r^2$. Asymptotic expansions in negative powers of r were assumed, and the Bondi mass M_B was defined as the integral over $\sin\vartheta\, d\vartheta\wedge d\varphi$ of a certain coefficient, giving a generalization of the Schwarzschild mass. In the Appendix it is shown that the Bondi mass may be written as

$$M_B=\lim_{r\to\infty}\frac{1}{8\pi}\left[\frac{A_r}{16\pi}\right]^{1/2}\int_{S_r}\mu h^{ac}h^{bd}C_{abcd}\ ,$$

where C_{abcd} is the Weyl tensor of g_{ab}. Here r labels the surfaces S_r, with area $A_r=4\pi r^2$, and dependence on u has been suppressed. The expression shows that the Bondi mass may be described as the average asymptotic Coulomb part of the gravitational field. The advantage of such an invariant expression in the present context is that the radius r does not appear explicitly except as the limit parameter, so that the limit could simply be removed to yield a quasilocal mass corresponding to M_B. Using the vacuum Gauss equation

$$\mathcal{R}+\theta\tilde{\theta}-\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab}=h^{ac}h^{bd}C_{abcd}\ ,$$

the Bondi mass may be rewritten in terms of the fundamental forms as

$$M_B=\lim_{r\to\infty}\frac{1}{8\pi}\left[\frac{A_r}{16\pi}\right]^{1/2}\int_{S_r}\mu(\mathcal{R}+\theta\tilde{\theta}-\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab})\ .$$

Comparing with the quasilocal energy E, the asymptotic behavior of the various terms may be translated from the spin-coefficient expressions in Sec. 9.8 of Penrose and Rindler [18]:

$$\mathcal{R}=2r^{-2}+O(r^{-3})\ ,\quad \theta\tilde{\theta}=-2r^{-2}+O(r^{-3})\ ,$$

$$\sigma_{ab}\tilde{\sigma}^{ab}=O(r^{-3})\ ,\quad \omega_a\omega^a=O(r^{-6})\ .$$

Hence the asymptotic limit of E exists, $E=O(1)$, and equals the Bondi mass:

$$M_B=\lim_{r\to\infty}E(S_r)=-\tfrac{1}{2}\lim_{r\to\infty}r\int_{S_r}\mathcal{H}\ ,$$

since the twist term tends to zero. Thus E satisfies the criterion of tending to the Bondi mass asymptotically, if the latter exists.

Note that the expression for Bondi-Sachs mass given by Penrose [19–21] is an asymptotic limit of the Hawking mass:

$$M_B=\lim_{\hat{r}\to\infty}M_H(S_{\hat{r}})\ ,$$

which appears to differ from the previous expression by a term in the shears. This is due to a different limit being taken, using a foliation based on an affine parameter $\hat{r}=r+O(r^{-1})$, as is explained in detail in the Appendix. The affine expression for the Bondi mass is used in most work on asymptotic flatness [18,22–26], though other expressions also exist [27,28]. Note also the "linkage" formulation of asymptotic mass and angular momentum [29–33]. Most positive-mass proofs use the affine version [34–36], as can be seen by comparing the expressions for mass with the previous definitions [37,38]. A positive-mass proof has been given by Schoen and Yau [39] using the original approach.

VI. SPATIAL INFINITY

Arnowitt *et al.* [3] defined an asymptotic mass $M_{\rm ADM}$ at spatial infinity, which is interpreted as the total mass of an asymptotically flat spacetime. The definition may be written in a more coordinate-independent way [40–42] as

$$M_{\rm ADM}=\lim_{r\to\infty}\frac{1}{8\pi}\left[\frac{A_r}{16\pi}\right]^{1/2}\int_{S_r}\mu h^{ac}h^{bd}C_{abcd}\ ,$$

with r now denoting the area radius of a family of spheres S_r which approach spatial infinity in a spatial three-surface (Appendix). As at null infinity, the twist term in E disappears in the limit, so that

$$M_{\rm ADM}=\lim_{r\to\infty}E(S_r)=-\tfrac{1}{2}\lim_{r\to\infty}r\int_{S_r}\mathcal{H}\ .$$

Thus M_B and $M_{\rm ADM}$ are null and spatial limits of the same quantity E, or equivalently of the average asymptotic Coulomb part of the gravitational field. This agreement has apparently not been noticed previously. Indeed, in relating $M_{\rm ADM}$ to the limit of M_B, it was thought necessary to require the vanishing at spatial infinity of the shear term that apparently distinguishes the area-radius definition from the affine version [42,43]. Note that it is not clear whether $M_{\rm ADM}$ is the limit of the Bondi mass M_B at spatial infinity, since the limits are from different directions and may not coincide. The situation is simply that if M_B or $M_{\rm ADM}$ exist, then they are the appropriate limits of E.

VII. ON UNIQUENESS

The results of Secs. IV–VI indicate that the freedom to choose a sensible quasilocal energy, even on purely geometrical grounds, is quite limited. More precisely, assume that the energy can depend only on the fundamental forms of the two-surface, including invariance under interchange and boosts between the null normals. Assume also that the energy is second degree in g_{ab}, i.e., a linear combination of second derivatives and quadratic first derivatives, cf. the discussion in Sec. 11.2 of Wald [24]. Then the only such functions are linear combinations of $\mathcal{R}$, $\theta\tilde{\theta}$, $\sigma_{ab}\tilde{\sigma}^{ab}$ and $\omega_a\omega^a$. The coefficients of $\mathcal{R}$

and $\theta\tilde{\theta}$ are determined by consideration of metric spheres in flat spacetime and in the Schwarzschild solution, and the coefficient of $\sigma_{ab}\tilde{\sigma}^{ab}$ is determined by demanding that the energy vanish in flat spacetime, as in Sec. IV. This automatically yields the Bondi mass at null infinity, as in Sec. V, and the ADM mass at spatial infinity, as in Sec. VI. This leaves only the coefficient of $\omega_a\omega^a$, which is not so clearly determined, but which can be interpreted as a contribution due to angular momentum [44]. The above assumptions are hardly compulsory but seem quite plausible, and it is sensible at least to exhaust such possibilities before resorting to more radical suggestions. It is certainly remarkable that E automatically satisfies these desiderata.

VIII. EXAMPLES

In calculating the energy for particular spacetimes, it is often most convenient to use the full Hamiltonian. Alternatively, in simple cases a transformation may be sought in which the line-element, evaluated at the two-surface S in question, takes the form

$$ds^2=-2d\xi\, d\eta+h_{ab}dx^a dx^b ,$$

where (x^1,x^2) are coordinates on S and (ξ,η) are affine parameters; this is referred to subsequently as the standard form. The fundamental forms may then be calculated by noting that $\mathcal{L}_u=\partial/\partial\xi$ and $\mathcal{L}_v=\partial/\partial\eta$ in the basis used. Also useful are the expressions $\mu\theta=\mu_\xi$ and $\mu\tilde{\theta}=\mu_\eta$.

In the particular case of spherical symmetry, a sphere of symmetry has zero twist and shears, and the same results are found as for the Penrose mass according to Tod [10] or the Hawking mass. Here spherical polar coordinates (r,ϑ,φ) may be taken such that $\mu=r^2\sin\vartheta\, d\vartheta\wedge d\varphi$, and consequently $A=4\pi r^2$, $\mathcal{R}=2r^{-2}$, $\theta=2r^{-1}r_\xi$, $\tilde{\theta}=2r^{-1}r_\eta$, and so

$$E=r(r_\xi r_\eta+\tfrac{1}{2}) .$$

This is the standard definition of gravitational energy in spherical symmetry, which was originally given in a different form by Misner and Sharp [5]. A few examples follow, with interpretive comments.

The Schwarzschild black-hole solution

$$ds^2=-\left[1-\frac{2m}{r}\right]dt^2+\left[1-\frac{2m}{r}\right]^{-1}dr^2+r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2)$$

can be put in standard form by taking

$$r_\xi=-r_\eta=-\frac{1}{\sqrt{2}}\left[1-\frac{2m}{r}\right]^{1/2},$$

$$t_\xi=t_\eta=\frac{1}{\sqrt{2}}\left[1-\frac{2m}{r}\right]^{-1/2},$$

and so $E=m$. It should be emphasized that an interior solution matched to the Schwarzschild exterior need not have total material mass m as measured on a spatial three-surface. In this sense E is not sensitive to the distribution of matter inside S, being only a single number, but rather measures the *effective* active gravitational energy as felt on S. Another way to see the necessity of considering the effective energy is to note that the maximally extended Schwarzschild solution has zero material mass, being a vacuum solution.

The Reissner-Nordström charged black-hole solution

$$ds^2=-\left[1-\frac{2m}{r}+\frac{e^2}{r^2}\right]dt^2+\left[1-\frac{2m}{r}+\frac{e^2}{r^2}\right]^{-1}dr^2+r^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2)$$

can be put in standard form by taking

$$r_\xi=-r_\eta=-\frac{1}{\sqrt{2}}\left[1-\frac{2m}{r}+\frac{e^2}{r^2}\right]^{1/2},$$

$$t_\xi=t_\eta=\frac{1}{\sqrt{2}}\left[1-\frac{2m}{r}+\frac{e^2}{r^2}\right]^{-1/2},$$

and so $E=m-\frac{1}{2}e^2r^{-1}$. Tod [10] obtained the same result for the Penrose mass, and noted that the term in e agrees with the linearized limit. If this is regarded as a correction to m which yields the effective energy, then there is the interesting result that the field becomes repulsive close to the $r=0$ singularity, a result which is also indicated by the behavior of geodesics, as in Sec. 5.5 of Hawking and Ellis [45]. For a nonextreme black hole, $e^2<m^2$, such negative E occurs inside the inner horizon, which is widely regarded as unstable due to an "infinite-blue shift" effect, in which case the inner region would not exist in practice. Nevertheless, the example indicates that a negative energy could be interpreted in a quasi-Newtonian way as a net repulsion of the two-surface, cf. [46].

The Robertson-Walker cosmological solutions are given by

$$ds^2=-dt^2+a^2[d\psi^2+f^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2)] ,$$

where $a(t)$ and

$$f(\psi)=\begin{cases}\sin\psi & \text{for } k=1 ,\\ \psi & \text{for } k=0 ,\\ \sinh\psi & \text{for } k=-1 ,\end{cases}$$

where k labels the spherical, flat, and hyperbolic cases, respectively. This may be put in standard form with radius $r=af$ by taking

$$\psi_\xi=-\psi_\eta=\frac{1}{a\sqrt{2}} , \quad t_\xi=t_\eta=\frac{1}{\sqrt{2}} .$$

The field equation $a_t^2=\frac{8}{3}\pi a^2\rho-k$, where ρ is the density, then yields $E=\frac{4}{3}\pi r^3\rho$. Recalling that $A=4\pi r^2$, E is just the product of density and the "area volume" associated with the area radius r, rather than the actual volume of a three-surface inside S. Again, this leads to the interpretation of E as the effective energy felt at S. Note that E is independent of the pressure, which could for instance be negative, as in the de Sitter and anti–de Sitter solutions.

Although quasilocal energy is usually considered only for spheres, the definition of E applies to any compact orientable two-surface, and it is interesting to compare the results obtained for a flat torus in a plane-symmetric spacetime with topological identifications. The general plane-symmetric line-element is given by Szekeres [47] as

$$ds^2=-2e^{-K}du\,dv+e^{-P}(e^{Q}\cosh W\,dx^2-2\sinh W\,dx\,dy+e^{-Q}\cosh W\,dy^2)\;,$$

where (K,P,Q,W) are functions of (u,v), and the toroidal identifications are

$$(x,y)=(x+x_0,y+y_0)\;.$$

The metric may be put in standard form by taking

$$u_\xi=v_\eta=e^{K/2}\;,\quad u_\eta=v_\xi=0\;,$$

and it is straightforward to calculate $\mu=e^{-P}dx\wedge dy$, $A=e^{-P}x_0y_0$, $\mathcal{R}=0$, $\theta\tilde{\theta}=e^{K}P_uP_v$, $\sigma_{ab}\tilde{\sigma}^{ab}=2e^{K}(Q_uQ_v\cosh^2W+W_uW_v)$, $\omega_a\omega^a=0$, and hence

$$E=\frac{1}{4}\left[\frac{x_0y_0}{4\pi}\right]^{3/2}e^{K-3P/2}(P_uP_v-Q_uQ_v\cosh^2W-W_uW_v)\;.$$

Plane-wave spacetimes are given by the above metric depending only on u, or only on v, so that $E=0$. This result is not surprising if E is interpreted as a measure of nonlinear gravitational energy, since plane waves propagate linearly.

Colliding-plane-wave spacetimes were introduced by Szekeres [47,48] and reviewed by Griffiths [49]. These have nonzero E, with no definite sign. As an example, the solution of Khan and Penrose [50], describing the collision of two impulsive gravitational waves, has $E<0$ after the collision, with E becoming unbounded at the singularity formed by the collision. (The exact expression is long and unilluminating.) The usual interpretation is that the two incoming waves interact nonlinearly, producing scattered radiation Ψ_2 whose magnitude grows without bound, a property dependent on the plane symmetry. The energy E provides a measure of the nonlinear interaction, being zero before the collision and becoming unbounded at the singularity. This illustrates both that E is not a preserved quantity, and that it is sensitive to gravitational interactions in vacuum. Similar results are obtained for generic colliding plane waves, and for asymptotically plane waves [51,52]: E equals $\mathrm{Re}\Psi_2$ up to a factor, with Ψ_2 generically becoming unbounded after the collision.

IX. ENERGY INEQUALITIES

It is sometimes suggested that quasilocal energy should be manifestly nonnegative. However, negative-energy matter can be described quite consistently in general relativity, and it seems reasonable to allow the quasilocal energy to become negative in such a situation. Additionally, a basic requirement of quasilocal energy is that it yield the Schwarzschild mass for the Schwarzschild solution, and this may be negative. The examples of Sec. VIII show that E is positive for certain physically familiar gravitational fields, but that negative E may occur in other circumstances, such as the negative-mass Schwarzschild solution. From purely quasilocal arguments, the fundamental forms are freely specifiable on any one two-surface, and so it is easy to construct two-surfaces with negative E. Nevertheless, it may be possible to find a positivity theorem based on global assumptions, such as asymptotic flatness, global hyperbolicity, etc., together with a local energy condition on the matter. A weaker possibility would be positivity for spacetimes sufficiently close to flat spacetime [25], or for sufficiently round spheres [53]. Conversely, E may be a suitable quantity to control the gravitational field in the context of global existence theorems or cosmic censorship. In this context it is interesting to note that in the two-dimensional dilaton gravity theory of Callan *et al.* [54], positive gravitational energy is associated with trapped spatial singularities, and negative energy with naked singularities [55]. In spherical symmetry, there is a similar relationship between the sign of E and the signature and trapping of singularities [6].

A related question is that of monotonicity, e.g., whether E increases as A increases along a foliation of two-surfaces. Here it may be useful to consider *well-oriented* surfaces such that $\theta\tilde{\theta}<0$, which have an orientation determined according to which expansion is negative and which positive. For such surfaces, the Hawking mass decreases internally and increases externally [8], which can be interpreted as a quasilocal generalization of the Bondi mass-loss result. A similar result for E would be of interest, but seems more difficult to obtain.

Similar comments apply to energy inequalities such as the Gibbons-Penrose isoperimetric inequality [56,57] and the Thorne hoop conjecture [58] for trapped surfaces. For a marginally trapped sphere, $\theta\tilde{\theta}=0$ and $g=0$,

$$\left[\frac{16\pi E^2}{A}\right]^{1/2}=1-\frac{1}{8\pi}\int_S\mu(\tfrac{1}{2}\sigma_{ab}\tilde{\sigma}^{ab}+2\omega_a\omega^a)\;,$$

and consequently the isoperimetric inequality $16\pi E^2\geq A$ can be violated by a suitable quasilocal choice of shears and twist. Again, global assumptions may be relevant. Note, however, that the twist and internal shear vanish on a Killing horizon [11], so that a *static* black hole satisfies the isoperimetric *equality* $16\pi E^2=A$ on the horizon, a more precise result than hitherto obtained [11,59–61]. The very triviality of this result suggests that E may be an appropriate definition for such purposes. Expressed equivalently, E is the irreducible mass [62] on a static horizon, i.e., the mass of a Schwarzschild black hole with the same area.

X. CRITERIA FOR QUASILOCAL ENERGY

As noted in the Introduction, there is now a plethora of suggested quasilocal energies, which disagree even in the Reissner-Nordström and Kerr cases [13]. The ambiguity can only be resolved by settling on generally agreed criteria, and I would like to make a few such suggestions. Most fundamentally, the energy should be *well defined* on any compact orientable spatial two-surface, or at least for the case of spherical topology. This firm but fair requirement disqualifies almost all of the contenders, which depend on particular symmetries [46,63–65], a preferred three-surface [17,66,67], global requirements [68,69] or the existence of solutions to particular equations [4,14,70]. To the best of my knowledge, this leaves only E and the Hawking mass as unambiguous quasilocal definitions.

The other basic criteria are that the energy vanishes in flat spacetime, reduces to the standard definition in spherical symmetry, and yields the Bondi mass at null infinity and the ADM mass at spatial infinity. It would also be of interest to investigate the Newtonian and linearized limits. Eardley [8] and Christodoulou and Yau [52] give a comparable list of criteria, of which positivity and monotonicity remains as open questions for E. Bergqvist [70] also gives a list of criteria, and shows that there are infinitely many possibilities which satisfy it. The problem here is that many of the suggested quasilocal energies are based on the introduction of spinor fields satisfying various particular equations, such as the twistor or Sen-Witten equations, and there is no natural way to choose between them. The embarrassment of possibilities can be reduced by disallowing spurious fields and instead demanding that the definition depend directly on the fundamental forms, as these are the only purely geometrical quantities associated with a two-surface. The fundamental forms are also the dynamically independent parts of the gravitational field on a two-surface, or equivalently the free gravitational data.

Even with such criteria, the arguments of Sec. VII show that a unique "correct" quasilocal energy will require further justification, presumably of a dynamical sort. The main point of this article is that the 2+2 Hamiltonian $\mathcal{H}$ is a dynamically preferred quantity which can justifiably be interpreted as the energy density of the Einstein gravitational field referred to a two-surface element. The fact that the associated quasilocal energy E has the agreeable properties described herein may be taken as firm supporting evidence. In view of this, are there any objections to using the 2+2 Hamiltonian of a field to *define* its quasilocal energy? Is E the long-sought gravitational energy?

Note added. Another desired property of quasilocal energy is that it should be related to the Hawking temperature in the context of black holes. For the Schwarzschild solution with mass m, Hartle and Hawking [76] obtain the temperature $1/8\pi m$ as measured by a constant-r detector. It is therefore desirable that a quasilocal energy coincide with m for constant-r spheres in the Schwarzschild solution, as would be the case for any quasilocal energy that coincided with the standard definition in spherical symmetry. It has been suggested that one might instead expect to obtain a different temperature, taking into account the blueshifting from infinite to finite radius [67]. However, one must also take into account the Unruh effect for an accelerating detector, since constant-r detectors are accelerating. The answer obtained from quantum field theory is that the temperature measured by such a detector is $1/8\pi m$ [76]. In other words, the blueshift effect and acceleration effect cancel in this case. This is discussed after Eq. (8.85) of Birrell and Davies [77].

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APPENDIX: THE BONDI MASS

When defining the Bondi mass as a limit of a quasilocal energy along a null foliation of two-surfaces S, there is an important subtlety, namely, that the limit depends on the foliation used, and in particular whether the foliation is based on an affine parameter or a luminosity parameter (area radius). Conversely, different quasilocal integrals must be used in order to obtain the same limit [71,72]. The following is based on a calculation of Frauendiener [71].

Assume a parameter s which is related to the area radius r by

$$r=s+\tfrac{1}{2}fs^{-1}+O(s^{-2})$$

for some f. Take a spin basis $(l,n,m,\bar{m})$ such that m spans S and $l=\partial/\partial s$. The Newman-Penrose convergence ρ is then given by $\rho=-s^{-1}+fs^{-3}+O(s^{-4})$.

The Bondi mass [1,2] was originally defined using the Bondi coordinates (u,r,ϑ,φ), where u is null and the area two-form of a constant-(u,r) surface is $\mu=r^2\sin\vartheta\, d\vartheta\wedge d\varphi$. Namely, the definition is

$$M_B=\frac{1}{4\pi}\int M\sin\vartheta\, d\vartheta\wedge d\varphi\ ,$$

where the mass aspect M is defined by the expansion

$$\frac{g^{rr}}{g^{ru}}=-1+2Mr^{-1}+O(r^{-2})\ ,$$

or equivalently by

$$M=\tfrac{1}{2}\lim_{r\to\infty} r\left[\frac{g^{rr}}{g^{ru}}+1\right]\ .$$

This may be translated in terms of the parameter s by

$$g^{rr}=g^{ss}(1-\tfrac{1}{2}fs^{-2})^2+g^{su}f_u s^{-1}+O(s^{-2})\ ,$$

$$g^{ru}=g^{su}(1-\tfrac{1}{2}fs^{-2})+O(s^{-3})\ .$$

Consider two special choices of s. First, take s to be the affine parameter $\hat{r}$ used in Sec. 9.8 of Penrose and Rindler [18], for which

$$f=-\hat{\sigma}^0\bar{\hat{\sigma}}^0\ ,\quad g^{\hat{r}u}=1\ ,\quad g^{\hat{r}\hat{r}}=-1-2\mathrm{Re}\hat{\Psi}_2^0\hat{r}^{-1}+O(\hat{r}^{-2})\ ,$$

where Re denotes the real part,

$$\hat{\sigma}=\hat{\sigma}^0\hat{r}^{-2}+O(\hat{r}^{-3})\ ,\quad \hat{\Psi}_2=\hat{\Psi}_2^0\hat{r}^{-3}+O(\hat{r}^{-4})\ ,$$

and $\hat{\sigma}$ and $\hat{\Psi}_2$ are the Newman-Penrose shear and complex "Coulomb" term, respectively. Hence the mass aspect is given by

$$M=-\mathrm{Re}\hat{\Psi}_2^0+\tfrac{1}{2}f_u$$
$$=-\mathrm{Re}(\hat{\Psi}_2^0+\hat{\sigma}^0\bar{\hat{\sigma}}^0_u)=-\mathrm{Re}(\hat{\Psi}_2^0-\hat{\sigma}^0\hat{\sigma}^{0\prime}) .$$

Alternatively, taking s to be the area radius r,

$$f=0 , \quad \frac{g^{rr}}{g^{ru}}=-1-2\mathrm{Re}\Psi_2^0 r^{-1}+O(r^{-2}) ,$$

and so simply

$$M=-\mathrm{Re}\Psi_2^0 .$$

There are various equivalent expressions [43], since

$$h^{ac}h^{bd}C_{abcd}=2e^{-m}h^{ac}l^b n^d C_{abcd}$$
$$=-2e^{-2m}l^a n^b l^c n^d C_{abcd}=-4\mathrm{Re}\Psi_2 ,$$

using the null normals l^a and n^a (Sec. II).

The above calculation shows that the "Coulomb" terms for the spin bases adapted to r and $\hat{r}$ are related by

$$\mathrm{Re}\Psi_2^0=\mathrm{Re}(\hat{\Psi}_2^0-\hat{\sigma}^0\hat{\sigma}^{0\prime}) ,$$

essentially because

$$r=\hat{r}-\tfrac{1}{2}\hat{\sigma}^0\bar{\hat{\sigma}}^0\hat{r}^{-1}+O(\hat{r}^{-2}) .$$

Consequently, the Bondi mass can be expressed in either of the forms

$$M_B=-\lim_{r\to\infty}\frac{1}{2\pi}\left[\frac{A_r}{16\pi}\right]^{1/2}\int_{S_r}\mu\mathrm{Re}\Psi_2$$
$$=-\lim_{\hat{r}\to\infty}\frac{1}{2\pi}\left[\frac{A_{\hat{r}}}{16\pi}\right]^{1/2}\int_{S_{\hat{r}}}\hat{\mu}\mathrm{Re}(\hat{\Psi}_2-\hat{\sigma}\hat{\sigma}') .$$

These are limits of the quasilocal energy E and the Hawking mass M_H, respectively:

$$M_B=\lim_{r\to\infty}E(S_r)=\lim_{\hat{r}\to\infty}M_H(S_{\hat{r}}) .$$

Thus the Bondi mass is the limit of the Hawking mass along the affine foliation used by Penrose and Rindler [18], and is also the limit of the quasilocal energy E along a foliation parametrized by area radius. The area radius or luminosity parameter was the original choice [1] and an affine parameter is used in most other work [18,22,26,73–75]. The area radius is in many ways the most natural choice, for instance from purely geometrical considerations, or from the link with the ADM mass (Sec. VI). Additionally, note that other affine parameters would also require different expressions, with the particular affine parameter $\hat{r}$ being chosen to agree with the area radius to the highest possible order.

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