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Supersymmetric homogeneous quantum cosmologies coupled to a scalar field

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Recent work on $N = 2$ supersymmetric Bianchi type IX cosmologies coupled to a scalar field is extended to a general treatment of homogeneous quantum cosmologies with explicitly solvable momentum constraints, i.e., Bianchi types I, II, VII, VIII in addition to the Bianchi type IX, and special cases, namely the Friedmann universes, the Kantowski-Sachs space, and Taub-NUT space. In addition to the earlier explicit solution of the Wheeler-DeWitt equation for Bianchi type IX, describing a virtual wormhole fluctuation, an additional explicit solution is given and identified with the "no-boundary state."

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I. INTRODUCTION

Quantum cosmology is the application of quantum mechanics to the earliest Universe. General relativity predicts that the earliest Universe is dominated by gravity, which is enhanced above all other interactions, up to a singularity, by the strong nonlinearity (i.e., gravitation of gravity) built into that theory. Quantum cosmology therefore must contain a theory of quantum gravity. Unfortunately a consistent quantum field theory of gravity has not yet been given. However, some of the questions of quantum cosmology (but, of course, not all of them, and possibly not the deepest ones) can already be raised in "toy models" of quantum gravity which bypass the unsolved problems of quantum gravitational field theory and instead require only the framework of quantum mechanics of a system with a finite number of degrees of freedom (see e.g., [1–7]).

Such toy models can be constructed by quantizing not the full theory of general relativity (or even larger theories in which it is contained as some limiting case) but only certain classes of its spatially homogeneous solutions. The restriction to spatial homogeneity implies, in principle, a dimensional reduction of the (1+3)-dimensional field theory of gravitation down to a (1+0)-dimensional quantum mechanical model. The configuration space of the reduced models is no longer the full function space (called superspace [1]) of all three-metrics on a spacelike three-surface embedded in space-time [2], but merely the finite-dimensional space (called minisuperspace [3]) spanned by the parameters of the considered class of homogeneous three-metrics. Such reduced models avoid the unsolved problems associated with the short-wavelength limit of quantum gravity; on the other hand they can, of course, also shed no light on these problems.

Spatially homogeneous three-metrics evolving along some trajectory through minisuperspace are special exact solutions of classical general relativity [4,5]. Whether corresponding exact solutions of quantum gravity exist cannot be answered with confidence before such a theory has been constructed. The dimensionally reduced quantum mechanical minisuperspace models cannot, therefore, be considered to be exact solutions like their classical counterparts, but must be considered as just models. Their usefulness and interest hinges on the fact that they provide a comparatively transparent framework in which some questions raised by quantum cosmology can be studied (see e.g., [1–7]).

The dimensional reduction from a field theory down to a finite number of degrees of freedom can, in principle, be carried out before or after quantizing the theory, and the result need not be the same. In the present paper we shall follow the first path: As we consider minisuperspace as a device to avoid the problems of the gravitational quantum field theory it would seem inconsistent with that reasoning to invoke the quantized field theory, if only formally, in an intermediate step. Even more important, from a practical point of view, is the fact that model building is easier and more direct starting from the classically reduced theory, and, as we have mentioned, we are not trying to construct solutions of quantum gravity, but models.

Spatially homogeneous three-geometries are three-manifolds on which one of the three-dimensional Lie groups acts transitively [4,5]. Thus, on the three-manifold there are three linearly independent Killing vectors ξ_i, $i = 1,2,3$, satisfying the Lie algebra $[\xi_i, \xi_j] = \mathcal{C}^k{}_{ij}\xi_k$ where $\mathcal{C}^k{}_{ij} = -\mathcal{C}^k{}_{ji}$ are the structure constants of the group. The three-dimensional Lie groups have been classified by Bianchi [8] into 9 different types (see e.g., [5]), according to their structure constants. A complete list of the structure constants, an invariant basis χ_i and its dual basis of one-forms ω^i, $i = 1,2,3$, and a complete set of Killing vectors ξ_i for all Bianchi types has been given by Taub [9] (see also [5]). The four-metric of a spatially homogeneous space-time can be written in the form

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$$ds^2 = -N^2(t)dt^2 + g_{ij}(t)\omega^i\omega^j, \tag{1.1}$$

where the basis one-forms ω^i satisfy

$$d\omega^i = \frac{1}{2}\mathcal{C}^i{}_{jk}\omega^j\Lambda\omega^k. \tag{1.2}$$

$N(t)$ is the lapse function which is an arbitrary positive function of time and reflects the reparametrization invariance of the time coordinate t. The elements of the tensor g_{ij} of the three-metric in the basis ω^i depend only on time and not on the spatial coordinates. Therefore, there are at most 6 independent elements of $g_{ij}(t)$, spanning the space of all allowed three-metrics, the minisuperspace. Here we shall restrict our attention to the case where $g_{ij}(t)$ can be consistently chosen to be diagonal. It can then be parametrized by three parameters α, β_+, β_- or $\beta^1, \beta^2, \beta^3$ via

$$g_{ij}(t) = \frac{1}{6\pi}e^{2\alpha(t)}\left(e^{2\beta(t)}\right)_{ij} = \frac{1}{6\pi}e^{2\beta^i}\delta_{ij} \tag{1.3}$$

with the diagonal traceless matrix $\boldsymbol{\beta}(t)$ in Misner's parametrization [3,10], $\boldsymbol{\beta}(t) =$ diag $(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$. Thus minisuperspace is spanned by merely three coordinates. The spatially homogeneous form (1.1) of the four-metric can be inserted into the vacuum equations of general relativity, $R_{\mu\nu} = 0$, where $R_{\mu\nu}$ is the Ricci tensor associated with the four-metric. The resulting equations of motion for the independent elements of $g_{ij}(t)$ are second order in time. For some (but not all [5]) of the Bianchi types these equations of motion may be coached into the framework of an unconstrained Hamiltonian system [3],

$$\dot{q}^\nu = \frac{\partial H(q,p)}{\partial p_\nu}, \quad \dot{p}_\nu = -\frac{\partial H(q,p)}{\partial q^\nu}, \tag{1.4}$$

moving on the surface of vanishing "energy" H,

$$H = 0. \tag{1.5}$$

In Eq. (1.4) the dot denotes differentiation with respect to a suitable parameter λ playing the role of time. q^ν is the chosen parametrization of the independent elements of $g_{ij}(t)$. The p_ν are the canonically conjugate momenta. $H(q,p)$ is the Hamiltonian which generates the required equations of motion. Choosing $d\lambda = \sqrt{3\pi/2}e^{-3\alpha}N(t)dt = \sqrt{3\pi/2}e^{-(\beta^1+\beta^2+\beta^3)}N(t)dt$, it takes the form [3–5]

$$\begin{aligned} H &= \frac{1}{2}(-p_\alpha^2 + p_+^2 + p_-^2) + V^{(0)}(\alpha, \beta_+, \beta_-) \\ &= \frac{3}{2}\left[(p_1)^2 + (p_2)^2 + (p_3)^2 - 2p_1p_2 - 2p_1p_3 - 2p_2p_3\right] \\ &\quad + V^{(0)}(\beta^1, \beta^2, \beta^3) \end{aligned} \tag{1.6}$$

with the potential

$$V^{(0)} = -12\pi^2\,{}^{(3)}g\,{}^{(3)}R, \tag{1.7}$$

where ${}^{(3)}g$ is the determinant and ${}^{(3)}R$ the scalar curvature of the three-metric. The condition $H = 0$ expresses the reparametrization invariance of the time parameter in Eq. (1.4).

In the cases where the unconstrained description (1.4) is valid it can be obtained most directly [11] from the Hamiltonian formulation of general relativity [12]. For later convenience we now list the Bianchi types where the unconstrained description (1.4) applies, together with their potentials [5]:

$$\text{type I: } \mathcal{C}^i{}_{jk} = 0\,, \quad V_1^{(0)} \equiv 0 \tag{1.8}$$

$$\text{type II: } \mathcal{C}^3_{12} = -\mathcal{C}^3_{21} = 1\,, \quad \text{all other} \quad \mathcal{C}^i{}_{jk} \text{ vanish;}$$

$$V_2^{(0)} = \frac{1}{6}e^{4\alpha}e^{-8\beta_+} = \frac{1}{6}e^{4\beta^3} \tag{1.9}$$

$$\text{type VII: } \mathcal{C}^2_{13} = -\mathcal{C}^2_{31} = \mathcal{C}^1_{32} = -\mathcal{C}^1_{23} = 1\,; \quad \text{all other} \quad \mathcal{C}^k{}_{ij} \text{ vanish;}$$

$$V_7^{(0)} = \frac{1}{3}e^{4\alpha}e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-) - 1] = \frac{1}{6}\left(e^{2\beta^1} - e^{2\beta^2}\right)^2 \tag{1.10}$$

$$\text{type VIII: } \mathcal{C}^3_{12} = -\mathcal{C}^3_{21} = \mathcal{C}^1_{32} = -\mathcal{C}^1_{23} = \mathcal{C}^2_{31} = -\mathcal{C}^2_{13} = 1;$$

$$\begin{aligned} V_8^{(0)} &= \frac{1}{6}e^{4\alpha}\{2e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-) - 1] + e^{-8\beta_+} + 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)\} \\ &= \frac{1}{6}\left[e^{4\beta^1} + e^{4\beta^2} + e^{4\beta^3} + 2e^{2\beta^3}(e^{2\beta 1} + e^{2\beta^2}) - 2e^{2\beta^1+2\beta^2}\right] \end{aligned} \tag{1.11}$$

$$\text{type IX: } \mathcal{C}^1_{23} = -\mathcal{C}^1_{32} = \mathcal{C}^2_{31} = -\mathcal{C}^2_{13} = \mathcal{C}^3_{12} = -\mathcal{C}^3_{21} = 1;$$

$$\begin{aligned} V_9^{(0)} &= \frac{1}{6}e^{4\alpha}\left[2e^{4\beta_+}(\cosh(4\sqrt{3}\beta_-) - 1) + e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)\right] \\ &= \frac{1}{6}\left[e^{4\beta^1} + e^{4\beta^2} + e^{4\beta^3} - 2e^{2\beta^1+2\beta^2} - 2e^{2\beta^1+2\beta^3} - 2e^{2\beta^2+2\beta^3}\right] \end{aligned} \tag{1.12}$$

The other Bianchi types and more general cases of type VII, in general do not give rise to unconstrained Hamiltonian systems.

We add to this list a few special cases where further symmetries are present and where an unconstrained Hamiltonian system with less than three degrees of freedom is obtained.

Friedmann-Robertson-Walker (FRW) universe. The closed ($k = 1$), open ($k = -1$), and flat ($k = 0$) FRW universes classically, are isotropic special cases of the Bianchi types IX, V ($\mathcal{C}^1_{13} = -\mathcal{C}^1_{31} = \mathcal{C}^2_{23} = -\mathcal{C}^2_{32} = 1$, all other $\mathcal{C}^k{}_{ij}$ vanish), and I, respectively, where $p_+ = p_- = 0$, $\beta_+ = \beta_- = 0$. Thus for the FRW universe without matter

$$\begin{aligned} H_{\rm FRW} &= -\frac{p_\alpha^2}{2} + V^{(0)}_{\rm FRW}(\alpha), \\ V^{(0)}_{\rm FRW} &= -\frac{k}{2}e^{4\alpha}. \end{aligned} \tag{1.13}$$

It should be noted that the closed FRW universe without matter cannot exist classically.

Kantowski-Sachs (KS) models [13]. These spaces have a four-dimensional symmetry group with a three-dimensional subgroup which is of Bianchi type IX [5]. However, the latter subgroup does not act transitively in three-space but only on two-dimensional surfaces foliating the three-space. The space-time metric may be written in the standard form (1.1), (1.3) with

$$\omega^3 = dr\ , \quad \omega^1 = d\theta\ , \quad \omega^2 = \sin\theta d\varphi,$$

$$\boldsymbol{\beta}(t) = \ \mathrm{diag}\ (\beta_+, \beta_+, -2\beta_+),$$

and

$$\begin{aligned} H_{\rm KS} &= -\tfrac{p_\alpha^2}{2} + \tfrac{p_+^2}{2} + V^{(0)}_{\rm KS}(\alpha,\beta_+) = -3p_1p_3 + \tfrac{3}{2}(p_3)^2 + V^{(0)}_{\rm KS}(\beta^1,\beta^3), \\ V^{(0)}_{\rm KS} &= -\tfrac{2}{3}e^{4\alpha}e^{-2\beta_+} = -\tfrac{2}{3}e^{2(\beta^1+\beta^3)}. \end{aligned} \tag{1.14}$$

Taub-NUT (Newman-Unti-Tamburino) space [14,9,15,5]. The Taub space [9] is of Bianchi type IX with a rotational symmetry around one spatial axis. Therefore we may take $\beta_- = 0$, $p_- = 0$ and have

$$\begin{aligned} H_{\rm T} &= -\frac{p_\alpha^2}{2} + \frac{p_+^2}{2} + V^{(0)}_{\rm T}(\alpha,\beta_+) = -3p_1p_3 + \frac{3}{2}(p_3)^2 + V^{(0)}_{\rm T}(\beta^1,\beta^3), \\ V^{(0)}_{\rm T} &= \frac{1}{6}e^{4\alpha}(e^{-8\beta_+} - 4e^{-2\beta_+}) = \frac{1}{6}\left(e^{4\beta^3} - 4e^{2\beta^1+2\beta^3}\right). \end{aligned} \tag{1.15}$$

The new parametrization

$$\gamma = e^{\alpha+\beta_+}, \quad g = e^{2\alpha-4\beta_+},$$

and the redefinition of the lapse function $N(t) \to N(t)/g(t)$ gives rise to the new Hamiltonian

$$H_{\text{T-NUT}} = 6(gp_g^2 - \gamma p_g p_\gamma) + \frac{1}{6}(g - 4\gamma^2), \tag{1.16}$$

which describes a dynamical system in which the variable γ remains positive if it is positive initially (see e.g. [5])and the variable g may pass through zero. The region $g > 0$ corresponds to Taub space [9], the region $g < 0$ corresponds to NUT space [15], in which ω^3 is timelike and dt spacelike. As is well known both spaces form different parts of the single Taub-NUT space-time [14,5].

In the present paper we shall be concerned with the construction of specific quantized versions of the dynamical systems (1.4)–(1.7). These quantum models will be constructed by coupling additional fermionic degrees of freedom to the purely gravitational systems (1.4)–(1.7) in such a way that the coupled system acquires a larger symmetry, namely supersymmetry. For this to be possible the potentials (1.8)–(1.16) must satisfy a certain condition; they must be derivable from an underlying (usually simpler) potential ϕ. This condition is verified for all the listed Bianchi types in Sec. II.

A further ingredient in the construction of our quantum models is the coupling of the supersymmetrized gravity model to a supersymmetric spatially homogeneous matter field, represented by a spatially homogeneous complex scalar field and its supersymmetric fermionic partner.

A part of the theory we shall describe here for all the Bianchi types listed has been presented for the Bianchi type IX in our earlier paper [16] which, in turn, builds on our earlier work in [17–19]. An application of the theory to the Bianchi type II (but without coupling to a matter field) has already been given in [20].

Dimensional reductions from (1+3)-dimensional supergravity down to (1+0)-dimensional supersymmetric theories have been presented for the Bianchi type I model without matter [21,22] the closed Friedmann model without [23] and with matter [24], the Taub model [25], and very recently also for Bianchi IX [26] and other Bianchi class A models [27]. From (1+3)-dimensional supergravity with a single conserved real spinorial supercharge ($N = 1$) a (1+0)-dimensional theory with $N = 4$ conserved real supercharges is obtained.

By contrast, the general supersymmetric extension of the Hamiltonian systems (1.4)–(1.7) we shall consider in this paper only leads to an $N = 2$ supersymmetry. In principle, it may be considered as a subsymmetry of the larger $N = 4$ supersymmetry obtained from supergravity, but we shall not attempt here to make that connection explicit. The further extension of our models to $N = 4$ supersymmetry is nontrivial and requires further work. For the case of the Bianchi type IX model without matter such an extension is given in [28].

II. SUPERSYMMETRIC EXTENSION OF BIANCHI TYPES

A. Supersymmetry condition of the Bianchi potentials

The geometrodynamics of the Bianchi types considered reduce, formally, to the Hamiltonian dynamics of a particle in a three-dimensional potential. (Eliminating the arbitrary time-parameter the description may even be further reduced to motion in a two-dimensional time-dependent potential, but we shall not make use of this possibility here). However, an important nonstandard feature in this picture is the nondefinite metric $G^{(0)}_{\mu\nu}$ in minisuperspace, whose line element in the parametrization by α, β_+, β_- or $\beta^1, \beta^2, \beta^3$ may be written as

$$dS^2 = G^{(0)}_{\mu\nu} dq^\mu dq^\nu, \tag{2.1}$$

with $G^{(0)}_{\mu\nu} =$ diag $(-1, 1, 1)$ or $G^{(0)}_{\mu\nu} = -\frac{1}{6}(1 - \delta_{\mu\nu})$, respectively. In fact, as emphasized by Misner [3], because of reparametrization invariance with respect to $\lambda, d\lambda = e^{2\Omega} d\lambda^{(0)}$, the metric in minisuperspace is fixed only up to an arbitrary conformal factor, here written as $\exp[2\Omega(q)]$:

$$G_{\mu\nu} = e^{2\Omega(q)} G^{(0)}_{\mu\nu}. \tag{2.2}$$

The inverse of this conformal factor appears in the potential of (1.6) i.e.

$$V(q) = e^{-2\Omega(q)} V_0(q). \tag{2.3}$$

The supersymmetric extension of particle motion in a potential well is treated by supersymmetric quantum mechanics invented by Nicolai [29], Witten [30] and developed further by many authors (see e.g., [31–36]). In particular, the particle dynamics in a potential on a curved manifold (configuration space) of arbitrary dimension with the metric

$$dS^2 = G_{\mu\nu}(q) dq^\mu dq^\nu \tag{2.4}$$

has been studied in the ($N = 2$)-supersymmetric σ model by a number of authors [34-36]. Supersymmetry requires that the potential $V(q)$ is derivable from a globally defined superpotential $\phi(q)$ via

$$V(q) = \frac{1}{2} G^{\mu\nu}(q) \frac{\partial\phi(q)}{\partial q^\mu} \frac{\partial\phi(q)}{\partial q^\nu}. \tag{2.5}$$

We shall demand that $\phi(q)$ solves Eq. (2.5) and has the same symmetries as H_0. Here $G^{\mu\nu}(q)$ is the inverse of the metric tensor in configuration space defined by the kinetic energy

$$T(q, \dot{q}) = \frac{1}{2} G_{\mu\nu}(q) \dot{q}^\mu \dot{q}^\nu. \tag{2.6}$$

We note that Eq. (2.5) is the Hamilton-Jacobi equation corresponding to Eqs. (1.5), (1.6) in Euclidean time; i.e., $\phi(q)$ is a Euclidean action of classical general relativity. Depending on the boundary conditions posed, there are different solutions of the Euclidean Hamilton-Jacobi equation. The physical interpretation of the Euclidean actions is related to quantum tunneling [37]: They are the actions required for a system to reach a classically inaccesible point from a given "initial point." The choice of the initial point depends on the physical question posed. For tunneling out of some equilibrium state the initial point will correspond to a local or global minimum of $\phi(q)$. For example, for cosmology initial points corresponding to minima of ϕ at fixed α in the limit of vanishing scale parameter $e^\alpha \to 0$ are of particular interest. Once $\phi(q)$ is given, the most likely path followed in the tunneling process is given by the solutions of the classical Euclidean equations

$$p_\nu = G_{\nu\mu} \frac{dq^\mu}{d\lambda} = \frac{\partial\phi}{\partial q^\nu}, \tag{2.7}$$

which must be solved under the condition that the path $q(\lambda)$ connects the chosen initial point with the given final point. For a given ϕ solving Eq. (2.5) the solutions of Eq. (2.7) may be used to give a physical interpretation.

If a final point is accessible from the initial point by a classically allowed path, ϕ becomes imaginary. While in Eq. (2.5) this has no immediate consequence, the supersymmetrically extended Hamiltonian then has unusual properties which seem to indicate that supersymmetric extensions cannot be based on imaginary or complex ϕ [see Eq. (2.30) below].

The new potential $\phi(q)$ is called the superpotential and appears like a potential in the superspace version (in the sense of supersymmetry) of the Lagrangean of the [(1+0)-dimensional] supersymmetric σ model. In order to extend the Hamiltonian (1.6) to a supersymmetric Hamiltonian we therefore have to solve Eq. (2.5) after inserting the potentials (1.8)–(1.16) and to identify a superpotential $\phi(q)$, in each case. It follows from Eqs. (2.2), (2.3), and (2.5) that a superpotential $\phi(q)$ is independent of the conformal factor. It is therefore sufficient to examine the case $\Omega(q) = 0$, where the metric $G_{\mu\nu} = G^{(0)}_{\mu\nu}$ is flat and $V(q)$ is given by $V^{(0)}(q)$.

We shall now solve Eq. (2.5) and construct the superpotentials for the Bianchi types of Sec. I. In the case of Bianchi type I, $\phi(q)$ should preserve the invariance of H_0 under arbitrary shifts $\delta\alpha, \delta\beta_+, \delta\beta_-$ or $\delta\beta^1, \delta\beta^2, \delta\beta^3$ which requires

$$\phi_1 \equiv 0 \tag{2.8}$$

(a constant can always be subtracted from ϕ).

For Bianchi type II only the invariance under independent shifts $\delta\beta^1, \delta\beta^2$ remains, which restricts the allowed solutions of Eq. (2.5) to a function of β^3 which is obtained as

$$\phi_2 = \frac{1}{6}e^{2\alpha-4\beta_+} = \frac{1}{6}e^{2\beta^3}. \tag{2.9}$$

For Bianchi type VII the symmetry operations are $\beta^1 \leftrightarrow \beta^2$, $\beta^3 \to \beta^3 + \delta\beta^3$; or $\beta_- \to -\beta_-$, $\beta_+ \to \beta_+ + \delta\beta_+$, $\alpha \to \alpha - \delta\beta_+$. These are respected by the solutions

$$\begin{aligned}\phi_7 &= \frac{1}{3}e^{2\alpha+2\beta_+}[\cosh 2\sqrt{3}\beta_- - A(\alpha+\beta_+)] \\ &= \frac{1}{6}\left(e^{2\beta^1} + e^{2\beta^2} - 2A[\tfrac{1}{2}(\beta_1+\beta_2)]e^{\beta^1+\beta^2}\right)\end{aligned} \tag{2.10}$$

where A is an arbitrary function of its argument.

For Bianchi type VIII the Hamiltonian H_0 is invariant merely under $\beta^1 \leftrightarrow \beta^2$. This symmetry is respected by the solutions

$$\begin{aligned}\phi_8 &= \frac{1}{6}e^{2\alpha}\left[2e^{2\beta_+}\cosh 2\sqrt{3}\beta_- - e^{-4\beta_+}\right] \\ &= \frac{1}{6}\left(+e^{2\beta^1} + e^{2\beta^2} - e^{2\beta^3}\right),\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\tilde{\phi}_8 &= \frac{1}{6}e^{2\alpha}\left[4e^{2\beta_+}\sinh^2\sqrt{3}\beta_- - e^{-4\beta_+} \mp 4ie^{-\beta_+}\cosh\sqrt{3}\beta_-\right] \\ &= \frac{1}{6}\left(e^{2\beta^1} + e^{2\beta^2} - e^{2\beta^3} - 2e^{\beta^1+\beta^2} \mp 2i(e^{\beta^1} + e^{\beta^2})e^{\beta^3}\right)\end{aligned} \tag{2.12}$$

However, the latter solution is complex indicating that the underlying trajectories are, in part, classically allowed. This corresponds to the fact that $V_8^{(0)}$ is not a binding potential.

Finally, for Bianchi type IX we have the three symmetries $\beta^i \leftrightarrow \beta^j (i \neq j)$. They are preserved by

$$\begin{aligned}\phi_9 &= \frac{1}{6}e^{2\alpha}\left[2e^{2\beta_+}\cosh 2\sqrt{3}\beta_- + e^{-4\beta_+}\right] \\ &= \frac{1}{6}\left(e^{2\beta^1} + e^{2\beta^2} + e^{2\beta^3}\right).\end{aligned} \tag{2.13}$$

This is the superpotential chosen in Ref. [17,19,27]. As a solution of the Euclidean Hamilton Jacobi equation it was obtained in Ref. [38]. As was also shown there a further solution with the required symmetry exists, which is given by

$$\begin{aligned}\tilde{\phi}_9 &= V_9^{(0)}\left(\frac{\alpha}{2}, \frac{\beta_+}{2}, \frac{\beta_-}{2}\right) \\ &= \frac{1}{6}\Bigg(e^{2\beta^1} + e^{2\beta^2} + e^{2\beta^3} - 2e^{\beta^1+\beta^2} \\ &\qquad -2e^{\beta^1+\beta^3} - 2e^{\beta^2+\beta^3}\Bigg).\end{aligned} \tag{2.14}$$

Further solutions have been given in [38] but they break the permutation symmetry $\beta^i \leftrightarrow \beta^j$ and are therefore not considered here.

B. Supersymmetry condition for important special cases

Now we determine Euclidean actions for the special cases (1.13)–(1.16) listed in Sec. I.

FRW universes.

$$G_{\mu\nu}^{(0)} = -1; \quad \phi_{\,\mathrm{FRW}} = \begin{cases} \frac{1}{2}\ e^{2\alpha}, & k=1, \\ 0, & k=0, \\ \frac{i}{2}\ e^{2\alpha}, & k=-1, \end{cases} \tag{2.15}$$

As the open FRW universe expands classically even if it is empty ϕ becomes imaginary in this case.

Kantowski Sachs space. The Hamiltonian H_0 is invariant under $\beta^1 \to \beta^1 + \delta\beta$, $\beta^3 \to \beta^3 - \delta\beta$ i.e. ϕ can only depend on $\beta^1 + \beta^3$. Thus, in the variables α, β_+,

$$G_{\mu\nu}^{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \phi_{\,\mathrm{KS}} = \frac{1}{3}e^{2\alpha-\beta_+}. \tag{2.16}$$

Taub space. We only consider solutions obtained by restrictions of the Bianchi type IX solutions to axial symmetry. In the variables α, β_+,

$$G_{\mu\nu}^{(0)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \phi_{\,\mathrm{T}} = \frac{1}{6}e^{2\alpha}\left(2e^{2\beta_+} + e^{-4\beta_+}\right), \tag{2.17}$$

and

$$\tilde{\phi}_{\,\mathrm{T}} = \frac{1}{6}e^{2\alpha}(e^{-4\beta_+} - 4e^{-\beta_+}). \tag{2.18}$$

Taub-NUT space.

$$G^{\mu\nu} = \begin{pmatrix} 12g & -6\gamma \\ -6\gamma & 0 \end{pmatrix}; \quad \phi_{\,\mathrm{T-NUT}} = \frac{1}{6}(2\gamma^2 + g), \tag{2.19}$$

and

$$\tilde{\phi}_{\mathrm{T-NUT}} = \frac{1}{6}(g - 4\gamma\sqrt{g}). \tag{2.20}$$

As ϕ and $\tilde{\phi}$ are invariant against conformal changes of the metric in minisuperspace the expression for Taub-NUT space follow from those for Taub space by a direct substitution of the coordinate change after Eq. (1.15).

C. Supersymmetric quantization

Having established the supersymmetry condition (2.5) in all cases we wish to consider, we are now in a position to quantize all models in a way which renders them ($N = 2$)-supersymmetric. We simply apply to this purpose the quantization rules of the supersymmetric σ model. These quantization rules may be stated as follows [34,35]:

A classical Hamiltonian system with the Hamiltonian

$$H_0 = \frac{1}{2} G^{\mu\nu}(q) \left(p_\nu p_\mu + \frac{\partial \phi}{\partial q^\mu} \frac{\partial \phi}{\partial q^\nu} \right) \tag{2.21}$$

is quantized by associating with H_0 a quantum Hamiltonian H, reducing to H_0 in the classical limit $\hbar \to 0$, of the form

$$2H = \tilde{Q}Q + Q\tilde{Q}, \tag{2.22}$$

where Q, $\tilde{Q}$ are linear operators satisfying

$$Q^2 = 0 = \tilde{Q}^2. \tag{2.23}$$

If the matrix $G_{\mu\nu}(q)$ is positive definite, i.e., the metric $G_{\mu\nu}$ Riemannian, Q and $\tilde{Q}$ are mutually adjoint. If the metric $G_{\mu\nu}$ is pseudo-Riemannian (as it is in minisuperspace) then Q and $\tilde{Q}$ cannot be mutually adjoint, but become so, if a suitable Wick rotation is performed rendering $G_{\mu\nu}$ Riemannian but keeping ϕ fixed. The operators Q and $\tilde{Q}$ have the explicit form

$$Q = \psi^a e_a{}^\nu(q) \left(\pi_\nu + i \frac{\partial \phi}{\partial q^\nu} \right),$$
$$\tilde{Q} = \bar{\psi}_a e^{a\nu}(q) \left(\pi_\nu - i \frac{\partial \phi}{\partial q^\nu} \right). \tag{2.24}$$

Here the following new quantities have been introduced. $e_a{}^\nu(q)$ is the vielbein associated with $G^{\nu\mu}(q)$ and satisfies

$$e_a{}^\nu(q) e_b{}^\mu(q) \eta^{ab} = G^{\nu\mu}(q), \tag{2.25}$$

where η^{ab} is the unit tensor, if $G_{\mu\nu}$ is Riemannian, and the Minkowski tensor $\eta^{ab} = \text{diag}\,(-1, 1, \ldots, 1)$ if one eigenvalue of $G_{\mu\nu}$ is negative. Latin indices are raised and lowered by the use of η^{ab} and η_{ab}. The ψ^a and their adjoint $\bar{\psi}_a$ are fermionic operators satisfying

$$\left[\psi^a, \psi^b\right]_+ = 0 = \left[\bar{\psi}_a, \bar{\psi}_b\right]_+,$$
$$\left[\psi^a, \bar{\psi}_b\right]_+ = \delta^a_b. \tag{2.26}$$

The π_ν are operators

$$\pi_\nu = -i\hbar \frac{\partial}{\partial q^\nu} + i\hbar \omega_\nu{}^b{}_c \bar{\psi}_b \psi^c, \tag{2.27}$$

where the spin connections $\omega_\nu{}^b{}_c$ are functions of the q^μ and defined by

$$\omega_\nu{}^b{}_c = -e^b{}_\mu e_{c;\nu}^{\mu} = -\omega_{\nu c}{}^b. \tag{2.28}$$

Here $e_{c;\nu}^{\mu}$ denotes the Riemann-covariant derivative of the vielbein fields. The $\omega_\nu{}^b{}_c$ vanish identically if the $G^{\mu\nu}(q)$ are independent of the q^λ. If the metric $G_{\mu\nu}(q)$ is conformal to a constant metric $G_{\mu\nu} = e^{2\Omega(q)} G^{(0)}_{\mu\nu}$ as in Eq. (2.2) with the parametrization $(\alpha, \beta_+, \beta_-)$ then

$$\omega_\nu{}^b{}_c = \frac{\partial \Omega}{\partial q^\lambda} \left(e^{b\lambda} e_{c\nu} - e_c{}^\lambda e^b{}_\nu \right). \tag{2.29}$$

We note that the special operator ordering in Eq. (2.24) with (2.27) is crucial to ensure that Q and $\tilde{Q}$ are mutually adjoint (with respect to the invariant measure $\sqrt{\det(G_{\mu\nu})} d^n q$) if $G_{\mu\nu}$ is Riemannian [34,35].

The explicit form of the Hamiltonian (2.22) follows from (2.24). In the special case where the metric (2.25) is flat and constant it takes the form

$$H = -\frac{\hbar^2}{2} G^{\nu\mu} \frac{\partial}{\partial q^\nu} \frac{\partial}{\partial q^\mu} + \frac{1}{2} G^{\nu\mu} \frac{\partial \phi}{\partial q^\nu} \frac{\partial \phi}{\partial q^\mu} + \frac{\hbar}{2} e_a{}^\nu e_b{}^\mu \frac{\partial^2 \phi}{\partial q^\nu \partial q^\mu} [\bar{\psi}^a, \psi^b]. \tag{2.30}$$

The last term in Eq. (2.30) makes it difficult to accommodate imaginary or complex ϕ which appear in the Bianchi type VIII case (2.12), the open FRW case (2.15) and the NUT case (2.20), where $g < 0$. We shall therefore not consider these cases here further.

The Hamiltonian (2.22) commutes with Q and $\tilde{Q}$:

$$[H, Q] = 0 = [H, \tilde{Q}]; \tag{2.31}$$

i.e., the theory is invariant under the supersymmetry transformation

$$\Omega \to \Omega + [\Omega, \tilde{\epsilon} Q] + [\tilde{Q} \epsilon, \Omega], \tag{2.32}$$

where ϵ and $\tilde{\epsilon}$ are arbitrary parameters, anticommuting among themselves and with all fermionic variables and commuting with bosonic variables.

D. Application to cosmological models

The Schrödinger equation is called the Wheeler-DeWitt equation [1–3] in the present case and is the quantum analog of (1.5). It reads

$$H|\psi\rangle = 0. \tag{2.33}$$

Like Eq. (1.5) it expresses the local reparametrization invariance of the arbitrary time parameter, which does not appear in (2.33). Supersymmetry is a local symmetry in supergravity, i.e., invariance under the transformation (2.32) must be required for arbitrary *time-dependent* $\tilde{\epsilon}(t)$ and $\epsilon(t)$ [39,18]. This imposes the constraints $Q = 0$, $\tilde{Q} = 0$ on the state vector $|\psi\rangle$:

$$Q|\psi\rangle = 0 = \tilde{Q}|\psi\rangle. \tag{2.34}$$

These constraints imply the Wheeler-DeWitt equation (2.33), but they are not equivalent to it, as Q and $\tilde{Q}$ are not mutually adjoint.

The fermion number

$$F = \bar{\psi}_a \psi^a \tag{2.35}$$

is conserved by H, $[H,F]=0$, and $[Q,F]=Q$, $[\tilde{Q},F]=-\tilde{Q}$. Therefore the sectors with fixed fermion numbers $F=f(0\leq f\leq n$, where n is the dimension of minisuperspace) can be considered separately. We define the fermion vacuum by

$$\psi^a|0\rangle = 0 \quad \text{for all} \quad a. \tag{2.36}$$

Then, a general state with $F=f$ takes the form

$$|\psi_f\rangle = \frac{1}{f!} f_{\mu_1\cdots\mu_f}(q)\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_f}|0\rangle \tag{2.37}$$

with completely antisymmetric $f_{\mu_1\cdots\mu_f}$, where

$$\bar{\psi}^\mu = e^{a\mu}\bar{\psi}_a. \tag{2.38}$$

In this representation π_μ is proportional to the covariant derivative operator $\pi_\mu = -i\nabla_\mu$

$$\pi_\mu|\psi_f\rangle = \frac{1}{f!}\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_f}|0\rangle(-i)f_{\mu_1\cdots\mu_f;\mu}. \tag{2.39}$$

Thus,

$$\begin{aligned} Q|\psi_f\rangle &= \frac{1}{(f-1)!}\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_{f-1}}|0\rangle\left(-i\hbar\nabla_\mu + i\frac{\partial\phi}{\partial q^\mu}\right) f^\mu{}_{\mu_1\cdots\mu_{f-1}}, \\ \tilde{Q}|\psi_f\rangle &= \frac{1}{(f+1)!}\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_{f+1}}|0\rangle\left[\left(-i\hbar\frac{\partial}{\partial q^{\mu_1}} - i\frac{\partial\phi}{\partial q^{\mu_1}}\right) f_{\mu_2\cdots\mu_{f+1}} + \text{cyclic permutations}\right] \end{aligned} \tag{2.40}$$

The constraints (2.34) therefore reduce to

$$\begin{aligned} &\left(-i\hbar\nabla_\mu + i\tfrac{\partial\phi}{\partial q^\mu}\right) f^\mu_{\mu_1\cdots\mu_{f-1}}(q) = 0 \\ &\epsilon^{\mu_1\cdots\mu_{f+1}}\left(-i\hbar\tfrac{\partial}{\partial q^{\mu_1}} - i\tfrac{\partial\phi}{\partial q^{\mu_1}}\right) f_{\mu_2\cdots\mu_{f+1}} = 0 \end{aligned} \tag{2.41}$$

in the f-fermion sector.

A similar representation can be built on the filled fermion state $|n\rangle$ defined by

$$\bar{\psi}^a|n\rangle = 0 \quad \text{for all} \quad a. \tag{2.42}$$

Thus,

$$\begin{aligned} |n\rangle &= \frac{1}{n!}\epsilon_{a_1\cdots a_n}\bar{\psi}^{a_1}\cdots\bar{\psi}^{a_n}|0\rangle \\ &= \frac{1}{n!}\sqrt{|\det(G_{\mu\nu})|}\epsilon_{\mu_1\cdots\mu_n}\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_n}|0\rangle \end{aligned} \tag{2.43}$$

In this representation the state $|\psi_{n-f}\rangle$ takes the form

$$|\psi_{n-f}\rangle = \frac{1}{f!}g_{\mu_1\cdots\mu_f}(q)\psi^{\mu_1}\cdots\psi^{\mu_f}|n\rangle. \tag{2.44}$$

The constraints (2.34) now take the form

$$\begin{aligned} &\epsilon^{\mu_1\cdots\mu_{f+1}}\left(-i\hbar\tfrac{\partial}{\partial q^{\mu_1}} + i\tfrac{\partial\phi}{\partial q^{\mu_1}}\right) g_{\mu_2\cdots\mu_{f+1}} = 0 \\ &\left(-i\hbar\nabla_\mu - i\tfrac{\partial\phi}{\partial q^\mu}\right) g^\mu_{\mu_1\cdots\mu_{f-1}} = 0; \end{aligned} \tag{2.45}$$

i.e., they have the same form as Eqs. (2.41) with $|\psi_f\rangle \Leftrightarrow |\psi_{n-f}\rangle$, $f_{\mu_1\cdots\mu_f} \Leftrightarrow g_{\mu_1\cdots\mu_f}$ and $\phi \Leftrightarrow -\phi$ interchanged.

Comparing with Eq. (2.41) we find, possibly up to an irrelevant sign,

$$f_{\mu_1\cdots\mu_f}(q) = \sqrt{|\det G_{\mu\nu}|}\epsilon_{\mu_1\cdots\mu_f\mu_{f+1}\cdots\mu_n}g^{\mu_{f+1}\cdots\mu_n}. \tag{2.46}$$

Thus if the solutions in the sectors $f=0,\ldots,[n/2]$ have been determined those in the other sectors can be inferred.

If a Wick rotation is performed in the Hamiltonian to render the metric in minisuperspace Riemannian but keeping ϕ fixed, then the classical Hamiltonian H_0, Eq. (1.6), becomes positive definite. Furthermore, then $\eta^{\nu\mu} =$ diag $(1,1,\ldots,1)$ holds and $e_a{}^\nu = e^{a\nu}$. Thus Q and $\tilde{Q}$ are then mutually adjoint and H is self-adjoint. In this case a nontrivial state $|\psi_f\rangle$ satisfying $\tilde{Q}|\psi_f\rangle = 0$, $Q|\psi_f\rangle = 0$ cannot be written as $|\psi_f\rangle = \tilde{Q}|\psi_{f-1}\rangle$ with another state $|\psi_{f-1}\rangle$ (which would necessarily be in the $F=f-1$ sector), or as $|\psi_f\rangle = Q|\psi_{f+1}\rangle$ with another state $|\psi_{f+1}\rangle$ (which would necessarily be in the $F=f+1$ sector), because otherwise

$$\langle\psi_f|\psi_f\rangle = \langle\psi_{f-1}|Q|\psi_f\rangle = 0 \tag{2.47}$$

or

$$\langle\psi_f|\psi_f\rangle = \langle\psi_{f+1}|\tilde{Q}|\psi_f\rangle = 0 \tag{2.48}$$

would follow (where now the scalar product also includes an integration over the q with their invariant measure in minisuperspace).

On the other hand, if $G_{\mu\nu}$ is pseudo-Riemannian these conclusions do not hold because then neither $(Q)^+|\psi\rangle$ nor $(\tilde{Q})^+|\psi\rangle$ need to vanish. A Wick rotation in minisuperspace therefore severely restricts the possible solutions in supersymmetric quantum cosmology.

Formal solutions in the sectors $f=0$ and $f=n$ can be immediately written down:

$$|\psi_0\rangle = \text{const} \times \exp(-\phi/\hbar)|0\rangle, \qquad |\psi_n\rangle = \text{const} \times \exp(+\phi/\hbar|n\rangle. \tag{2.49}$$

These solutions are the only ones in these sectors. They apply to all the cosmological models we have described in the previous sections and exist independent of whether or not a Wick rotation has been performed. However, they are formal, because no boundary conditions have been posed so far and no criteria have been given allowing to decide whether these solutions are physically acceptable or have to be rejected. Such criteria could only be derived from a definite physical interpretation of the wave function. Unfortunately this point is far from being well understood and shall be discussed further below. However, for the time being we shall demand that a physically acceptable state (2.49) is normalizable for fixed α (scale parameter) [17].

As $G_{\mu\nu}$ is pseudo-Riemannian (no Wick rotation), it is possible to construct solutions in all other fermion sectors as well [19]. This is different from $N = 4$ supersymmetric minisuperspace models [22,26–28], in which an additional internal rotational symmetry inherited from the Lorentz invariance of supergravity rules out all states except those in the empty and filled fermion sectors.

As in all examples so far we have $n \leq 3$, it is enough to consider the sector with $f = 1$. With the general ansatz

$$|\psi_1\rangle = f_\nu(q)e^{-\phi(q)/\hbar}\bar{\psi}^\nu|0\rangle, \tag{2.50}$$

we obtain the system of equations

$$\hbar f^\nu{}_{;\nu} - 2\phi_{;\nu}f^\nu = 0, \qquad \frac{\partial f_\nu}{\partial q^\mu} - \frac{\partial f_\mu}{\partial q^\nu} = 0. \tag{2.51}$$

Because of the second equation there is a function $f(q)$ by which f_ν can be represented as $f_\nu = -i\hbar\frac{\partial f}{\partial q^\nu}$. Then the ansatz (2.50) takes the form

$$|\psi_1\rangle = \tilde{Q}f(q)e^{-\phi(q)/\hbar}|0\rangle, \tag{2.52}$$

which automatically satisfies $\tilde{Q}|\psi_1\rangle = 0$. As we have already mentioned in Eqs. (2.47), (2.48), this form of $|\psi_1\rangle$ is not permitted if $G_{\nu\mu}$ has been Wick-rotated to a Riemannian form. It is permitted, however, for the pseudo-Riemannian minisuperspace metric. If $f(q) \equiv$ const, $|\psi_1\rangle$ vanishes identically; i.e., this case has to be excluded.

Why Wick rotation (i.e. a complexification of the scale parameter) should lead to the same restriction of physical states as the requirement of the rotational symmetry in the $N = 4$ supersymmetric models is not clear. However it seems suggestive in this context that the scale parameter in our models must in fact have two complex supersymmetric partners as soon as a massive matter field is added (see Sec. III A). This would seem to make it natural to treat the scale parameter as a complex variable. If no matter field is present, the second supersymmetric partner of the scale parameter decouples and may therefore be present, but is then not needed in our models. Two complex supersymmetric partners of the scale parameter are, of course, automatically present in $N = 4$ models.

The remaining condition $Q|\psi_1\rangle = 0$ now leads to the condition

$$\left(\hbar\nabla_\mu - 2\frac{\partial\phi}{\partial q^\mu}\right)G^{\mu\nu}\frac{\partial}{\partial q^\nu}f(q) = 0 \tag{2.53}$$

which is a version of the Wheeler-DeWitt equation. All its solutions $f(q) \not\equiv$ const give rise to states in the one-fermion sector. The ansatz (2.52) automatically solves one of the supersymmetry constraints, but it also sacrifices the first-order form of the resulting wave equation. One advantage, however, remains: there are no ambiguities about the appropriate operator ordering in the second-order wave equation. Such ambiguities have already been resolved in the explicit expressions for Q and $\tilde{Q}$.

E. Physical meaning of the superpotentials

In the preceding section we have seen that simple solutions (2.49) of the supersymmetry constraints (2.34) exist in the empty and filled fermion sectors. The semiclassical form of these solutions offers the possibility to discuss the meaning of the two different choices of the superpotential ϕ and $\tilde{\phi}$ found in Secs. II B and II C. We shall give this discussion first for the case of Bianchi type IX, where the solutions (2.49) are

$$|\psi_0\rangle = \text{const} \times \exp(-\phi_9/\hbar)|0\rangle, \qquad |\tilde{\psi}_0\rangle = \text{const} \times \exp(-\tilde{\phi}_9/\hbar)|0\rangle, \tag{2.54}$$

and

$$|\psi_3\rangle = \text{const} \times \exp(+\phi_9/\hbar)|3\rangle, \qquad |\tilde{\psi}_3\rangle = \text{const} \times \exp(+\tilde{\phi}_9/\hbar)|3\rangle. \tag{2.55}$$

The state $|\psi_0\rangle$ was first obtained in [51] and [17]. The states $|\psi_3\rangle$ and $|\tilde{\psi}_3\rangle$ both diverge for $|\beta_\pm| \to \infty$ at fixed α and are therefore ruled out as acceptable physical solutions. The two remaining solutions (2.54) can be interpreted semiclassically: ϕ_9 and $\tilde{\phi}_9$ are two different, classical extremal Euclidean actions of two different regular Riemannian space-times, both having the three-geometry g_{ij} of Eq. (1.1) with given parameters α, β_+, β_- as a boundary. As discussed in Sec. II A it is possible to reconstruct these Riemannian space-times from their extremal action, using the canonical relations

$$p_\alpha = -\frac{d\alpha}{d\lambda} = \frac{\partial\phi}{\partial\alpha}, \qquad p_+ = \frac{d\beta_+}{d\lambda} = \frac{\partial\phi}{\partial\beta_+}, \qquad p_- = \frac{d\beta_-}{d\lambda} = \frac{\partial\phi}{\partial\beta_-}. \tag{2.56}$$

For the choice $\phi = \phi_9$, these equations can be integrated in order to determine $\alpha, \beta_+, \beta_-, N$ as a function of a suitable affine parameter (denoted by ρ below) and three constants of integration [40]. The result is the Bianchi type IX space-time [40]

$$ds^2 = \frac{d\rho^2}{\sqrt{F(\rho)}} + \frac{1}{4}\rho^6\sqrt{F(\rho)} \times \left[\frac{\omega^1\omega^1}{\rho^4 - a_1^4} + \frac{\omega^2\omega^2}{\rho^4 - a_2^4} + \frac{\omega^3\omega^3}{\rho^4 - a_3^4}\right], \quad (2.57)$$

with

$$F(\rho) = \left(1 - \frac{a_1^4}{\rho^4}\right)\left(1 - \frac{a_2^4}{\rho^4}\right)\left(1 - \frac{a_3^4}{\rho^4}\right). \quad (2.58)$$

Here a_1, a_2, a_3 are constants of integration. They have to be chosen in such a way that the given three-metric of the Bianchi type IX space-time with parameters α, β_+, β_- becomes the three-metric of Eq. (2.57) for a suitable choice of the parameter ρ, with $\rho^2 > \max(a_1^2, a_2^2, a_3^2)$. The four-metric (2.57) is regular for $\rho^2 > \max(a_1^2, a_2^2, a_3^2)$ and becomes flat Euclidean for $|\rho| \to \infty$. Therefore, the given three-metric must form the *inner* boundary of the four-metric (2.57). These boundary conditions correspond to a wormhole [41,26] with the given three-metric arising by a quantum fluctuation from an asymptotically Euclidean metric. The associated wave function vanishes rapidly in the same limit of three-geometries (wormholes), with large volume ($\alpha \to \infty$), i.e., the probability amplitudes for paths of all three-geometries larger than and collapsing to the given one, whose sum builds up the wave function $\sim \exp(-\phi_9/\hbar)$ in a path integral, interfere destructively for large three-geometries but constructively for small three-geometries (on the Planck scale). Because of the singularity of the four-metric (2.57) for small ρ^2 [$\rho^2 = \max(a_1^2, a_2^2, a_3^2)$] it is not possible to choose the given three-geometry with α, β_+, β_- as the outer boundary of (2.57). Therefore the state $\sim \exp(-\phi_9/\hbar)$ cannot describe a cosmological quantum state arising by a sum over probability amplitudes of paths of three-geometries *expanding* into the given one.

Now we turn to the state $\sim \exp(-\tilde{\phi}_9/\hbar)$, where we have to choose $\phi = \tilde{\phi}_9$ in Eq. (2.56). The integration of these equations (see [52]) now gives rise to Riemannian space-times with three-metrics which are regular for small three-geometries, and the given three-geometry must be imposed as the *outer* boundary of these space-times. This corresponds to a cosmological solution, a quantum fluctuation "from nothing." In fact, in a construction via a path integral the wave-function $\exp(-\tilde{\phi}_9/\hbar)$ may be seen as the result of a sum over the probability amplitudes of all paths of compact three-geometries evolving from a point to the given three-geometry. This is precisely the prescription given by Hartle and Hawking [6] for the no-boundary state. Thus we arrive at the remarkable conclusion that the no-boundary state for the Bianchi IX model has an explicit and very simple form (see Note added in proof).

Let us now discuss the solutions for the other Bianchi types in a similar manner.

For type II Eqs. (2.56) become

$$\frac{d\alpha}{d\lambda} = -2\phi_2, \; \frac{d\beta_+}{d\lambda} = -4\phi_2, \; \beta_- = \text{const} \quad (2.59)$$

and λ is related to Euclidean standard time by

$$\frac{d\lambda}{dt} = \sqrt{\frac{3\pi}{2}}e^{-3\alpha}. \quad (2.60)$$

The solutions yield the Riemannian metrics

$$ds^2 = dt^2 + t^2\left(c_1^2(\omega^1)^2 + c_2^2(\omega^2)^2 + \frac{c_3^2}{t^4}(\omega^3)^2\right) \quad (2.61)$$

which describe the collapse from a disc-shaped to a pencil-shaped and the expansion back to a disc-shaped three-geometry from $t = -\infty$ to $t = 0$ to $t = +\infty$. The quantum fluctuation of *cosmological* interest described by this solution is therefore the expansion of a pencil-shaped three-geometry with vanishing three-volume at $t = 0$ to a given Bianchi II three-metric forming the outer boundary.

For type VII Eqs. (2.56) are

$$\frac{d\alpha}{d\lambda} = -\frac{d\beta_+}{d\lambda} = -2\phi_7 - \frac{1}{3}e^{2\alpha+2\beta_+}A'(\alpha+\beta_+),$$
$$\frac{d\beta_-}{d\lambda} = \frac{2}{\sqrt{3}}e^{2\alpha+2\beta_+}\sinh 2\sqrt{3}\beta_-. \quad (2.62)$$

From the first equations it follows that $\alpha + \beta_+ = \text{const}$. The last equation yields

$$\beta_- = \frac{1}{\sqrt{3}}\,\text{arctanh}\,[Ce^{4\lambda e^{2(\alpha+\beta_+)}}],$$

where C is a constant of integration. Therefore the value of $\alpha + \beta_+$ is determined by the initial condition and any given value of β_- can be reached after making an appropriate choice of C. However, the trajectories of α and β_+ depend on the unspecified function $A(\alpha + \beta_+)$, i.e., the general solution still permits a large variety of different behavior.

ϕ_8 could be discussed similarly. However, it does not seem possible to construct a wave function of the form $\exp(\mp\phi_8)$ which is normalizable for fixed α. Therefore we do not enter that discussion here. In fact, of all the Bianchi types discussed only the Bianchi type IX appears to give results which are of real physical interest.

Turning to the special cases, we note first that for the closed FRW model Eq. (2.56) reduces to

$$\frac{d\alpha}{d\lambda} = e^{2\alpha}, \; \frac{d\lambda}{dt} = \sqrt{\frac{3\pi}{2}}e^{-3\alpha}, \quad (2.63)$$

which gives the metric

$$ds^2 = dt^2 + \frac{1}{4}t^2[(\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2] \quad (2.64)$$

describing a tunneling solution from vanishing scale parameter (at $t = 0$) to a given final value of the scale parameter. It also describes the fluctuation from an asymptotically Euclidean three-metric at $t = -\infty$ to the final value of the scale parameter. The first case may lead to the quantum initiation of an isotropic universe. The second case may again be interpreted as a virtual wormhole.

The Taub case may be discussed as the special case of the Bianchi type IX solutions for $\beta_- = 0$. Thus $a_1 = a_2$ in Eqs. (2.57), (2.58). It is then apparent that the (0,0) and the (3,3) components of the Euclidean metric both change sign when ρ^4 drops below a^4. This signals the transition from the Taub regime to the NUT regime. This transition occurs in the regime where Eq. (2.57) is applicable only if $a_1 = a_2 > a_3$. In the NUT regime the analytic continuation from the pseudo-Riemannian to a Riemannian space-time metric has to be changed, because the t coordinate becomes spacelike and is not rotated in the complex plane, while the ω^3 direction becomes timelike and must be analytically continued to the Riemannian regime. Thus, Eq. (2.57) in the Taub-NUT regime becomes

$$ds^2 = \frac{d\rho^2}{|1 - \frac{a_1^4}{\rho^4}|\sqrt{1 - \frac{a_3^4}{\rho^4}}} + \frac{1}{4}\sqrt{\rho^4 - a_3^4}\left(\omega^1\omega^1 + \omega^2\omega^2 + \frac{|\rho^4 - a_1^4|}{\rho^4 - a_3^4}\omega^3\omega^3\right), \tag{2.65}$$

for $\rho^2 > \max(a_1, a_3)$. This solution corresponds to an axially symmetric virtual wormhole.

For $\tilde{\phi}_{\rm T}$ Eq. (2.56) becomes

$$\frac{d\alpha}{d\lambda} = -2\tilde{\phi}_{\rm T}\,, \quad \frac{d\lambda}{dt} = \sqrt{\frac{3\pi}{2}}e^{-3\alpha},$$
$$\frac{d\beta_+}{d\alpha} = -\frac{1}{2}\frac{\partial}{\partial\beta_+}\ln|e^{-4\beta_+} - 4e^{-\beta_+}|. \tag{2.66}$$

$\tilde{\phi}_{\rm T}$ and the argument of the log vanish for $\beta_+ = -\frac{2\ln 2}{3}$. It follows from the last of (2.66) that an initial point at $\alpha = -\infty$, $\beta_+ = 0$ may either tunnel to positive β_+ with increasing α to approach a finite positive limiting value of $g = e^{2\alpha - 4\beta_+}$ while α keeps increasing, or it may tunnel to negative values of β_+ with α increasing until $\beta_+ = -\frac{2\ln 2}{3}$ and decreasing afterwards, if β_+ tunnels to values below $-\frac{2\ln 2}{3}$. Thus $\tilde{\phi}_{\rm T}$ is the solution of cosmological relevance corresponding to the Hartle-Hawking state. It appears that the NUT region cannot be reached in this state.

F. Inclusion of a cosmological constant

A cosmological term with a cosmological constant Λ may be included in the supersymmetric models, as was shown in [18]. The Hamiltonian acquires the additional term

$$H_{\rm cos} = \Lambda \tag{2.67}$$

where Λ is the cosmological constant. We assume that the conformal factor in $G_{\mu\nu}$ has been chosen in such a way that (2.67) is just constant [18,19]. The supercharges (2.24) get additional contributions

$$Q_{\rm cos} = i\sqrt{2|\Lambda|}\psi^\Lambda,$$
$$\tilde{Q}_{\rm cos} = \mp i\sqrt{2|\Lambda|}\bar{\psi}_\Lambda, \tag{2.68}$$

where the upper (lower) sign applies to positive (negative) Λ, and where ψ^Λ and its adjoint $\bar{\psi}_\Lambda$ anti-commute with the other ψ^a, $\bar{\psi}_a$ and satisfy

$$(\psi^\Lambda)^2 = 0 = (\bar{\psi}_\Lambda)^2,$$
$$[\psi^\Lambda, \bar{\psi}_\Lambda]_+ = 1. \tag{2.69}$$

There is still a conserved fermion number F which is now given by

$$F = \bar{\psi}_a\psi^a + \bar{\psi}_\Lambda\psi^\Lambda. \tag{2.70}$$

A state in the sector with $F = f$ can be written as

$$|\psi_f\rangle = \frac{1}{f!}f^{(0)}_{\mu_1\cdots\mu_f}(q)\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_f}|0\rangle + \frac{1}{(f-1)!}f^{(1)}_{\mu_1\cdots\mu_{f-1}}(q)\bar{\psi}_\Lambda\bar{\psi}^{\mu_1}\cdots\bar{\psi}^{\mu_{f-1}}|0\rangle. \tag{2.71}$$

The constraint $Q|\psi_f\rangle = 0$ is expressed by the equations

$$\left(\hbar\nabla_\mu - \frac{\partial\phi}{\partial q^\mu}\right)f^{(1)\mu}{}_{\mu_1\cdots\mu_{f-2}} = 0,$$
$$\left(\hbar\nabla_\mu - \frac{\partial\phi}{\partial q^\mu}\right)f^{(0)\mu}{}_{\mu_1\cdots\mu_{f-1}} + \sqrt{2|\Lambda|}f^{(1)}{}_{\mu_1\cdots\mu_{f-1}} = 0, \tag{2.72}$$

and $\tilde{Q}|\psi_f\rangle = 0$ takes the explicit form

$$\epsilon^{\mu_1\cdots\mu_{f+1}}\left(\hbar\frac{\partial}{\partial q^{\mu_1}} + \frac{\partial\phi}{\partial q^{\mu_1}}\right)f^{(1)}{}_{\mu_2\cdots\mu_f} \mp\sqrt{2|\Lambda|}f^{(0)}{}_{\mu_1\cdots\mu_f} = 0 \tag{2.73}$$

where the doubled sign refers to the choice in Eq. (2.68).

G. A new conserved probability current?

One hope connected with supersymmetric quantization is to repeat the success of Dirac's construction of a conserved probability current which became possible after taking the "square root" of the Klein-Gordon equation. This hope may be idle; in any case, it has not yet been satisfied. In order to shed some light on the problems one encounters we shall discuss here some attempts at the construction of a conserved current with a positive density. For simplicity we shall consider the case $G_{\mu\nu} = G^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and ϕ real. Furthermore we shall put $\hbar = 1$. It will be useful to consider, together with the state vector $|\psi\rangle$, also its adjoint with respect to the fermionic variables, (but not the q variables), which we denote by $\langle\psi|$. The scalar product $\langle\psi|\psi\rangle$ then involves only a summation over the discrete fermionic components, not an integration over the variables q, i.e., by construction $\langle\psi|\psi\rangle$ is positive and q-dependent. The supersymmetry constraints are written as

$$(Q + \tilde{Q})|\psi\rangle = 0,$$
$$i(Q - \tilde{Q})|\psi\rangle = 0. \tag{2.74}$$

We introduce fermionic operator ξ^ν, χ^ν by

$$\xi^\nu = \psi^\nu + \bar{\psi}^\nu, \quad \chi^\nu = i(\psi^\nu - \bar{\psi}^\nu), \tag{2.75}$$

with the properties

$$\begin{aligned} &(\xi^0)^+ = -\xi^0, \quad (\chi^0) = -\chi^0, \\ &(\xi^i)^+ = \xi^i, \quad (\chi^i) = -\chi^i \quad i = 1,2, \\ &[\xi^\nu, \xi^\mu]_+ = 2\eta^{\nu\mu} = [\chi^\nu, \chi^\mu]_+, \\ &[\xi^\nu, \chi^\mu]_+ = 0. \end{aligned} \tag{2.76}$$

Multiplying Eqs. (2.74) by ξ^0 and χ^0 from the left, respectively, we obtain

$$\begin{aligned} &\left(-\partial_0 + \xi^0 \xi^j \partial_j + i\xi^0 \chi^\nu \phi_{|\nu}\right)|\psi\rangle = 0, \\ &\left(-\partial_0 + \chi^0 \chi^j \partial_j - i\chi^0 \xi^\nu \phi_{|\nu}\right)|\psi\rangle = 0 \end{aligned} \tag{2.77}$$

and the adjoint equations

$$\begin{aligned} &-\partial_0 \langle\psi| - \partial_j \langle\psi|\xi^j \xi^0 + i\phi_{|\nu} \langle\psi|(\chi^\nu)^+ \xi^0 = 0, \\ &-\partial_0 \langle\psi| - \partial_j \langle\psi|\chi^j \chi^0 - i\phi_{|\nu} \langle\psi|(\xi^\nu)^+ \chi^0 = 0. \end{aligned} \tag{2.78}$$

Here ∂_0, ∂_j and $\phi_{|0}$, $\phi_{|\nu}$ denote derivatives. Multiplying the first of Eqs. (2.77) with $\langle\psi|A$ from the left and the first of Eqs. (2.78) with $A|\psi\rangle$ from the right, where A is any operator which is independent of the coordinates, and adding both equations we obtain

$$\begin{aligned} &-\partial_0 \langle\psi|A|\psi\rangle + \langle\psi|A\xi^0\xi^j\partial_j|\psi\rangle \\ &+(\partial_j\langle\psi|)\xi^0\xi^j A|\psi\rangle + i\phi_{|0}\langle\psi|[A,\xi^0\chi^0]_+|\psi\rangle \\ &+i\phi_{|j}\langle\psi|[A,\xi^0\chi^j]_-|\psi\rangle = 0. \end{aligned} \tag{2.79}$$

In order to obtain a conservation law the operator A must satisfy the conditions

$$[A, \xi^0\xi^j]_- = 0 = [A, \xi^0\chi^j]_- = [A, \xi^0\chi^0]_+. \tag{2.80}$$

For example, the choices $A = i\chi^0$, or $A = i\xi^0\xi^1\xi^2\chi^1\chi^2$ satisfy these conditions.

The conservation law following from Eq. (2.79) then reads

$$\partial_0\langle\psi|A|\psi\rangle + \partial_j\left(-\langle\psi|A\xi^0\xi^j|\psi\rangle\right) = 0. \tag{2.81}$$

Similarly, a second class of conservation laws can be derived from the second of Eqs. (2.77) and (2.78). It reads

$$\partial_0\langle\psi|B|\psi\rangle + \partial_j\left(-\langle\psi|B\chi^0\chi^j|\psi\rangle\right) = 0, \tag{2.82}$$

with an operator B independent of the coordinates satisfying

$$[B, \chi^0\chi^j]_- = [B, \chi^0\xi^j]_- = 0 = [B, \chi^0\xi^0]_+. \tag{2.83}$$

Possible choices are here $B = i\xi^0$ or $B = i\chi^0\chi^1\chi^2\xi^1\xi^2$. Unfortunately, among all these conserved currents there is none with a positive density $\langle\psi|A|\psi\rangle$ or $\langle\psi|B|\psi\rangle$ because Eqs. (2.80) and (2.82) imply

$$\text{Tr } A = - \text{ Tr } (\chi^0\xi^0A\xi^0\chi^0) = - \text{ Tr } A, \tag{2.84}$$

and the same equation for B; i.e., the operators A and B or any linear conbination cannot be positive.

Chosing $A = B = 1$ we obtain the balance equations

$$\partial_0\langle\psi|\psi\rangle + \partial_j(-\langle\psi|\xi^0\xi^j|\psi\rangle) = i\phi_{|0}\langle\psi|\xi^0\chi^0|\psi\rangle \tag{2.85}$$

and

$$\partial_0\langle\psi|\psi\rangle + \partial_j(-\langle\psi|\chi^0\chi^j|\psi\rangle) = i\phi_{|0}\langle\psi|\xi^0\chi^0|\psi\rangle. \tag{2.86}$$

The α dependence of ϕ is seen to spoil the conservation laws with a positive probability density. Thus, if $\phi = 0$ as in Bianchi type I models, one has conserved currents with a positive density. On the other hand, subtracting Eqs. (2.85), (2.86) we find the general transversality condition

$$\partial_j\langle\psi|(\xi^0\xi^j - \chi^0\chi^j)|\psi\rangle = 0. \tag{2.87}$$

H. Interpretation of the wave function

In order to make contact between the equations of motion and physical reality it is necessary to fix a physical interpretation of the wave function. Unfortunatley, there is as yet no generally accepted interpretation like the usual statistical intepretation of the common quantum mechanics. Instead several alternatives have been proposed in the literature (see e.g., [1–7,42–44]), some of which we now discuss briefly.

1. Semiclassical interpretation

As always in quantum mechanics the wave function can only be intepreted with respect to a given setup for a measurement whose role is, among others, to lift the measured phenomenon from the microscopic quantum level to a macroscopic level which can be treated classically. In the case where the wave function refers to the entire (microscopic) Universe there is no *separate* measurement device, and the only possibility is apparently that the Universe itself acts as its own measurement device. In this view, therefore, a direct physical meaning can be given to the wave function of the Universe only after it has reached a form in which the classical features of the universe have become apparent. By contrast, the role of the wave function in the genuine quantum domain is reduced to a mere mathematical device allowing to calculate the interpretable semiclassical wave function. This is perhaps the most conservative attitude towards the problem of interpretation. It is contained, in the semiclassical limit, in all alternative and stronger proposals trying to give some meaning to the wave function even in the quantum domain. The desire to do this arises from the fact that not all solutions of the Wheeler DeWitt equation reach the semiclassical regime.

2. Probability current from the time-independent Wheeler-DeWitt equation

The (usual time-independent) Wheeler-DeWitt equation based on the (non-supersymmetric) quantization of

H, Eq. (1.6), allows to define a conserved current, like in the Klein-Gordon equation, with generally nonpositive density [2,3]. It can be argued [3] that sufficiently close to the semiclassical limit the conserved density is positive to a very good approximation and can therefore be made the basis of a statistical interpretation. This interpretation has often been used (e.g., [2,3,7,48,49]). In the supersymmetric framework of the present paper there are also fermionic degrees of freedom on which the wave function depends. The definition of the conserved current of the Wheeler-DeWitt equation must then be extended and reads

$$j^\nu = \frac{i}{2} G^{\nu\mu}(q) \left\{ \langle\psi| \frac{\partial}{\partial q^\mu} |\psi\rangle - \left(\frac{\partial}{\partial q^\mu} \langle\psi| \right) \psi\rangle \right\}, \qquad (2.88)$$

with the scalar product $\langle\psi|\psi\rangle$ taken with respect to the fermionic variables only, as defined in Sec. II G. According to Misner [3] the direction (with the unit vector n_ν) of the flow of time in minisuperspace should be defined by the condition $n_\nu j^\nu > 0$ which can be satisfied, close to the semiclassical limit, at least for states with a well-defined classical limit (i.e. states not containing superpositions of macroscopically different quantum states).

3. *Probability current from the time-dependent Wheeler-DeWitt equation*

It is possible to introduce a cosmological time T into the time-independent Wheeler-DeWitt equation with a cosmologial term. To this end one chooses the conformal gauge in which the cosmological term is constant [cf. Sec. II F)] and then interprets this term as a constant of integration, i.e., an energy eigenvalue in a time-independent Schrödinger equation. It is then natural to look at the associated time-dependent equation

$$i\hbar \frac{\partial|\psi\rangle}{\partial T} = H|\psi\rangle, \qquad (2.89)$$

which is the time-dependent Wheeler-DeWitt equation. It has been shown that T is given by the elapsed spacetime volume (see [42] and references given there). Equation (2.89) has a positive conserved invariant measure [42]:

$$dP = \langle\psi|\psi\rangle \sqrt{|\det(G_{\mu\nu})|} d^n q, \qquad (2.90)$$

with the scalar product $\langle\psi|\psi\rangle$ again taken with respect to the fermionic variables only as defined in Sec. II G. dP is therefore a natural choice for a probability measure (see e.g., [44]). It applies also to solutions of the time-independent Wheeler-DeWitt equation to which we shall confine ourselves in the present paper. This statistical interpretation is also frequently used as, e.g., in [6,43,44,46–48]. A discussion of conditional probabilities and entropy in quantum cosmology based on this interpretation has been given in [44]. In view of the possibility to define a consistent unitary "single particle" theory of wave equations of Klein-Gordon type based on this type of interpretation [43] it appears to be the most convincing one, at present.

III. SUPERSYMMETRIC COUPLING TO A SCALAR FIELD

A. Expressions for the supercharges in one conformal gauge

Matter can be introduced into the models we are considering by adding to the Hamiltonian (1.6) a matter term H_M. For the case of a spatially homogeneous complex field $z(t)$ with a conventional form of the kinetic energy $T_M = |\dot{z}|^2$ the form of the Hamiltonian H_M is fixed by supersymmetry up to an arbitrary potential $W(z)$ which is an analytic function of z. H_M takes the form [45]

$$\begin{aligned} H_M &= |p_z|^2 + V_M(z, z^*), \\ V_M &= e^{6\alpha + |z|^2} \left(|DW(z)|^2 - 3|W(z)|^2 \right). \end{aligned} \qquad (3.1)$$

Here $p_z = \dot{z}^*$ is the canonically conjugate momentum of z, and $DW(z) = \frac{dW}{dz} + z^* W$.

For the closed Friedmann universe an $N = 4$ supersymmetric minisuperspace model obtained from supergravity with matter has been studied in [24], where the coupled system of equations satisfied by the wave function was obtained. Here we shall treat $N = 2$ supersymmetric anisotropic models whose greater simplicity allows us to obtain some explicit solutions in a Born-Oppenheimer approximation.

In the following we shall choose that conformal gauge of the minisuperspace metric in which the prefactor $\exp(6\alpha + |z|^2)$ in the potential term of Eq. (3.1) is cancelled. Thus

$$\begin{aligned} H_0 = \frac{1}{2} G^{\mu\nu}(q) \left(p_\nu p_\mu + \frac{\partial\phi}{\partial q^\nu} \frac{\partial\phi}{\partial q^\mu} \right) + e^{-6\alpha - |z|^2} |p_z|^2 \\ + \left(|DW(z)|^2 - 3|W(z)|^2 \right) \end{aligned} \qquad (3.2)$$

with $G_{\mu\nu}(q) = e^{6\alpha + |z|^2} G^{(0)}_{\mu\nu}$. In this case the supercharges $Q, \tilde{Q}$ may be extended by matter terms preserving their property $Q^2 = 0 = \tilde{Q}^2$, so that again a supersymmetric quantization (2.22) is obtained for the matter-extended Hamiltonian. The construction of $Q, \tilde{Q}$ has been given in Ref. [16] in the context of the Bianchi type IX. However, as the explicit form of ϕ never had to be used there, the same construction immediately carries over to the general case. Thus, we can simply state the result. To be explicit we shall take the case with configuration space $(\alpha, \beta_+, \beta_-, z)$. The supercharges may be written as

$$\begin{aligned} Q &= Q_0 + Q_K + Q_P, \\ \tilde{Q} &= \tilde{Q}_0 + \tilde{Q}_K + \tilde{Q}_P. \end{aligned} \qquad (3.3)$$

Here $Q_0, \tilde{Q}_0$ is the pure gravitational term (without cosmological constant) considered up to here, and reads, explicitly,

$$Q_0 = ie^{-3\alpha-\frac{|z|^2}{2}}\left\{\psi^0\left(-\hbar\frac{\partial}{\partial\alpha}+\frac{\partial\phi}{\partial\alpha}\right)+\psi^1\left(-\hbar\frac{\partial}{\partial\beta_+}-3\hbar\bar{\psi}_1\psi^0+\frac{\partial\phi}{\partial\beta_+}\right)+\psi^2\left(-\hbar\frac{\partial}{\partial\beta_-}-3\hbar\bar{\psi}_2\psi^0+\frac{\partial\phi}{\partial\beta_-}\right)\right\}$$

$$\tilde{Q}_0 = ie^{-3\alpha-\frac{|z|^2}{2}}\left\{-\bar{\psi}_0\left(-\hbar\frac{\partial}{\partial\alpha}-\frac{\partial\phi}{\partial\alpha}\right)+\bar{\psi}_1\left(-\hbar\frac{\partial}{\partial\beta_+}-3\hbar\bar{\psi}_0\psi^1-\frac{\partial\phi}{\partial\beta_+}\right)+\bar{\psi}_2\left(-\hbar\frac{\partial}{\partial\beta_-}-3\hbar\bar{\psi}_0\psi^2-\frac{\partial\phi}{\partial\beta_-}\right)\right\}. \quad (3.4)$$

$Q_K, \tilde{Q}_K$ in (3.3) are kinetic terms due to the scalar field. Q_K contains a first new fermionic field χ^1 associated with z, and its adjoint $\bar{\chi}_1$ with fermionic commutation relations, the derivative operators $\partial/\partial z$, and spin-connection terms following from the extension of minisuperspace by z and the fact that in our conformal gauge $G_{\mu\nu}$ depends also on z (while z^* merely plays the role of a parameter in Q_K). Thus,

$$Q_K = ie^{-3\alpha-\frac{|z|^2}{2}}\chi^1\left(-\sqrt{2}\hbar\frac{\partial}{\partial z}-3\hbar\bar{\chi}_1\psi^0-\hbar\frac{z^*}{\sqrt{2}}\psi^a\bar{\psi}_a\right),$$

$$\tilde{Q}_K = ie^{-3\alpha-\frac{|z|^2}{2}}\bar{\chi}_1\left(-\sqrt{2}\hbar\frac{\partial}{\partial z^*}-3\hbar\bar{\psi}_0\chi^1-\hbar\frac{z}{\sqrt{2}}\bar{\psi}_a\psi^a\right). \quad (3.5)$$

Finally, Q_P and $\tilde{Q}_P$ in (3.3) are potential terms due to the scalar field. Q_P contains a second new fermionic field χ^2 associated with z and $\tilde{Q}_P$ contains its adjoint $\bar{\chi}_2$. In addition a new second fermionic variable χ^0 associated with $q^0=\alpha$ also appears in Q_P, together with its adjoint $\bar{\chi}_0$. It is made necessary by the negative term in the matter potential of (3.1). Explicitly,

$$Q_P = i\sqrt{2}\chi^2\left\{[DW(z)]^*+\frac{\hbar}{\sqrt{3}}e^{-3\alpha-\frac{|z|^2}{2}}\bar{\chi}_0\chi^1\right\}$$
$$+i\sqrt{6}\chi^0[W(z)]^*,$$

$$\tilde{Q}_P = i\sqrt{2}\bar{\chi}_2\left(-DW(z)+\frac{\hbar}{\sqrt{3}}e^{-3\alpha-\frac{|z|^2}{2}}\bar{\chi}_1\chi^0\right)$$
$$+i\sqrt{6}\bar{\chi}_0W(z). \quad (3.6)$$

There is still a conserved fermion number

$$F=\bar{\psi}_a\psi^a+\bar{\chi}^0\chi_0+\bar{\chi}_1\chi^1+\bar{\chi}_2\chi^2, \quad (3.7)$$

but now there are seven sectors $F=f$ with $f=0,1,2,3,4,5,6$.

Nontrivial solutions in the sectors $f=0$ and $f=6$ exist only if $W(z)\equiv 0$ and are then given by

$$|\psi_0\rangle = f(z)e^{-\phi/\hbar}|0\rangle,$$
$$|\psi_6\rangle = g(z^*)e^{\phi/\hbar}|n\rangle, \quad (3.8)$$

with arbitrary analytical functions $f(z)$, $g(z^*)$.

The other fermion sectors can be considered like in Sec. II D. Thus it is sufficient to consider $f=1,2,3$.

In the Appendix we show that all solutions in the one-fermion sector are of the form (2.52). The ansatz $|\psi_1\rangle=\tilde{Q}f(q,z,z^*)e^{-\phi/\hbar}|0\rangle$ satisfies $\tilde{Q}|\psi_1\rangle=0$, and the remaining condition $Q|\psi_1\rangle=0$ leads to the Wheeler-DeWitt equation for $f(q,z,z^*)$

$$\frac{\hbar}{2}\eta^{\mu\nu}\left(\hbar\frac{\partial}{\partial q^\mu}-2\frac{\partial\phi}{\partial q^\mu}\right)\frac{\partial f}{\partial q^\nu}-3\hbar^2\frac{\partial f}{\partial\alpha}+\hbar^2\frac{\partial^2 f}{\partial z\partial z^*}+\hbar^2 z^*\frac{\partial f}{\partial z^*}=e^{6\alpha+|z|^2}(|DW|^2-3|W|^2)f. \quad (3.9)$$

For $f=2$ the ansatz

$$|\Psi_2\rangle=\tilde{Q}\left(f^\mu\bar{\psi}_\mu+g_1\bar{\chi}_1+g_2\bar{\chi}_2\right)|0\rangle \quad (3.10)$$

can be made, with five undetermined functions f^μ, g_1, g_2. Terms with $\tilde{Q}\bar{\chi}_0$ have been eliminated from (3.10) by subtracting the vanishing state $0=\tilde{Q}^2h(q)|0\rangle$ with appropriately chosen $h(q)$. We obtain a system of four Wheeler-DeWitt equations, i.e., second-order wave equations, and two auxiliary equations which are of first order. One of the auxiliary equations is not independent, however, and may be dropped. These equations look tedious, and we shall not record them here.

Similarly for $f=3$ the ansatz

$$|\Psi_3\rangle=\tilde{Q}\left(f^{\mu\nu}\bar{\psi}_\mu\bar{\psi}_\nu+g_1^\mu\bar{\psi}_\mu\bar{\chi}_1+g_2^\mu\bar{\psi}_\mu\bar{\chi}_2+g_{12}\bar{\chi}_1\bar{\chi}_2\right)|0\rangle \quad (3.11)$$

can be made with 10 undetermined functions $f^{\mu\nu}=-f^{\nu\mu}, g_1^\mu, g_2^\mu, g_{12}$. Again we made use of $\tilde{Q}^2=0$ to eliminate the five terms involving $\tilde{Q}\bar{\chi}_0$. Now a system of 6 second-order wave equations is obtained together with 9 auxiliary equations of first order, five of which are not independent and may be dropped.

These systems of wave equations are similar in their general form to Eq. (3.9) and also to the system of wave equations obtained in [24] for the supersymmetric Friedmann universe with matter. In the latter case there also appear two fermionic partners for the scale parameter and two fermionic partners for the complex scalar field. However, a detailed comparison of our wave equations imposing $N = 2$ supersymmetry constraints in anisotropic models with those of [24] imposing $N = 4$ supersymmetry and rotational invariance in isotropic models would be difficult.

The wave equation (3.9) is the simplest one obtained in any of the nontrivial fermion sectors. To avoid the singularity for $\phi \to \infty$ we demand that $\partial f/\partial q^\nu \to 0$ in the same limit. The semiclassical limit $\hbar \to 0$ coincides with the limit $e^{6\alpha+|z|^2} \to \infty$ in dimensionless units. In the semiclassical limit one may write

$$f e^{-\phi} \sim e^{iS(q,z,z^*)/\hbar}, \tag{3.12}$$

to obtain

$$\frac{1}{2}\eta^{\mu\nu}\frac{\partial S}{\partial q^\mu}\frac{\partial S}{\partial q^\nu} + \frac{\partial S}{\partial z}\frac{\partial S}{\partial z^*} + V(\alpha,\beta_+,\beta_-) + V_M(z,z^*) = 0 \tag{3.13}$$

which is the Hamilton-Jacobi equation of the classical model. Thus for $\alpha \to \infty$ a wave packet moving along trajectories of the classical system is obtained.

The classical trajectories will start with "initial" conditions in that region of configuration space where the wave function makes its transition from an exponentially decaying or growing behavior $\sim e^{-\phi/\hbar}$ or $e^{-\tilde{\phi}/\hbar}$ to an oscillatory behavior ($\sim e^{iS/\hbar}$) [6,46,7]. For example, if a Bianchi type IX space is considered (without invoking supersymmetry this was done in [47–50]), and if the potential $W(z)$ is chosen in such a way that the transition from exponential to oscillatory behavior occurs for sufficiently large $e^{2\alpha} >> 1$, then the corresponding classical initial values of β_+,β_- and $\partial\beta_+/\partial\alpha$, $\partial\beta_-/\partial\alpha$ will be very small, providing a possible explanation for the observed isotropy of the Universe.

B. Choice of flat minisuperspace metric

The choice of a prefactor in $G_{\mu\nu}$ is just a matter of convenience [3]. Therefore it is of interest to explore other choices than the one made in the preceeding section. The simplest and most obvious one seems to be the choice $G_{\mu\nu}(q) = G^{(0)}_{\mu\nu} = \eta_{\mu\nu}$. In the present section we consider the coupling to a scalar field for this case and derive expressions for Q, $\tilde{Q}$ corresponding to Eqs. (3.3)–(3.6). The Hamiltonian H_0 now takes the form

$$H_0 = \frac{1}{2}\eta^{\mu\nu}\left(p_\mu p_\nu + \frac{\partial\phi}{\partial q^\mu}\frac{\partial\phi}{\partial q^\nu}\right) + H_M, \tag{3.14}$$

where H_M is given by Eq. (3.1). The supercharges are again written as in Eq. (3.3). The expressions for Q_0, $\tilde{Q}_0$ are now simpler than in Eq. (3.4) because no connection terms appear for a flat metric. Thus

$$Q_0 = i\left\{\psi^0\left(-\hbar\frac{\partial}{\partial\alpha} + \frac{\partial\phi}{\partial\alpha}\right) + \psi^1\left(-\hbar\frac{\partial}{\partial\beta_+} + \frac{\partial\phi}{\partial\beta_+}\right) + \psi^2\left(-\hbar\frac{\partial}{\partial\beta_-} + \frac{\partial\phi}{\partial\beta_-}\right)\right\},$$

$$\tilde{Q}_0 = i\left\{\bar{\psi}_0\left(\hbar\frac{\partial}{\partial\alpha} + \frac{\partial\phi}{\partial\alpha}\right) + \bar{\psi}_1\left(-\hbar\frac{\partial}{\partial\beta_+} - \frac{\partial\phi}{\partial\beta_+}\right) + \bar{\psi}_2\left(-\hbar\frac{\partial}{\partial\beta_-} - \frac{\partial\phi}{\partial\beta_-}\right)\right\}. \tag{3.15}$$

Also the kinetic matter terms Q_K, $\tilde{Q}_K$ now lack all three-fermion terms, because the extension of minisuperspace by z and z^* leaves the extended metric flat. Thus

$$Q_K = i\sqrt{2}\chi^1\left(-\hbar\frac{\partial}{\partial z}\right),$$
$$\tilde{Q}_K = i\sqrt{2}\bar{\chi}_1\left(-\hbar\frac{\partial}{\partial z^*}\right). \tag{3.16}$$

However, the potential terms Q_P, $\tilde{Q}_P$ now become more complicated. The additional dependence of H_M on α and $|z|^2$ entails the necessity of more three-fermion terms which do not have any obvious interpretation in terms of spin connections. Thus we write

$$Q_P = i\sqrt{2}e^{3\alpha+|z|^2/2}\{\chi^2[DW(z)]^* + \sqrt{3}\chi^0[W(z)]^*\} + i\hbar T,$$

$$\tilde{Q}_P = i\sqrt{2}e^{3\alpha+|z|^2/2}\{-\bar{\chi}_2[DW(z)] + \sqrt{3}\bar{\chi}_0[W(z)]\} - i\hbar\tilde{T}, \tag{3.17}$$

where T and $\tilde{T}$ denote three-fermion terms. They can be found by compensating all terms in $Q^2 = 0 = \tilde{Q}^2$.The result is

$$T = \sqrt{\frac{2}{3}}\bar{\chi}_0\chi^1\chi^2 + 3\bar{\chi}_2\psi^0\chi^2 + 3\bar{\chi}_0\psi^0\chi^0 + \frac{z^*}{\sqrt{2}}\left(\bar{\chi}_0\chi^1\chi^0 + \bar{\chi}_2\chi^1\chi^2\right),$$

$$\tilde{T} = -\sqrt{\frac{2}{3}}\bar{\chi}_2\bar{\chi}_1\chi^0 - 3\bar{\chi}_2\bar{\psi}_0\chi^2 - 3\bar{\chi}_0\bar{\psi}_0\chi^0 + \frac{z}{\sqrt{2}}\left(\bar{\chi}_0\bar{\chi}_1\chi^0 + \bar{\chi}_2\bar{\chi}_1\chi^2\right). \tag{3.18}$$

In the zero- and one-fermion sector the three-fermion terms do not contribute. Proceeding as in Eqs. (3.8) and (3.9) the wave-equation in the one-fermion sector is obtained as

$$\frac{1}{2}\hbar\eta^{\mu\nu}\left(\hbar\frac{\partial}{\partial q^{\mu}}-2\frac{\partial\phi}{\partial q^{\mu}}\right)\frac{\partial f}{\partial q^{\nu}}+\hbar^{2}\frac{\partial^{2} f}{\partial z\partial z^{*}}$$

$$=e^{6\alpha+|z|^{2}}(|DW|^{2}-3|W|^{2})f. \quad (3.19)$$

It is somewhat simpler in form than Eq. (3.9) but leads to the same conclusions. In particular, the semiclassical solutions (3.12), (3.13) are the same.

C. Approximate solutions in the one-fermion sector

For α negative and sufficiently large the right-hand side of Eq. (3.19) is negligible and also ϕ approaches zero. Then

$$f\simeq e^{ik_{\nu}q^{\nu}+\frac{i}{\sqrt{2}}(kz+k^{*}z^{*})}, \quad (3.20)$$

with

$$k_0^2=|k|^2+k_1^2+k_2^2 \quad (3.21)$$

and

$$|\psi_1\rangle\simeq e^{-\phi}e^{ik_{\nu}q^{\nu}+\frac{i}{\sqrt{2}}(kz+k^{*}z^{*})}(\hbar k_{\nu}\bar{\psi}^{\nu}+\hbar k^{*}\bar{\chi}_1)|0\rangle. \quad (3.22)$$

For sufficiently large α (see below) Eq. (3.19) may be solved in a Born-Oppenheimer approximation assuming that the scalar field adjusts itself quasi-instantaneously to the gravitational field. We consider the case where $W(z)$ has a quadratic stationary point z_0 with $W(z_0)=0$,

$$W(z)=\frac{c}{2}(z-z_0)^2+\cdots,$$
$$DW=c(z-z_0)+\cdots,\quad c\neq 0. \quad (3.23)$$

Thus

$$V_M\simeq e^{6\alpha+|z_0|^2}\left(|c|^2|z-z_0|^2+\cdots\right). \quad (3.24)$$

For this potential and for fixed α we can solve the eigenvalue problem

$$-\hbar^2\frac{\partial\tilde{f}}{\partial z\partial z^*}+V_M f=E(\alpha)\tilde{f} \quad (3.25)$$

in terms of eigenfunctions and eigenvalues of the two-dimensional harmonic oscillator. The energy E depends on α like

$$E(\alpha)=E_0e^{3\alpha}. \quad (3.26)$$

The constant prefactor E_0 depends on the two quantum numbers of the harmonic oscillator which will not change in the adiabatic regime. It then remains to solve the reduced problem

$$\frac{\hbar}{2}\eta^{\mu\nu}\left(\hbar\frac{\partial}{\partial q^{\mu}}-2\frac{\partial\phi}{\partial q^{\mu}}\right)\frac{\partial g}{\partial q^{\nu}}=E_0e^{3\alpha}g, \quad (3.27)$$

where the right-hand side gives the energy of the matter field. The assumption of adiabaticity is satisfied if the *relative* change of the frequency $\omega(\alpha)=2|c|e^{3\alpha+|z_0|^2/2}$ of the scalar field over a period $2\pi/\omega(\alpha)=T$ remains small. Using α itself as a parameter to measure time we have, for the change $\Delta\alpha$ of α over one period,

$$\int_0^{\Delta\alpha}\omega(\alpha)d\alpha=2\pi,$$
$$\Delta\alpha\simeq\frac{3\pi}{|c|}e^{-3\alpha-|z_0|^2/2}, \quad (3.28)$$

and the adiabatic condition becomes

$$3\Delta\alpha<<1, \quad (3.29)$$

which will be satisfied in the limit of a large scale-parameter.

The total wave function in this limit then is approximately

$$f(z,z^*,q)\simeq\tilde{f}(z,z^*|\alpha)g(\alpha,\beta_+,\beta_-), \quad (3.30)$$

where $\tilde{f}$ solves Eq. (3.25) and g solves Eq. (3.27). For concreteness let us assume now that we are dealing with the Bianchi type IX case. In the limit where the adiabatic approximation is valid the prefactor $e^{-\phi/\hbar}$ or $e^{-\tilde{\phi}/\hbar}$ of f will then be very sharply peaked at $\beta_+=0=\beta_-$. Therefore we may put $\beta_+=\beta_-=0$ in f and Eq. (3.27) is reduced, respectively, to

$$-\frac{\hbar^2}{2}\frac{\partial^2 g}{\partial\alpha^2}+\hbar e^{2\alpha}\frac{\partial g}{\partial\alpha}=E_0e^{3\alpha}g\quad\text{for }\phi=\phi_9 \quad (3.31)$$

or

$$-\frac{\hbar^2}{2}\frac{\partial^2 g}{\partial\alpha^2}-\hbar e^{2\alpha}\frac{\partial g}{\partial\alpha}=E_0e^{3\alpha}g\quad\text{for }\phi=\tilde{\phi}_9. \quad (3.32)$$

In the semiclassical limit the solutions are given by

$$g\simeq(2E_0e^{3\alpha}-e^{4\alpha})^{1/2}$$
$$\times\exp\left[\frac{1}{2}e^{2\alpha}-i\int^{\alpha}d\alpha(2E_0e^{3\alpha}-e^{4\alpha})^{1/2}\right] \quad (3.33)$$

for $\phi=\phi_9$ or

$$g\simeq(2E_0e^{3\alpha}-e^{4\alpha})^{1/2}$$
$$\times\exp\left[-\frac{1}{2}e^{2\alpha}-i\int^{\alpha}d\alpha(2E_0e^{3\alpha}-e^{4\alpha})^{1/2}\right] \quad (3.34)$$

for $\phi=\tilde{\phi}_9$, which are both outgoing waves as long as

$$e^{\alpha}<2E_0. \quad (3.35)$$

This condition in the present model defines the maximum radius of the Universe, where the outgoing wave is reflected to become a standing wave. In the physical Universe, long before this event happens, the coherence of the wave function has been spread over so many degrees of freedom (not contained in our minisuperspace) that this coherence becomes completely irrelevant for any conceivable physical process and a classical description is required.

Note added in proof. After this paper had been finished the state $|\tilde{\psi}_0\rangle$ of (2.54) was obtained as a solution of the quantum constraints of $N = 1$ supergravity restricted to homogeneous spatial three-geometries of Bianchi type IX and homogeneous Rarita-Schwinger fields transforming according to the spin-$\frac{3}{2}$ representation of the homogeneity group [R. Graham and H. Luckock, "The Hartle-Hawking State for the Bianchi IX Model in Supergravity," report (unpublished)].

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APPENDIX

In this appendix we examine the general form of the solution of $\tilde{Q}|\psi_1\rangle = 0$ in the one-fermion sector with $\tilde{Q} = \tilde{Q}_0 + \tilde{Q}_K + \tilde{Q}_P$ given by Eqs. (3.4)–(3.6). With $|\psi_1\rangle = (f^\nu \bar{\psi}_\nu + g^\nu \bar{\chi}_\nu)|0\rangle$ we obtain that the 15 coefficients of $\bar{\psi}_\nu \bar{\psi}_\mu$, $\bar{\chi}_\nu \bar{\psi}_\mu$ and $\bar{\chi}_\nu \bar{\chi}_\mu$ must vanish. Using the abbreviations

$$\Omega = e^{-3\alpha - \frac{|z|^2}{2}}, \qquad D_\nu = -\hbar\partial_\nu - \partial\phi/\partial q^\nu, \tag{A1}$$

the conditions that the coefficient of $\bar{\psi}_\nu \bar{\psi}_\mu$ vanish read

$$\begin{aligned} -D_\alpha f^1 - D_{\beta_+} f^0 + 3\hbar f^1 &= 0, \\ -D_\alpha f^2 - D_{\beta_-} f^0 + 3\hbar f^2 &= 0, \\ D_{\beta_+} f^2 - D_{\beta_-} f^1 &= 0. \end{aligned} \tag{A2}$$

Defining the functions $\tilde{f}^\nu$ by

$$f^\nu = e^{-3\alpha - \phi/\hbar} \tilde{f}^\nu, \tag{A3}$$

Eqs. (A2) become

$$\begin{aligned} \partial_\alpha \tilde{f}^1 + \partial_{\beta_+} \tilde{f}^0 &= 0, \\ \partial_\alpha \tilde{f}^2 + \partial_{\beta_-} \tilde{f}^0 &= 0, \\ \partial_{\beta_+} \tilde{f}^2 - \partial_{\beta_-} \tilde{f}^1 &= 0, \end{aligned} \tag{A4}$$

which imply

$$\tilde{f}^0 = -\frac{\partial F}{\partial \alpha}, \; \tilde{f}^1 = \frac{\partial F}{\partial \beta_+}, \; \tilde{f}^2 = \frac{\partial F}{\partial \beta_-} \tag{A5}$$

or

$$f^\nu = i\Omega D^\nu \tilde{F}, \tag{A6}$$

with

$$\tilde{F} = \frac{i}{\hbar} e^{|z|^2/2} e^{-\phi/\hbar} F. \tag{A7}$$

Here F and $\tilde{F}$ are yet undetermined functions of $\alpha, \beta_+, \beta_-, z, z^*$. Equation (A6) implies

$$f^\nu \bar{\psi}_\nu |0\rangle = \tilde{Q}_0 \tilde{F} |0\rangle. \tag{A8}$$

Next we consider the conditions that the coefficients of $\bar{\chi}_\nu \bar{\chi}_\mu$ vanish. They take the form

$$\begin{aligned} i\sqrt{2}\hbar\Omega\partial_{z^*} g^0 + i\sqrt{6} W g^1 &= 0, \\ i\sqrt{2} D W g^0 + i\sqrt{6} W g^2 &= 0, \\ -i\sqrt{2}\hbar\Omega\partial_{z^*} g^2 + i\sqrt{2} D W g^1 - i\sqrt{\frac{2}{3}}\hbar\Omega g^0 &= 0. \end{aligned} \tag{A9}$$

From the first two of these equations it follows (for $W \neq 0$) that

$$\begin{aligned} g^1 &= -\frac{\hbar}{\sqrt{3}} \frac{\Omega}{W} \partial_{z^*} g^0, \\ g^2 &= -\frac{1}{\sqrt{3}} \frac{DW}{W} g^0, \end{aligned} \tag{A10}$$

and the last of Eqs. (A9) is then automatically satisfied. With

$$\tilde{g}^0 = -\frac{i}{\sqrt{6}} \frac{1}{W} g^0, \tag{A11}$$

Eqs. (A10) imply

$$g^\nu \bar{\chi}_\nu |0\rangle = (\tilde{Q}_K + \tilde{Q}_P) \tilde{g}^0 |0\rangle. \tag{A12}$$

Finally we equate to zero the coefficients of $\bar{\chi}_0 \bar{\psi}_\nu$ which yields

$$i\Omega D^\nu g^0 - i\sqrt{6} W f^\nu = 0 \tag{A13}$$

and implies with Eqs. (A11), (A6) that

$$D^\nu (\tilde{g}^0 - \tilde{F}) = 0. \tag{A14}$$

Hence, we may rewrite Eq. (A8) as

$$f^\nu \bar{\psi}_\nu |0\rangle = \tilde{Q}_0 \tilde{g}^0 |0\rangle, \tag{A15}$$

and Eqs. (A12), (A15) together imply

$$|\psi_1\rangle = (f^\nu \bar{\psi}_\nu + g^\nu \bar{\chi}_\nu)|0\rangle = (\tilde{Q}_0 + \tilde{Q}_K + \tilde{Q}_P)\tilde{g}^0|0\rangle = \tilde{Q}\tilde{g}^0|0\rangle. \tag{A16}$$

Because of the form (A16) of $|\psi_1\rangle$ the coefficients of the remaining terms $\bar{\chi}_\mu \bar{\psi}_\nu$, $(\mu \neq 0)$ now automatically vanish. Nontrivial solutions in the one-fermion sector therefore must all have the form (A16), which is permitted only for a nondefinite signature of the metric in mini-superspace. For a Riemannian signature of the metric there are no states in the one-fermion sector.

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