

Strings in cosmological and black hole backgrounds: Ring solutions

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The string equations of motion and constraints are solved for a ring-shaped ansatz in cosmological and black hole spacetimes. In FRW universes with arbitrary power behavior [$R(X^0) = a|X^0|^\alpha$], the asymptotic form of the solution is found for both $X^0 \rightarrow 0$ and $X^0 \rightarrow \infty$ and we plot the numerical solution for all times. Right after the big bang ($X^0 = 0$), the string energy decreases as $R(X^0)^{-1}$ and the string size grows as $R(X^0)$ for $0 < \alpha < 1$ and as X^0 for $\alpha < 0$ and $\alpha > 1$. Very soon [$X^0 \sim 1$], the ring reaches its oscillatory regime with frequency equal to the winding and constant size and energy. This picture holds for all values of α including string vacua (for which, asymptotically, $\alpha = 1$). In addition, an exact nonoscillatory ring solution is found. For black hole spacetimes (Schwarzschild, Reissner-Nordström, and stringy), we solve for ring strings moving towards the center. Depending on their initial conditions (essentially the oscillation phase), they are absorbed or not by Schwarzschild black holes. The phenomenon of particle transmutation is explicitly observed (for rings not swallowed by the hole). An effective horizon is noticed for the rings. Exact and explicit ring solutions inside the horizon(s) are found. They may be interpreted as strings propagating between the different universes described by the full black hole manifold.

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I. INTRODUCTION

The systematic investigation of string dynamics in curved spacetimes started in [1] has shown a variety of new physical phenomena [2]. These results are relevant both for fundamental (quantum) strings and for cosmic strings, which behave in an essentially classical way [3].

String propagation has been studied in nonlinear gravitational plane waves [4–6] and shock waves [7,8], conical spacetimes [9,10], black holes [11], and cosmological spacetimes [2,12,13].

Among the cosmological backgrounds, de Sitter spacetime occupies a special place. On one hand, it is the relevant inflationary geometry, and on the other, string propagation turns out to be special there [1,12,13]. Moreover the classical string equations of motion (plus the string constraints) happen to be integrable in D -dimensional de Sitter universe [16–18]. More precisely, they are equivalent to a σ model on the Grassmannian $SO(D,1)/O(D)$ with periodic boundary conditions (for closed strings) or Neumann boundary conditions (for open strings).

For generic Friedmann-Robertson-Walker (FRW) cosmological spacetimes, the propagation of strings is certainly a nonintegrable problem. Therefore, one is faced with a rather formidable set of coupled nonlinear partial differential equations. Analogous difficulties arise in other nonintegrable backgrounds, such as Schwarzschild black holes. To grasp basic physical properties, we consider in the present paper solutions for the motion of classical closed test strings for which the τ and σ dependence are separated. (Hence, we have to solve just nonlinear ordinary differential equations.) That is, we

make separable ring-shaped ansätze sharing the symmetry of the background geometry.

As is well known, the effective action for string theory corresponds to a modification of Einstein-Hilbert action (see for instance the review [20] for the tree-level effective action). This leads to cosmological spacetimes that are solutions of the variational equations for this action. It is therefore interesting to investigate ringlike solutions in these stringy cosmological spacetimes as well, and to compare their features with those encountered in our study of FRW spacetimes.

We consider cosmological spacetimes with the metric

$$ds^2 = (dX^0)^2 - R(X^0)^2 \sum_{i=1}^{D-1} (dX^i)^2 \quad (1.1)$$

for FRW universes $R(X^0) = a|X^0|^\alpha$, where α is a constant, α is 1/2 for radiation-dominated universes and 2/3 for matter-dominated universes. However, we shall consider arbitrary real values of α . In particular, for $\alpha = 1$ we have a tree-level string vacuum [23]. String vacua may give more complicated functions $R(X^0)$, but $R(X^0)$ usually grows as X^0 for large X^0 [20]. In terms of the conformal time η ,

$$\eta = \int^{X^0} \frac{dX^0}{R(X^0)}.$$

The metric (1.1) takes the form

$$ds^2 = R(\eta)^2 \left[(d\eta)^2 - \sum_{i=1}^{D-1} (dX^i)^2 \right],$$

where $R(\eta)^2 = A\eta^k$ and $k = 2\alpha/(1-\alpha)$.

The ring-shaped ansatz for the string solution is

defined by

$$\begin{aligned} X^0 &= X^0(\tau), \\ X^1 &= f(\tau)\cos\sigma, \\ X^2 &= f(\tau)\sin\sigma, \\ X^i &= \text{const}, \quad i \geq 3, \end{aligned} \quad (1.2)$$

and leads to two coupled ordinary nonlinear differential equations on two functions: the ring radius $f(\tau)$ (in comoving coordinates) and the cosmic time $X^0 = X^0(\tau)$. The other relevant physical quantities here are the invariant string size $S(\tau) \equiv f(\tau)R(X^0(\tau))$ and the string energy $E(X^0) = (1/\alpha')\dot{X}^0(\tau)$. Here τ stands for the proper string time and α' for the string tension.

We can summarize our results as follows for all power expanding universes [$R(X^0) \sim a|X^0|^\alpha$]. At the time $X^0=0$ singularity (big bang and/or big crunch), the string energy is infinite, and its size is zero. Right after that, the energy redshifts as:

$$E(X^0) \sim \text{const} \times \frac{1}{a'R(X^0)}, \quad (1.3)$$

and the invariant size grows as follows

$$S = \begin{cases} \text{const} \times R(X^0) & \text{for } X^0 \rightarrow 0 \text{ when } k > 0, \\ \text{const} \times X^0 & \text{for } X^0 \rightarrow 0 \text{ when } k < 0. \end{cases} \quad (1.4)$$

Notice that in the second case (power inflation) the string size is proportional to the particle horizon ($\sim X^0$). In the first case (FRW cosmologies), the string grows as the expansion factor. This is exactly like the case of strings falling into a singular plane wave [4–6]. Stringy universes are special ($\alpha=1$) and we find (Sec. III) that

$$S = R(X^0) \ln \left[\frac{1}{R(X^0)} \right] \quad \text{for } R(X^0) \rightarrow 0. \quad (1.5)$$

Very soon [$X^0 \sim 1$], the ring string solution reaches an oscillatory regime with [see Eqs. (2.23), (2.33), and (3.4)]

$$f(\tau) \sim C \frac{1}{\tau^\alpha} \cos(\tau + \varphi) \quad (1.6)$$

with X^0 proportional to $C^{1-\alpha}\tau$. Here, C and φ are arbitrary constants, amplitude and phase, respectively. This describes for large X^0 a string of constant invariant size that oscillates with unit frequency. Quantum mechanically, Eq. (1.6) corresponds to an excitation of graviton or dilaton type. Multiple wound ring-shaped strings oscillate with arbitrary integer frequency [see Eq. (2.33)]. Notice that simple string oscillatory behaviors such as (1.6) are absent in inflationary cosmologies as the de Sitter universe.

In conclusion, when the Universe expands as any finite power of X^0 , the ring string oscillates similarly to flat spacetime. Only the oscillation amplitude varies with time in such a way that the invariant size, $S(\tau)$, and the energy remain constant. Near the big bang (or big crunch) singularities the string collapses [Eqs. (1.4), (1.5)].

We find in addition an exact ring solution with radius

$f(\tau)$ proportional to conformal time:

$$\eta(\tau) = K e^{\pm\tau/\sqrt{-k-1}}, \quad f(\tau) = \frac{K}{\sqrt{-k}} e^{\pm\tau/\sqrt{-k-1}}, \quad (1.7)$$

where K is an arbitrary constant. This solution is real for $k+1 < 0$. For $-1 < k < 0$, it is real for imaginary τ . For $k > 0$, this solution is real for imaginary conformal time η and imaginary string time τ . Hence, it can be considered as an *instanton* for $k > -1$.

Since Schwarzschild black holes are asymptotically flat spacetimes, we consider these ring string solutions starting very far from the black hole. They asymptotically behave as Minkowski solutions with a momentum p directed towards the black hole center, an oscillation amplitude m and an energy e with $e^2 = p^2 + m^2$. This energy is in fact proportional to the string energy at all times t , $E(t) = e/\alpha'$. Notice that the oscillation amplitude coincides with the classical string mass. For ring strings wounded n times, the mass turns to be nm . When the ring string approaches the black hole, its oscillations do not qualitatively change, as can be seen from Fig. 14. Then, the ring is swallowed or not by the black hole, depending on how much the string approaches the singularity. There exists an effective horizon within which the ring string is always absorbed by the black hole. We find that the sphere $r = \frac{3}{2}R_S$ (where $R_S =$ Schwarzschild radius) is completely inside this effective horizon. Numerical results indicate that the effective horizon here is close to the effective horizon for massless geodesics $\frac{3}{2}\sqrt{3}r_S = 2.5980 \dots R_S$.

Absorption by the black hole may occur for any value of the initial amplitude and momentum. The oscillation phase determines whether or not the ring will be swallowed. When the ring is not absorbed it gets out after turning an even or odd number of π 's around the center. For an even (odd) number we have back- (forward) scattering. The outgoing oscillation amplitude, m' is generically different from the initial amplitude m . This is an explicit illustration of the phenomenon of particle transmutation noticed in Ref. [14].

Several black hole perturbative string vacua are now known (for a recent review, see [15]). In particular there exist rotationally invariant solutions which are generalizations of Reissner-Nordström spacetime.

We show how a generic rotationally symmetric spacetime admits ringlike solutions to the string equations of motion, and apply this to the aforementioned black hole solutions. The analysis of the possibility of collapse of the string onto the singularity is radically different from the behavior present in Schwarzschild's background. Whenever the region close to the singularity is static (as would be the case for Reissner-Nordström spacetime), the singularity is strongly repulsive of the strings, similarly to what happens with point particles. Only if the usually present inner horizon is absent or coincides with the singularity will we get collapsing string with the world sheet lacking analyticity at the collapse point. The asymptotic region $r \gg |\theta|$, which corresponds to the asymptotically Minkowskian region of all these space-

times, does indeed reproduce the expected oscillatory behavior.

An exact solution for the string equations of motion is obtained for Reissner-Nordström and stringy black holes. It can be understood as the propagation of the string from one asymptotically flat region to another, crossing both the outer and the inner horizon present in these black holes. This is clearly analogous to the possibility that timelike curves can connect the different asymptotically flat regions.

The corresponding solution for Schwarzschild's background does exist, but remains always inside the horizon. Thus no violation of causality can be caused by the ring-like strings under consideration.

II. STRINGS PROPAGATING IN FRIEDMANN-ROBERTSON-WALKER COSMOLOGICAL SPACETIMES

We consider in this section closed test strings propagating in FRW spacetimes with the metric

$$\begin{aligned} ds^2 &= (dX^0)^2 - R(X^0)^2 \sum_{i=1}^{D-1} (dX^i)^2 \\ &= R(\eta)^2 \left[(d\eta)^2 - \sum_{i=1}^{D-1} (dX^i)^2 \right], \end{aligned} \quad (2.1)$$

where X^0 stands for cosmic time and η for conformal time. These times are related by

$$\eta = \int^{X^0} \frac{dX^0}{R(X^0)}. \quad (2.2)$$

The function $R(\eta)^2$ will be assumed of power type:

$$R(\eta)^2 = A\eta^k, \quad (2.3)$$

where A is a constant. For $k=-2$ we have de Sitter spacetime, for $k=2$ a radiation-dominated FRW universe, and for $k=4$ a matter-dominated universe [21].

For $k \neq -2$ we get, from Eqs. (2.2) and (2.3),

$$X^0 = \frac{2\sqrt{A}}{k+2} \eta^{(k+2)/2} \quad \text{and} \quad \alpha = \frac{k}{k+2}. \quad (2.4)$$

The Lagrangian function for a bosonic string in the geometry (2.1) takes the form

$$\mathcal{L} = \frac{1}{2} [(\partial_\mu X^0)^2 - R(X^0)^2 (\partial_\mu X^i)^2], \quad (2.5)$$

where ∂_μ are derivatives with respect to the world sheet coordinates x_μ , $x_0 = \tau$, and $x_1 = \sigma$ (we use the conformal gauge throughout). This yields the equations of motion

$$\begin{aligned} \partial^2 X^0 - R(X^0) \frac{dR}{dX^0} \sum_{i=1}^{D-1} (\partial_\mu X^i)^2 &= 0, \\ \partial_\mu (R^2 \partial^\mu X^i) &= 0, \quad 1 \leq i \leq D-1, \end{aligned} \quad (2.6)$$

and the constraints

$$T_{\pm\pm} = (\partial_\pm X^0)^2 - R(X^0)^2 (\partial_\pm X^i)^2 = 0. \quad (2.7)$$

Equations (2.6) and (2.7) turn to be integrable for the de Sitter case ($k=-2$) [16]. They are not integrable for gen-

eric k . We shall therefore proceed to analyze a simple ansatz where variables σ and τ separate. We choose a ring configuration whose radius depends on τ :

$$\begin{aligned} X^0 &= X^0(\tau), \\ X^1 &= f(\tau) \cos \sigma, \\ X^2 &= f(\tau) \sin \sigma, \\ X^i &= \text{const}, \quad i \geq 3. \end{aligned} \quad (2.8)$$

We then find from Eqs. (2.6) and (2.7) the following set of ordinary differential equations for $X^0(\tau)$ and $f(\tau)$:

$$\begin{aligned} \frac{d}{d\tau} \left[R^2 \frac{df}{d\tau} \right] + R^2 f &= 0, \\ \frac{d^2 X^0}{d\tau^2} + R \frac{dR}{dX^0} (\dot{f}^2 - f^2) &= 0, \\ \left[\frac{dX^0}{d\tau} \right]^2 - R^2 (\dot{f}^2 + f^2) &= 0, \end{aligned} \quad (2.9)$$

where \dot{f} stands for $df/d\tau$.

The string energy can be easily computed from the spacetime string energy-momentum tensor:

$$\begin{aligned} \sqrt{-G} T^{AB}(X) &= \frac{1}{2\pi\alpha'} \int d\sigma d\tau (\dot{X}^A \dot{X}^B - X'^A X'^B) \\ &\quad \times \delta^{(D)}(X - X(\sigma, \tau)). \end{aligned} \quad (2.10)$$

Therefore, whenever $X^0 = X^0(\tau)$, the string energy at a time X^0 is given by

$$E(X^0) = \int d^{D-1} X \sqrt{-G} T^{00}(X) = \frac{1}{\alpha'} \frac{dX^0}{d\tau}. \quad (2.11)$$

For the restriction to power-type metrics, Eqs. (2.3) and (2.9) take a particularly simple form for the conformal time $\eta = \eta(\tau)$:

$$\eta \dot{\eta} + k \dot{f}^2 = 0, \quad (2.12)$$

$$\eta \ddot{f} + k \dot{\eta} \dot{f} + \eta f = 0, \quad (2.13)$$

$$\dot{\eta}^2 - \dot{f}^2 - f^2 = 0. \quad (2.14)$$

The last equation is due to the string constraint. Notice that the first two equations guarantee that $[\dot{\eta}^2 - \dot{f}^2 - f^2]$ is a constant of the motion. Hence the constraint (2.14) will be satisfied for all times if it is satisfied by the initial conditions. We have therefore only two independent differential equations in the set (2.12)–(2.14), with one constraint on the possible initial conditions. In addition to this, the equations are invariant under the scaling transformation

$$\eta(\tau) \rightarrow \lambda \eta(\tau), \quad f(\tau) \rightarrow \lambda f(\tau), \quad (2.15)$$

where λ is an arbitrary constant. It thus follows that the space of solutions is parametrized by two initial conditions and a scaling factor.

The invariant spacetime interval (2.1) measured for these string solutions takes the form

$$ds^2 = \eta^k f^2 (d\tau^2 - d\sigma^2), \tag{2.16}$$

where we have used Eq. (2.14). We can then interpret

$$S(\tau)^2 = \eta^k f^2 \tag{2.17}$$

as the string size squared [2].

Equations (2.12)–(2.14) admit a pair of simple solutions that can be written in closed form:

$$\eta(\tau) = e^{\pm\tau/\sqrt{-k-1}}, \quad f(\tau) = \frac{1}{\sqrt{-k}} e^{\pm\tau/\sqrt{-k-1}}. \tag{2.18}$$

Of course, these two solutions can be scaled according to (2.15). They are real for $k+1 < 0$, and describe a string that explodes or collapses for large τ , which can be seen from the form taken in this case by the string size $S(\tau)$:

$$S(\tau)^2 = -\frac{1}{k} \exp\left[\pm \frac{k+2}{\sqrt{-k-1}} \tau\right]. \tag{2.19}$$

Clearly, the string explodes, for $\tau \rightarrow +\infty$, for the choice of plus sign of the exponent, $k+2$ being larger than zero ($-1 > k > -2$), or the choice of the minus sign whenever $k+2 < 0$. The converse situation, i.e., string collapse for $\tau \rightarrow +\infty$, occurs for the choice of the minus sign and $k+2 > 0$ ($-1 > k > -2$), or plus sign and $k+2 < 0$. Notice that, for these solutions,

$$\frac{S(\tau)}{R(\tau)} = \frac{1}{\sqrt{-k}} \exp\left[\pm \frac{\tau}{\sqrt{-k-1}}\right], \tag{2.20}$$

giving an exponential collapse and/or explosion of the string with respect to the evolution of the spacetime.

Solution (2.18) is real for $k+1 < 0$. For $-1 < k < 0$, it is real for imaginary τ . For $k > 0$, this solution is real for imaginary conformal time η and string time τ . Hence, it can be considered as a string *instanton* for $k > -1$. For $k > 0$, it is a real solution in a Universe with Euclidean signature.

The case $k = -2$ (de Sitter) is special. There, Eq. (2.18) yields the solution $q_0(\tau)$ found in Ref. [17]. In this case the string size remains constant.

Notice that (2.18) (for $k \neq -2$) collapses or explodes irrespective of the initial string size. On the other hand, for de Sitter space, the string equations of motion and of constraint reduce to a sinh-Gordon equation with potential $-2\cosh\alpha$ [16], where the string size squared is $S(\tau)^2 = \exp[\alpha(\sigma, \tau)] / (2H^2)$. Therefore, configurations with positive (negative) α are driven towards $+\infty$ ($-\infty$) α . That is, strings such that for any given initial moment they are of size bigger (smaller) than the horizon ($1/\sqrt{2}H$) tend to explode (collapse).

The peculiarity of de Sitter spacetime in this context can be clearly seen as soon as one realizes that Eqs. (2.12)–(2.14) can be cast in a Hamiltonian form with a constraint. Define

$$H = \frac{1}{2} \eta^{-k} (\Pi_\eta^2 - \Pi_f^2 - \eta^{2k} f^2), \tag{2.21}$$

and Poisson commutators the canonical ones,

$$\{\eta, \Pi_\eta\} = \{f, \Pi_f\} = 1,$$

the rest of the commutators between f, η, Π_f, Π_η being

equal to zero.

The equations of motion previously written down are then obtained from $\dot{g} = \{g, H\}$ for any g in the algebra \mathcal{A} generated by f, η, Π_f, Π_η . The equation of constraint is equivalent to the dynamical constraint $H \approx 0$.

We can write the following element $Q \in \mathcal{A}$, $Q = \eta \Pi_\eta + f \Pi_f$. It will be the generator of dilation of the string size. Its time derivative is easily computed to be

$$\{Q, H\} = (k+2)(H + \eta^k f^2) = (k+2)(H + S^2), \tag{2.22}$$

whence for $k = -2$ we have an additional (quadratic) conserved quantity. Notice that S^2 itself is not a conserved quantity in general, even for de Sitter spacetime, since $\{S^2, H\} = k f^2 \eta^{-1} \Pi_\eta - 2f \Pi_f$.

The existence of this additional conserved quantity for this ansatz and de Sitter spacetime reflects the fact that the motion of classical test strings in de Sitter spacetimes is integrable [16,17].

Let us now derive the asymptotic behavior of η and f analytically from (2.12)–(2.14), for solutions other than the previously written exponentials.

We find, for $\tau \rightarrow +\infty$ (and $k \neq -2$),

$$\begin{aligned} \eta(\tau) &\sim \tau^{2/(k+2)}, \\ X^0(\tau) &\sim \frac{2\sqrt{A}}{k+2} \tau, \\ f(\tau) &\sim \frac{2}{k+2} \tau^{-k/(k+2)} \cos(\tau + \varphi), \end{aligned} \tag{2.23}$$

where φ is a constant phase and oscillation amplitude has been normalized using the scaling (2.15). The size of the set of solutions with this asymptotic behavior is asymptotically a constant times an oscillating term:

$$S(\tau) \sim \left[\frac{2}{k+2} \right] |\cos(\tau + \varphi)| \sim \frac{\sqrt{2}}{k+2}. \tag{2.24}$$

That is, (2.23) describes the asymptotic behavior of a string whose size oscillates with unit frequency. Quantum mechanically, this corresponds to an excitation of graviton or dilaton type. Notice that this behavior holds for all $k \neq -2$.

The result (2.23) holds for large R and is therefore valid for any universe where Eq. (2.3) is valid *asymptotically*. We want to stress that all cosmological geometries exhibit simple oscillatory string behavior for large R except when $R(X^0)$ grows faster than any power of X^0 . That is, simple oscillatory string behavior does not appear in inflationary universes like de Sitter.

The contraction and dilation of the Universe in this limit, $\tau \rightarrow \infty$, is now governed by k : for $-2 < k < 0$ this corresponds to a contracting Universe; otherwise it will be expanding.

In summary, the asymptotic behavior of the string given by (2.23) is similar to the usual $|n|=1$ modes in Minkowski spacetime. It corresponds to “stable behavior” as defined in [12,13] (the special case $k = -2$ (de Sitter spacetime) is analyzed in Ref. [17]).

Behaviors of the type (2.23) for $\alpha > 0$ were called “string stretching” in Ref. [19] referring to the fact that the string amplitude $f(\tau)$ stretches when the universe expands. This is actually a coordinate-dependent effect since the invariant string size stays constant [Eq. (2.24)]. We prefer to use the term “string stretching” in de Sitter universe where the *invariant* string size in a whole class of string solutions grows as fast as the universe [12,13,17,18].

Before describing our numerical study of Eqs. (2.12)–(2.14), let us consider the singular points where η vanishes. Assuming that $\eta(\tau_0) = 0$, we find, for $k > 0$,

$$\begin{aligned} \eta(\tau) &\sim (\tau - \tau_0)^{1/(k+1)}, \\ f(\tau) &\sim f_0 \pm (\tau - \tau_0)^{1/(k+1)}. \end{aligned} \tag{2.25}$$

For the case $0 > k > -1$, f_0 must be set equal to 0. Expression (2.25) also holds as an asymptotic behavior for $k < -1$ as $\tau \rightarrow \tau_0$, but in such a case, both $\eta(\tau)$ and $f(\tau)$ diverge at $\tau = \tau_0$. We have again made use of the scaling freedom given by (2.15). The string size squared results to be

$$\begin{aligned} S(\tau)^2 &\sim (\tau - \tau_0)^{(k+2)/(k+1)} \text{ for } k < 0, \\ S(\tau)^2 &\sim f_0^2 (\tau - \tau_0)^{k/(k+1)} \text{ for } k > 0. \end{aligned} \tag{2.26}$$

In this regime, the Universe contracts (big crunch) as

$$R(\tau)^2 \sim (\tau - \tau_0)^{k/(k+1)} \rightarrow 0, \tag{2.27}$$

whereas the string collapses for $k > 0$ or $k < -2$, but explodes for $-2 < k < -1$.

Notice, that the string size $S(\tau)$ grows as $X_0(\tau)$ grows for $k < 0$. That is, the string size is proportional to the particle horizon in this regime. For $k > 0$, the string size behaves here as $R(X_0(\tau))$.

From Eqs. (2.11)–(2.25) we find that the string energy $E(X^0) = (1/\alpha') \dot{X}^0$ behaves near the big-bang ($X^0 = 0$) singularity as

$$E(X^0) \sim \frac{\text{const}}{\alpha'} \tau^{-k/(k+1)} \sim \frac{\text{const}}{\alpha'} \frac{1}{R(X^0)}. \tag{2.28}$$

Notice that $E(X^0)$ decreases for growing small X^0 with the gravitational redshift factor $1/R(X^0)$. On the contrary, when the particle horizon ($\sim X^0$) is much larger than the ring size, the string energy is no more redshifted and tends to a constant [Eq. (2.23)].

Another possible behavior is given by

$$\begin{aligned} \eta(\tau) &\sim \tau - \tau_0, \\ f(\tau) &\sim 1 - \frac{(\tau - \tau_0)^2}{2(k+1)}. \end{aligned} \tag{2.29}$$

For $0 > k > -1$, a more likely behavior is given by

$$\begin{aligned} \eta(\tau) &\sim \tau - \tau_0, \\ f(\tau) &\sim 1 + a(\tau - \tau_0)^{1-k}, \end{aligned} \tag{2.30}$$

with a a constant. Both sets of equations correspond to a collapsing (exploding) string for $k > 0$ ($k < 0$), with size $S(\tau) \sim (\tau - \tau_0)^{k/2}$. The behavior (2.29) is however subdominant when compared to (2.25) for $k > 0$. For $k < -1$ (2.25) is singular, and, since, as we shall later prove, η must vanish for some finite value of τ , it follows that (2.29) will be present. Similarly for $-1 < k < 0$.

The equations of motion (2.12)–(2.14) being invariant under time reversal ($\tau \rightarrow -\tau$), both asymptotic behaviors, (2.23) and (2.25), may describe initial or final situations of the string. The numerical analysis, carried out using MATHEMATICA [22], precisely shows how to connect such behaviors.

Equation (2.12) tells us that $\eta\ddot{\eta}$ has a definite sign, precisely that of $-k$. Therefore, if we start from a positive $\eta(0)$, and $k > 0$, $\ddot{\eta}(\tau)$ will be negative for as long as $\eta(\tau) > 0$. Then, if $\dot{\eta}(0) < 0$, $\dot{\eta}(\tau)$ will grow in absolute value for as long as $\eta(\tau) > 0$. In conclusion, η will vanish for some finite value of $\tau = \tau_0$. On the other hand, if $\dot{\eta}(0) > 0$ (always for $k > 0$), $\eta(\tau)$ need not vanish. The numerical analysis supports these conclusions.

In Fig. 1 we plot $\eta(\tau)$ and $f(\tau)$ for $k=2$ with initial conditions such that $\eta(0)\dot{\eta}(0) < 0$. We observe that the behavior (2.25) is reproduced with reasonable accuracy.

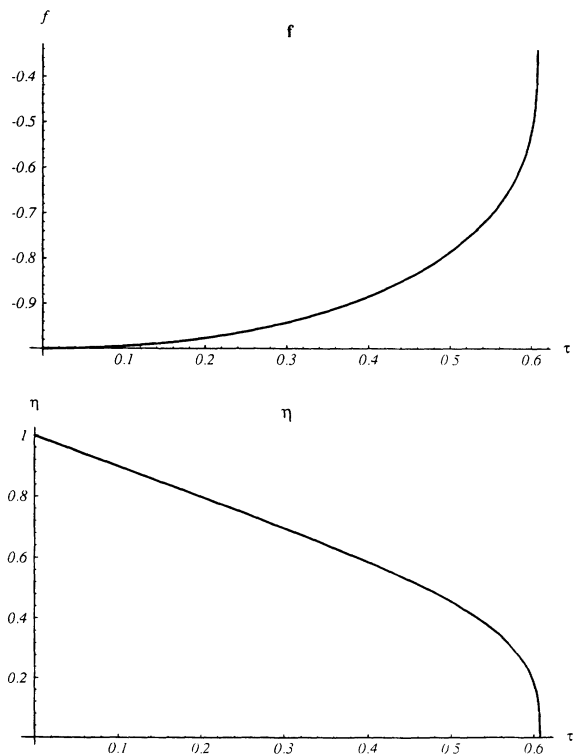


FIG. 1. Cosmological spacetime with $k=2$: numerical solution to the ringlike equations of motion for initial conditions such that $\eta(0)\dot{\eta}(0) < 0$.

In Fig. 2 we again consider $\eta(0)\dot{\eta}(0) < 0$, but this time for $k=4$.

Figure 3 depicts $\eta(\tau)$ and $f(\tau)$ for $k=2$ with initial conditions such that $\eta(0)\dot{\eta}(0) > 0$. We then find that the asymptotic regime (2.23) is very quickly reached. Similarly for $k=4$ (Fig. 4).

In Figure 5 we portray a solution for $k=2$ which starts from nothing at the big crunch $\eta=\tau=0$ and grows steadily up to an oscillatory solution, which is to be understood as a graviton or dilaton created *ex nihilo*.

The next figure, Fig. 6, presents a numerical solution for $k=-3/2$, close to a vanishing point of η , in order to show that the analysis is general. It is clear that it obeys equation (2.29) (modulo an unimportant scaling factor). We proceed to show in Fig. 7 a numerical solution, also for $k=-3/2$, but this time illustrating the asymptotic regime (2.23).

There are other possible modes of the string, and in fact our whole analysis easily generalizes to higher winding modes: take the ansatz

$$\begin{aligned} \eta &= \eta(\tau) , \\ X^1 &= f_n(\tau) \cos n\sigma , \\ X^2 &= f_n(\tau) \sin n\sigma , \\ X^i &= \text{const}, \quad i \geq 3 . \end{aligned} \tag{2.31}$$

Apart from the exponential solutions, which take the form

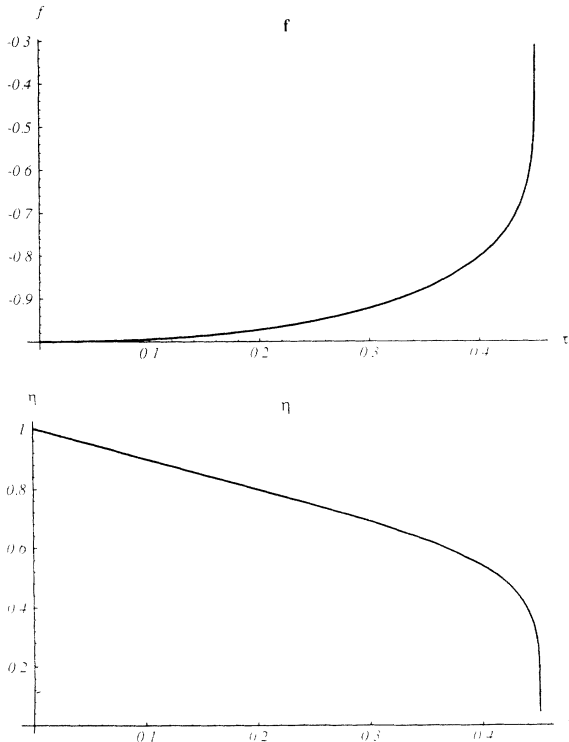


FIG. 2. Cosmological spacetime with $k=4$: numerical solution, for initial conditions such that $\eta(0)\dot{\eta}(0) < 0$.

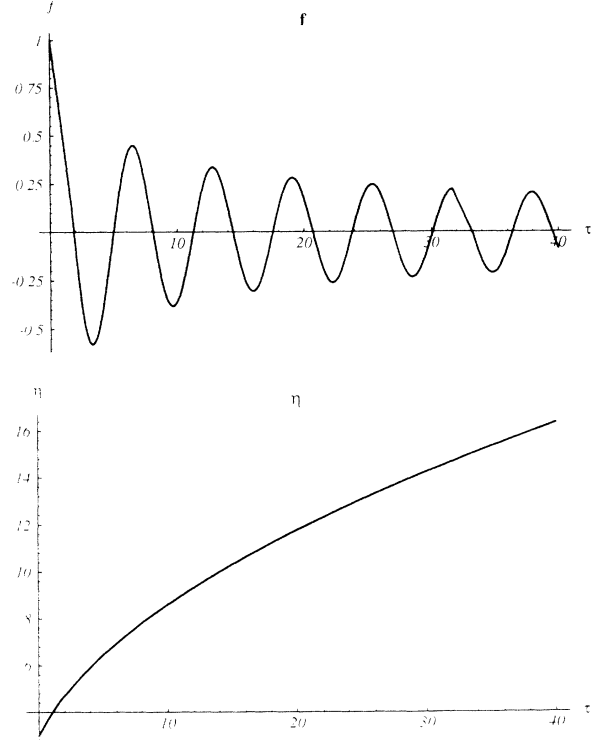


FIG. 3. Cosmological spacetime with $k=2$: initial conditions such that $\eta(0)\dot{\eta}(0) > 0$.

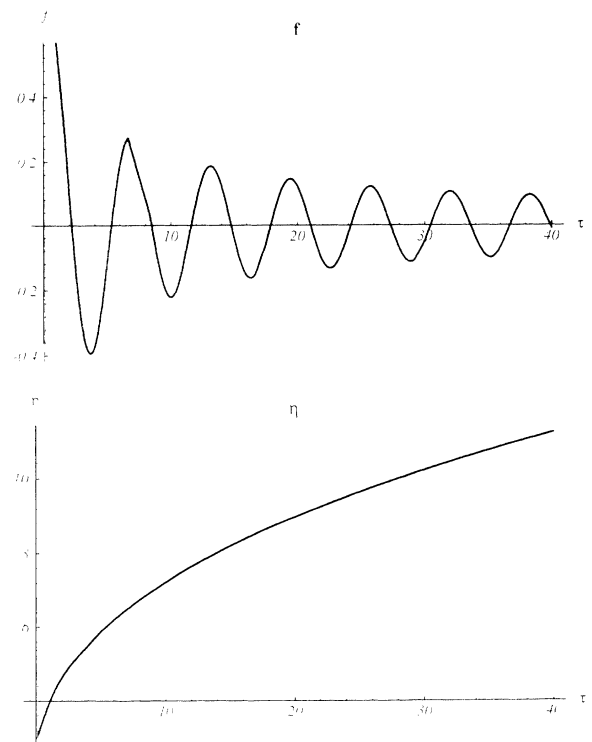


FIG. 4. Cosmological spacetime with $k=4$: initial conditions such that $\eta(0)\dot{\eta}(0) > 0$.

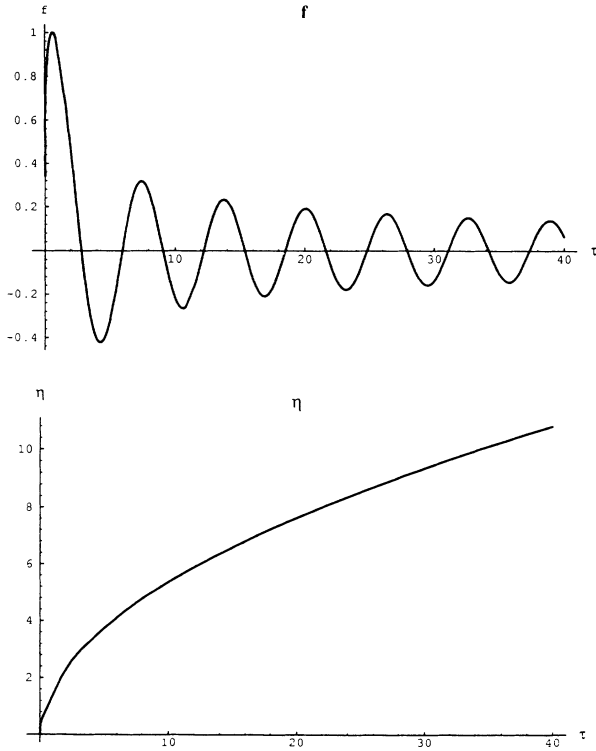


FIG. 5. Cosmological spacetime with $k=2$: both asymptotic regimes are present here.

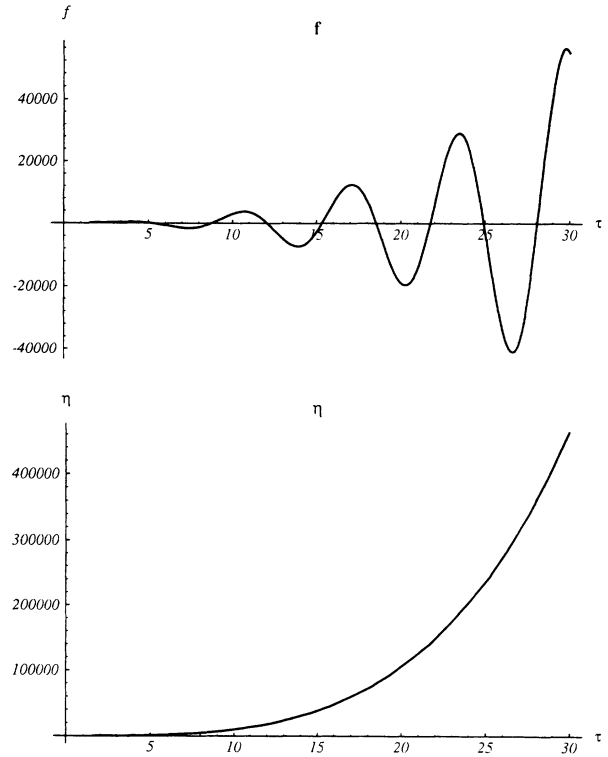


FIG. 7. Cosmological spacetime with $k=-3/2$: The large τ asymptotic regime.

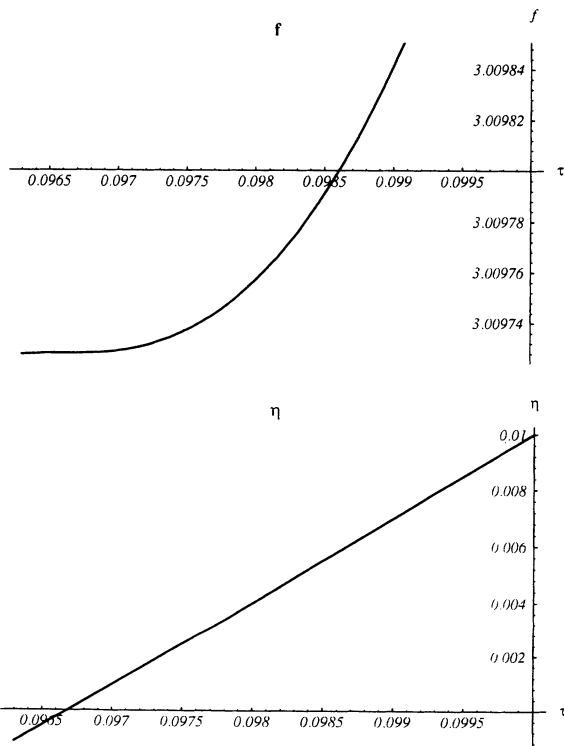


FIG. 6. Cosmological spacetime with $k=-3/2$: close to the vanishing point for η . Note the analyticity of the solution.

$$\eta(\tau) = e^{\pm n\tau/\sqrt{-k-1}}, \quad f(\tau) = \frac{1}{\sqrt{-k}} e^{\pm n\tau/\sqrt{-k-1}}, \quad (2.32)$$

only the $\tau \rightarrow \infty$ asymptotic behavior is slightly modified in the context of this generalized ansatz. We find that the analogue of (2.23) is

$$\begin{aligned} \eta(\tau) &\sim \tau^{2/(2+k)}, \\ f(\tau) &\sim \frac{2}{n(2+k)} \tau^{-k/(2+k)} \cos(n\tau + \varphi). \end{aligned} \quad (2.33)$$

III. STRINGS PROPAGATING IN STRINGY COSMOLOGICAL SPACETIMES

There are a number of interesting string vacua of this kind to be found in the literature [see (20) for a review]. These geometries can be either considered in the “string frame” or in the “Einstein frame.” The physics *changes* with the frame. We choose to work in the Einstein frame. The reason being that in such frame the spacetime metric in the effective field theory action is the *same* metric that appears in the string action for test strings [12]. In the string frame, the metric in the effective action contains the dilaton field whereas the dilaton never couples with classical test strings.

We shall assume $D=4$ uncompactified dimensions for this stringy universe as it was before the case for the

FRW geometries (Sec. II). In addition, we assume some extra compactified degrees of freedom contributing with an amount $c < 0$ to the central charge. No further assumptions will be made. In fact the value of $|c|$ gets absorbed in the choice of unit of length.

We consider in particular cosmological backgrounds in maximally symmetric, conformally flat space, that is to say, with metric of the form (2.1). Making the same ansatz as before (2.8) we again have equations (2.9) as equations of motion and constraint. Notice that the invariant string size is given by

$$S(\tau) = f(\tau)R(X^0(\tau)). \quad (3.1)$$

Let us first examine the linear dilaton, flat string frame metric solution of Myers [23], in Einstein's frame [12,20], (with the correct constant rescaling of the spatial components)

$$\begin{aligned} ds_E^2 &= (dX^0)^2 - (X^0)^2 \sum_{i=1}^{D-1} (dX^i)^2 \\ &= e^{2\eta} \left[d\eta^2 - \sum_{i=1}^{D-1} (dX^i)^2 \right], \end{aligned} \quad (3.2)$$

whence the equations of motion and constraint

$$\begin{aligned} \dot{f} + 2\dot{\eta}f + f &= 0, \\ \dot{\eta} + 2\dot{f}^2 &= 0, \\ \dot{\eta}^2 - \dot{f}^2 - f^2 &= 0, \end{aligned} \quad (3.3)$$

for ringlike solutions, following the by now usual ansatz. It has to be observed that η does not enter these equations save through its derivatives. We thus see that η can be shifted by an arbitrary constant. Disregarding this constant, we are actually restricted to a single nonlinear second-order ordinary differential equation for f , up to the choice of a sign for $\dot{\eta}$. There is no scaling freedom in this case.

In an analogous manner to the analysis carried out before, we obtain that the asymptotic behavior of the solutions for $\tau \rightarrow +\infty$ is of the form

$$\begin{aligned} f(\tau) &\sim \frac{1}{\tau} \cos(\tau + \varphi) + O(1/\tau^2), \\ \eta(\tau) &\sim \eta_0 + \ln \tau + O(1/\tau). \end{aligned} \quad (3.4)$$

Here, the oscillation amplitude for $f(\tau)$ is fixed for all ring solutions. Since the asymptotic string energy is proportional to e^{η_0} , it is then this arbitrary parameter which effectively plays the role of asymptotic string amplitude.

The string size is

$$S(\tau) = e^{\eta(\tau)} f(\tau), \quad (3.5)$$

and asymptotically, for $\tau \rightarrow +\infty$,

$$S(\tau) \sim e^{\eta_0} |\cos(\tau + \varphi)| \sim \frac{1}{\sqrt{2}} e^{\eta_0}. \quad (3.6)$$

The string is of a bounded size, whereas the Universe is expanding in this regime.

Observe that (a) $\ddot{\eta}$ is always negative and (b) $\dot{\eta} = 0$

(which implies $f = 0, \dot{f} = 0$) is a critical point of this two-dimensional dynamical system. This means that $\dot{\eta} > 0$ and $\dot{\eta} < 0$ are disconnected regions of configuration space, as was pointed out before. On the other hand, time reversal invariance means that they can be mapped into each other, so we concentrate on $\dot{\eta} > 0$. Because of (a), $\dot{\eta}$ will tend towards zero as τ grows, which corresponds to (3.4). Backwards in time, though, it will grow indefinitely. This growth might be up to some finite τ_0 , in which case, there will be a divergence at this point, of the form

$$\begin{aligned} f(\tau) &\sim \frac{1}{2} \ln(\tau - \tau_0), \\ \eta(\tau) &\sim \eta_0 + \frac{1}{2} \ln(\tau - \tau_0). \end{aligned} \quad (3.7)$$

The string collapses in this case, but more slowly than the spacetime:

$$S(\tau) \sim \frac{\sqrt{\tau - \tau_0}}{2} |\ln(\tau - \tau_0)|. \quad (3.8)$$

Both types of asymptotic behaviors, (3.4) and (3.7), are observed in the numerical solution depicted in Fig. 8.

The string energy (2.11) tends to $(1/\alpha')e^{\eta_0}$ for $X^0 \rightarrow \infty$ and it is redshifted as

$$\frac{e^{2\eta_0}}{2\alpha'R(X^0)}$$

near the big bang singularity as in Eq. (2.28).

As a conclusion, the behavior of ringlike string solu-

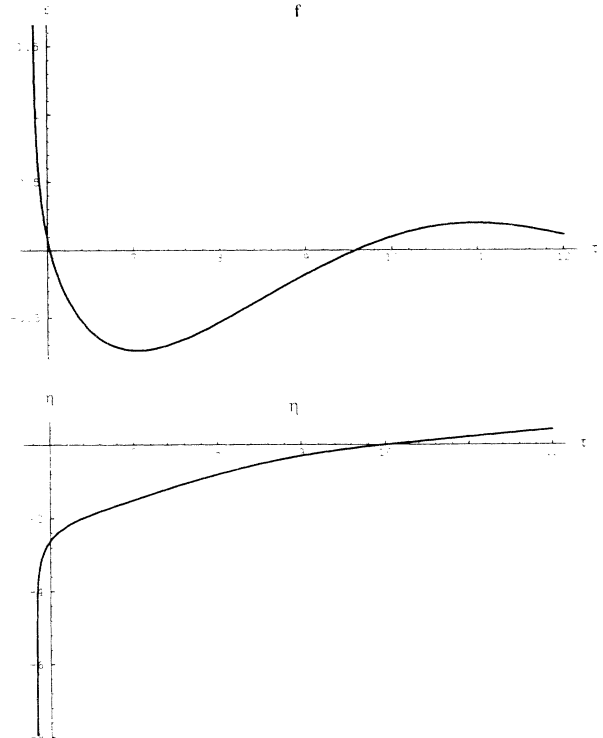


FIG. 8. Myers' spacetime: both asymptotics are observed.

tions in this spacetime is analogous to that found for FRW spacetimes, by making k tend formally towards $+\infty$. An important difference between this spacetime and the FRW solutions of general relativity lies in the fact that the “big crunch” singularity for the case at hand is a null singularity [29], whereas it is spacelike for FRW spacetimes. The collapse of the strings into these singularities is also of different kinds: while $S(\tau)/R(\tau)$ tends powerlike to zero for FRW spacetimes ($k > 0$), it is logarithmically divergent for the stringy solution under consideration.

Another interesting spacetime is the isotropic case of the one obtained by Mueller [24], which is asymptotically identical to the previous one for $X^0 \rightarrow +\infty$, but which possesses simple polynomial curvature singularities for two values of X^0 . In Einstein’s frame, it reads

$$\begin{aligned}
 ds_E^2 &= (dX^0)^2 - R(X^0, a)^2 \sum_{i=1}^{D-1} (dX^i)^2 \\
 &= (dX^0)^2 - R_0^2 (X^0 - a)^{(\sqrt{3}+1)/\sqrt{3}} \\
 &\quad \times (X^0 + a)^{(\sqrt{3}-1)/\sqrt{3}} \sum_{i=1}^{D-1} (dX^i)^2, \tag{3.9}
 \end{aligned}$$

for $D=4$. Here $c < 0$ as explained above and a is an arbitrary constant. Observe that

$$R(\lambda X^0, a) = \lambda R(X^0, a/\lambda),$$

from which we can fix $a=1$ without loss of generality, and R_0 can also be set to 1 by an adequate constant rescaling of the spatial coordinates. Making again the ring-like ansatz (2.8), the equations of motion (2.9) are consequently:

$$\begin{aligned}
 \ddot{X}^0 + 2 \left[\frac{c_1}{X^0 - 1} + \frac{c_2}{X^0 + 1} \right] \dot{X}^0 \dot{f} + f &= 0, \\
 \ddot{X}^0 + (X^0 - 1)^{2c_1} (X^0 + 1)^{2c_2} \left[\frac{c_1}{X^0 - 1} + \frac{c_2}{X^0 + 1} \right] \\
 \times (\dot{X}^0)^2 - f^2 &= 0, \\
 (\dot{X}^0)^2 - (X^0 - 1)^{2c_1} (X^0 + 1)^{2c_2} (\dot{X}^0)^2 + f^2 &= 0. \tag{3.10}
 \end{aligned}$$

Here,

$$c_1 = \frac{1}{2} \left[1 + \frac{1}{\sqrt{3}} \right], \quad c_2 = \frac{1}{2} \left[1 - \frac{1}{\sqrt{3}} \right]. \tag{3.11}$$

In D spacetime dimensions one just has to replace $\sqrt{3}$ in Eqs. (3.9)–(3.11) by $\sqrt{D-1}$.

From the expression of the metric one can conclude that in the asymptotic region $X^0 \rightarrow +\infty$ the behavior is analogous to that of Myers’ spacetime. That is to say,

$$\begin{aligned}
 X^0 &\stackrel{\tau \rightarrow +\infty}{\sim} p\tau, \\
 f &\stackrel{\tau \rightarrow +\infty}{\sim} \frac{1}{\tau} \cos(\tau + \varphi), \tag{3.12}
 \end{aligned}$$

where p is an arbitrary positive parameter proportional to e^{η_0} . On the other hand, it is clear that $X^0 = \pm 1$ are singularities of the metric.

The curvature scalar tends to ∞ as those points are approached, and the metric is imaginary for $-1 < X^0 < 1$. It is also evident that $X^0 = \pm 1$ are critical planes for the set of ordinary differential equations (3.10). Equations (3.10) are autonomous, so there is invariance under time translation, and we can choose $\tau = \tau_0$ as the point for which $X^0 = \pm 1$. In such case, the following asymptotic behavior is obtained by demanding that the solutions be regular for $X^0 \rightarrow 1^+$ when $\tau \rightarrow \tau_0$ (we choose for simplicity $\tau_0 = 0$):

$$\begin{aligned}
 X^0(\tau) &\stackrel{\tau \rightarrow 0}{\sim} 1 + b_1 \tau^{(1/c_2)}, \\
 f(\tau) &\stackrel{\tau \rightarrow 0}{\sim} \left[\frac{b_1}{2} \right]^{c_2} \frac{1}{c_2} [1 + O(\tau^2)]. \tag{3.13}
 \end{aligned}$$

This leads to a string size squared that tends to zero at the same rate as the spacetime.

Allowing for singular solutions, we obtain, for $X^0(\tau) \rightarrow 1^+$ as $\tau \rightarrow 0$,

$$\begin{aligned}
 X^0(\tau) &\stackrel{\tau \rightarrow 0}{\sim} 1 + b_1 \tau^{1/(c_1+1)}, \\
 f(\tau) &\stackrel{\tau \rightarrow 0}{\sim} \left[\frac{b_1}{2} \right]^{c_2} \frac{1}{c_2} \tau^{c_2/(c_1+1)} [1 + O(\tau^2)], \tag{3.14}
 \end{aligned}$$

in which case the string collapses as $\tau \rightarrow 0^+$ ($b > 0$) with

$$\frac{S(\tau)}{R(\tau)} \stackrel{\tau \rightarrow 0}{\sim} \tau^{c_2/(c_1+1)}. \tag{3.15}$$

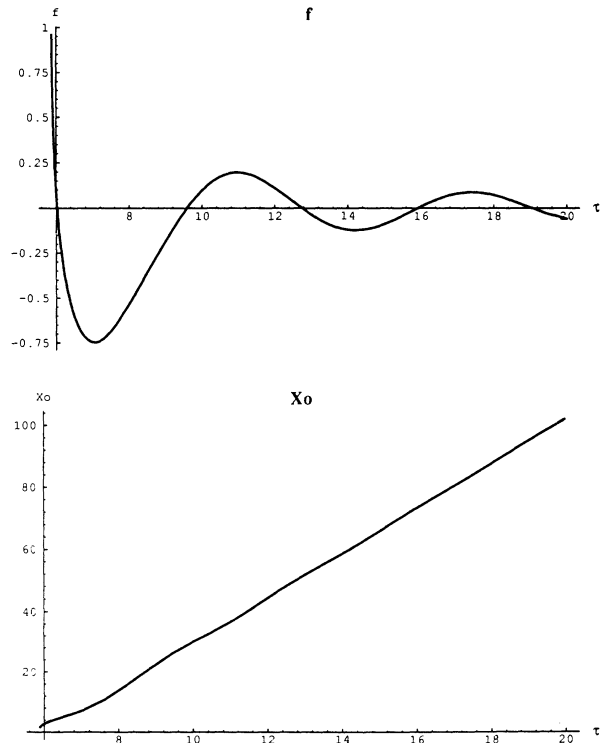


FIG. 9. Mueller’s spacetime: The large τ asymptotic behavior is clearly observed in this numerical solution.

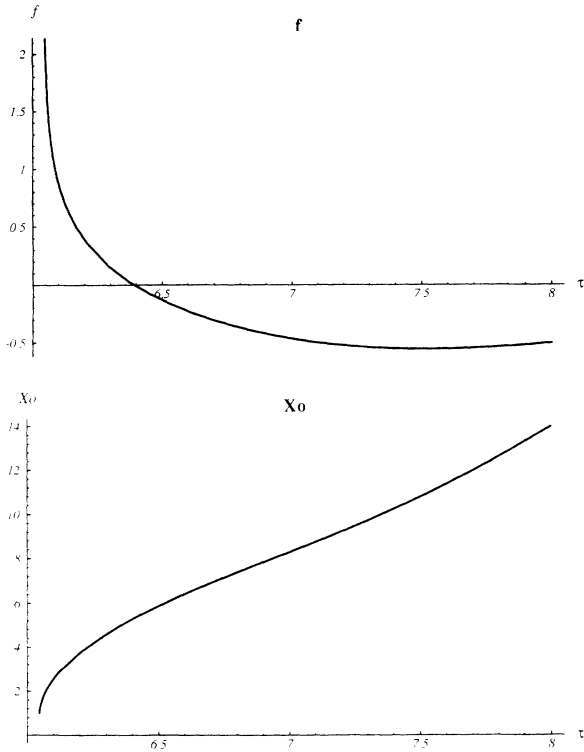


FIG. 10. Mueller's spacetime: A closer look to the limit $X^0 \rightarrow 1^+$.

Alternatively, and when considering $X^0=1$ as the big bang (and $X^0=-1$ as the big crunch), the string is coming out of the big bang faster than the expansion rate of the Universe.

$$r_\sigma t_\sigma - r_\tau t_\tau + r(r-1)(t_{\sigma\sigma} - t_{\tau\tau}) = 0,$$

$$r \sin\theta(\phi_{\tau\tau} - \phi_{\sigma\sigma}) + 2r \cos\theta(\phi_\tau \theta_\tau - \phi_\sigma \theta_\sigma) + 2(r_\tau \phi_\tau - r_\sigma \phi_\sigma) = 0,$$

$$2r(\theta_{\tau\tau} - \theta_{\sigma\sigma}) + 4(r_\tau \theta_\tau - r_\sigma \theta_\sigma) - r \sin 2\theta(\phi_\tau^2 - \phi_\sigma^2) = 0,$$

$$\frac{2r}{1-r}(r_{\tau\tau} - r_{\sigma\sigma}) - \frac{1}{r^2}(t_\tau^2 - t_\sigma^2) + 2r(\theta_\tau^2 - \theta_\sigma^2) + 2r \sin^2\theta(\phi_\tau^2 - \phi_\sigma^2) + \frac{1}{(r-1)^2}(r_\tau^2 - r_\sigma^2) = 0.$$

The constraints are

$$\frac{1-r}{r}(t_\sigma^2 + t_\tau^2) + \frac{r}{r-1}(r_\tau^2 + r_\sigma^2) + r^2(\theta_\tau^2 + \theta_\sigma^2) + r^2 \sin^2\theta(\phi_\tau^2 + \phi_\sigma^2) = 0,$$

$$\frac{1-r}{r}t_\tau t_\sigma + \frac{r}{r-1}r_\tau r_\sigma + r^2\theta_\tau \theta_\sigma + r^2 \sin^2\theta \phi_\tau \phi_\sigma = 0.$$

A. Rotationally symmetric, static 2+1 spacetime

Consider for a while constant θ in the equations above. This case is similar to the motion of a string in a conformally flat, rotationally symmetric, static 2+1 spacetime:

Passing to $X^0 \rightarrow -1$, we see that there are solutions of the same type as above, given by the exchange of c_1 with c_2 , and such that $X^0 \rightarrow -1^-$.

To illustrate these results, we portray in Fig. 9 a numerical solution in which the behavior for $\tau \rightarrow +\infty$ is particularly clear, and next in Fig. 10 a closer look at the limit $X^0 \rightarrow 1^+$, for a different numerical solution. It is seen that the singular behavior (3.14) is obeyed.

IV. STRINGS PROPAGATING IN BLACK HOLE SPACETIMES

The motion of classical test strings in a black hole background metric is also particularly interesting. Previous works on classical strings in black hole backgrounds considered infinite strings [25], static solutions [26], fluctuations around the center-of-mass motion [11], charged strings [27], and perturbations on static solutions [28].

We start with the Schwarzschild metric

$$ds^2 = R_S^2 \left[- \left[1 - \frac{1}{r} \right] dt^2 + \frac{dr^2}{1-1/r} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right], \quad (4.1)$$

where $R_S = 2m$ is the Schwarzschild radius, and the radial and "temporal" coordinates are $R = rR_S$ and $T = tR_S$. The equations of motion of a classical test string in such a background, and in the conformal gauge, are

$$d\bar{s}^2 = A^2(\rho)(-dt^2 + d\rho^2 + \rho^2 d\phi^2). \quad (4.4)$$

If we assume, as is natural, that $\phi = n\sigma$, then either $A(\rho) \propto 1/\rho$, or $\rho = \rho(\tau)$. In the second case, $\rho = \rho(\tau)$, we have that either $t = t(\tau)$ or $t = t(\sigma)$. Both these cases can be solved by quadratures. The less physical one (which

would correspond to compactified t coordinate), $t = t(\sigma)$, produces the solution

$$\begin{aligned} \rho &= (\lambda/n) \cos(n\tau + \nu) , \\ t &= \gamma\sigma + \delta , \end{aligned} \tag{4.5}$$

with γ, ν and δ arbitrary constants.

The more interesting case $\rho = \rho(\tau)$, $t = t(\tau)$, has the following solution ($\omega = \ln A$)

$$\begin{aligned} \int^\rho \frac{d\rho}{(c^2 e^{-4\omega} - n^2 \rho^2)^{1/2}} &= \pm \tau + d , \\ t &= c \int e^{-2\omega} d\tau + g , \end{aligned} \tag{4.6}$$

again with arbitrary constants c, d, g .

Our problem of interest, in a Schwarzschild background, cannot be put in this form. However, now that the analysis has been carried out for the conformally flat case, it is straightforward to generalize it to applicable expressions.

Consider then the metric

$$d\hat{s}^2 = A^2(\rho)(-dt^2 + d\rho^2 + b(\rho)^2 \rho^2 d\phi^2) . \tag{4.7}$$

If we assume, as before, that $\phi = n\sigma$, we have that either $Ab \propto 1/\rho$, or $\rho = \rho(\tau)$, in which event, either $t = t(\sigma)$ or $t = t(\tau)$, as before, and the solutions are also given by quadratures: either

$$\begin{aligned} \int^\rho \frac{d\rho}{(\gamma^2 - n^2 b^2 \rho^2)^{1/2}} &= \pm \tau + \nu , \\ t &= \gamma\sigma + \delta , \end{aligned} \tag{4.8}$$

with γ, ν and δ arbitrary constants, or

$$\begin{aligned} \int^\rho \frac{d\rho}{(c^2 e^{-4\omega} - n^2 b^2 \rho^2)^{1/2}} &= \pm \tau + d , \\ t &= c \int e^{-2\omega} d\tau + g , \end{aligned} \tag{4.9}$$

again with arbitrary constants c, d, g .

These solutions can now be applied to the case of equatorial motion of a string in a Schwarzschild black hole background. As a matter of fact, assume just that θ is constant. This implies that either $\theta = \pi/2$ (since θ is restricted between 0 and π), or $\phi_\tau^2 = \phi_\sigma^2$. Taking the axisymmetric solutions ($\phi = n\sigma, n = \text{integer}$), it is clear that only the equatorial motion makes sense, as is also physically evident. Let us concentrate on such an equatorial, axisymmetric motion. Applying formula (4.9), and the restriction that r cannot be smaller than 0, we are led to

$$\begin{aligned} t &= t_0 = \text{const} , \\ r &= \cos^2 \left[\frac{n\tau + \delta}{2} \right] . \end{aligned} \tag{4.10}$$

This equation describes a string propagating inside the Schwarzschild horizon $r=1$. Let us remember that r takes on a timelike character inside the horizon, whereas t is spacelike. It is thus better to study this solution in Kruskal-Szekeres (KS) coordinates [21]:

$$\begin{aligned} u &= \sqrt{1-r} e^{r/2} \sinh(t/2) , \\ v &= \sqrt{1-r} e^{r/2} \cosh(t/2) . \end{aligned} \tag{4.11}$$

The metric takes in these coordinates the form

$$ds^2 = \frac{4}{r} R_S^2 e^{-r} (-dv^2 + du^2) + R_s^2 r^2 (d\theta^2 + \sin^2 \theta d\phi^2) . \tag{4.12}$$

The coordinate v is a timelike coordinate, and u spacelike. We find from Eqs. (4.10) and (4.11):

$$\begin{aligned} r &= \cos^2 \left[\frac{n\tau + \delta}{2} \right] , \\ u &= \tanh(t_0/2) v . \end{aligned} \tag{4.13}$$

That is, the string falls into the singularity $r=0$, with constant speed $\tanh(t_0/4) < 1$ with respect to KS coordinates (see Fig. 11). Solution (4.13) (for $n=1$ and $\delta=0$) starts at, say, $\tau=0$, on the horizon, and bounces back to $r > 0$ after r has vanished for $\tau=\pi$. This behavior may be interpreted as follows: The motion outwards the singularity is unphysical. Which is to say that the string motion ends when the singularity $r=0$ is reached at $\tau=\pi$. Moreover, the invariant string size vanishes as $(\tau-\pi)^4$ when the singularity is approached.

The other (unphysical) solution (4.8), leads to

$$\begin{aligned} t &= \gamma\sigma + \delta , \\ \int^r \frac{rdr}{(r-1)^{1/2} [\gamma^2(r-1) - n^2 r^3]^{1/2}} &= \pm \tau + \beta , \end{aligned} \tag{4.14}$$

which is not so interesting given the periodicity we impose on σ .

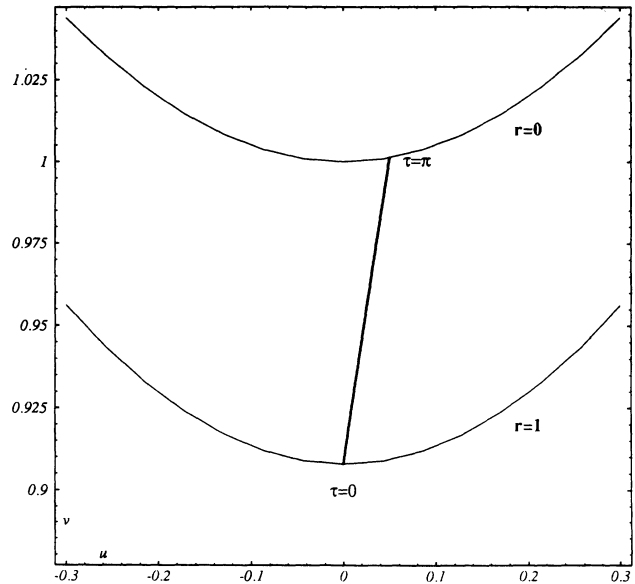


FIG. 11. Exact solution to the equations of motion inside the horizon of a Schwarzschild black hole, in Kruskal-Szekeres coordinates: the string world sheet covers all the equatorial section of the interior of the horizon.

B. Axisymmetric ansatz

The Schwarzschild manifold not being a symmetric space, the string equations of motion and of constraint, (4.2) and (4.3), are not integrable there. In order to separate the equations into ordinary differential equations, we make a simple ansatz with a symmetry compatible with the time evolution. Our concrete ansatz is to choose an axisymmetric (ring) configuration as follows:

$$\phi = n\sigma, \quad \theta = \theta(\tau), \quad t = t(\tau), \quad r = r(\tau), \quad (4.15)$$

where $n = \text{integer}$. This ansatz inserted in Eqs. (4.2) and (4.3) produces the following set of equations:

$$\begin{aligned} \dot{r} - (r - 3/2)\dot{\theta}^2 + n^2 \sin^2 \theta (r - 1/2) &= 0, \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} + n^2 r \sin \theta \cos \theta &= 0, \end{aligned} \quad (4.16)$$

with a conserved quantity

$$e^2 = \dot{r}^2 + r(r - 1)(\dot{\theta}^2 + n^2 \sin^2 \theta), \quad (4.17)$$

and t given by

$$i = \frac{er}{r - 1}. \quad (4.18)$$

Note that these equations are invariant under the change $\tau \rightarrow -\tau, t \rightarrow -t$. Since $i > 0$ outside the horizon, $t(\tau)$ is a monotonic function, and we can use either τ or t to study the time evolution for $r > 1$.

The string energy is found to be, in this case,

$$E(t) \equiv -P_0 = -\frac{G_{00}}{\alpha'} \frac{dX^0}{d\tau} = \frac{R_S e}{\alpha'}, \quad (4.19)$$

where we used Eq. (2.10)

$$P^0 = \int d^{D-1} X \sqrt{-G} T^{00}(X),$$

and α' is the string tension.

The invariant length of the string in this case is

$$ds^2 = R_S^2 n^2 r^2 \sin^2 \theta (-d\tau^2 + d\sigma^2). \quad (4.20)$$

A useful equation, satisfied by solutions of these equations, is

$$\left[\frac{d^2}{d\tau^2} + n^2 \right] (r \sin \theta) = \frac{1}{2} (n^2 \sin^2 \theta - 3\dot{\theta}^2) \sin \theta. \quad (4.21)$$

Let us now examine the possible asymptotic behaviors of these equations in different regimes. A first interesting question is the existence of collapsing solutions and the corresponding critical exponents. On computation, we find two possible collapsing behaviors from (4.16) and (4.17), with the adequate choice of origin for τ for $\tau \rightarrow 0$:

$$\begin{aligned} r &\sim \alpha \tau^{2/5}, \\ \theta &\sim \theta_f + 2\sqrt{\alpha} \tau^{1/5}, \end{aligned} \quad (4.22)$$

where α is a constant and

$$\begin{aligned} r &\sim e\tau + \frac{n^2 \sin^2 \theta_f}{4} \tau^2 - \frac{n^2 e \sin^2 \theta_f}{6} \tau^3 + O(\tau^4), \\ \theta &\sim \theta_f - \frac{n^2 \sin(2\theta_f)}{12} \tau^2 + \frac{n^2 \sin^3 \theta_f \cos \theta_f}{726} \tau^3 + O(\tau^4). \end{aligned} \quad (4.23)$$

This last one is obviously subdominant with respect to Eq. (4.22).

Consider now the regime given by large r and θ^2/r small. From Eqs. (4.17) and (4.21), we have that, for $|\tau| \rightarrow +\infty$,

$$\begin{aligned} r &\sim p|\tau|, \\ \theta &\sim \theta_0 - \frac{m}{p} \frac{\cos(n\tau + \varphi_0)}{\tau}, \end{aligned} \quad (4.24)$$

together with

$$e^2 = p^2 + n^2 m^2,$$

and $t \sim e\tau$. Here θ_0 is such that $\sin \theta_0 = 0$, i.e., $\theta = l\pi$ with l an integer. We could here understand p as an asymptotic radial momentum, e the energy, and nm the mass of the string. The latter is determined by the amplitude of the string oscillations.

We find for large τ that

$$\begin{aligned} x &= r \sin \theta \cos \phi = (-1)^{l+1} m \cos(n\tau + \varphi_0) \cos n\sigma, \\ y &= r \sin \theta \sin \phi = (-1)^{l+1} m \cos(n\tau + \varphi_0) \sin n\sigma. \end{aligned} \quad (4.25)$$

In this region, spacetime is Minkowskian, and we can recognize (4.25) as the n th excitation mode of a closed string. For $|n|=1$ this corresponds at the quantum level to a graviton and/or a dilaton. Notice that m is the amplitude of the string oscillations.

The string size is here $S(\tau) = R_S r(\tau) |\sin \theta(\tau)|$. We find from Eqs. (4.22)–(4.24) that

$$\begin{aligned} S(\tau) &\sim R_S m |\cos(\tau + \varphi)| \sim \frac{R_S m}{\sqrt{2}}, \\ S(\tau) &\sim R_S \alpha \sin \theta_f \tau^{2/5}. \end{aligned} \quad (4.26)$$

Whenever the string is not swallowed by the black hole, Eq. (4.24) describes both the incoming and outgoing regions $\tau \rightarrow \pm \infty$. However, the mass m , the momentum p , and the phase φ_0 are, in general, different in the two asymptotic regions. This is an illustration of a rather general phenomenon noticed at the quantum level: particle transmutation [14]. This means that the excitation state of a string changes in general when it is scattered by an external field like a black hole. Within our classical ansatz (4.15), the only possible changes are in amplitude (mass), momentum, and phase. It can be seen numerically that the excitation state is indeed modified by the interaction with the black hole.

Because of the structure of our ansatz, the string, if it is not absorbed for some finite τ , may return to $z = +\infty$ [$\theta_f = 0 \pmod{2\pi}$], where it started at $\tau = -\infty$, or go past the black hole towards $z = -\infty$ [$\theta_f = \pi \pmod{2\pi}$].

A special case of interest is that of solutions such that

$r(\tau) = r(-\tau)$. It follows that $\theta^2(\tau) = \theta^2(-\tau)$, from which $\theta(\tau) = \Delta - \epsilon\theta(-\tau)$, with ϵ a sign. We then see that Δ is restricted to multiples of π if the string does not fall into the black hole. If it is an odd multiple, we understand that the string has circled around the black hole a number of times and then has continued to infinity, whereas when it is an even multiple, the string bounces back after some dithering around the black hole. This analysis can be extended to all solutions.

Let us now analyze the absorption of the string by the black hole. If r starts at $+\infty$ for $\tau = -\infty$ and decreases ($\dot{r} < 0$), \dot{r} must change sign at the periastron at time τ_0 . Otherwise the singularity at $r = 0$ will be reached. Furthermore, $\ddot{r}(\tau_0) > 0$. We see from the first equation in the set (4.16) that this implies that $r(\tau_0) > 3/2$. In other words, if the string penetrates the $r < 3/2$ region, it will necessarily fall into the singularity. In yet another paraphrase, there is an effective horizon for ring string solutions. The surface $r = 3/2$ is necessarily contained within this horizon. Let us recall that for massless geodesics the effective horizon is a sphere of radius $r = \frac{3}{2}\sqrt{3}$.

To illustrate these points, we adjunct some figures. They depict the motion of ringlike strings described by Eqs. (4.15)–(4.18). We numerically integrate Eqs. (4.16) from large negative τ , where the asymptotic behavior (4.24) holds. We choose $\theta_0, n = 1$, and vary the values of p, m and φ_0 . Depending on this last set of three values the string is absorbed or not by the black hole.

In Fig. 12 we show an example of direct fall (i.e., with no bobbing around the black hole). In order to compare with this one, we next portray (Fig. 13) a case where the string goes past the black hole before returning to it and collapsing. The clearest view of this event is given by Fig. 13(d), which depicts $z = r \cos\theta$ as a function of $\rho = r \sin\theta$.

Figure 14 is dedicated to a nonfalling string. It is particularly interesting to point out that the excitation state has been changed by scattering by the black hole, as can be clearly seen from the third graph in this figure, which depicts $r \sin\theta$ as a function of τ . We see that the oscillation amplitude is larger after the collision than before. This means that the outgoing string mass is larger than the ingoing string mass. Hence, particle transmutation in the sense of Ref. [14] takes place here.

In the fourth of this series, Fig. 14(d), we portray $z = r \cos\theta$ against $\rho = r \sin\theta$. It is to be remarked that the string bounces (the lower end of the picture), then oscillates around the black hole, and finally escapes to infinity.

That the string be absorbed or not by the black hole is dictated by whether it comes or not within the effective horizon, as mentioned above. This, in turn, is crucially dependent on the phase φ_0 chosen as part of the initial data ($\tau \rightarrow -\infty$). Whatever value the mass (amplitude) m and the momentum p take, there is always some interval of values of φ_0 for which the string will be absorbed by the black hole.

In addition to numerical experiments, this behavior follows from the simple fact that a change of the initial phase φ_0 would displace the string worldsheet, thus possibly bringing it closer to the black hole.

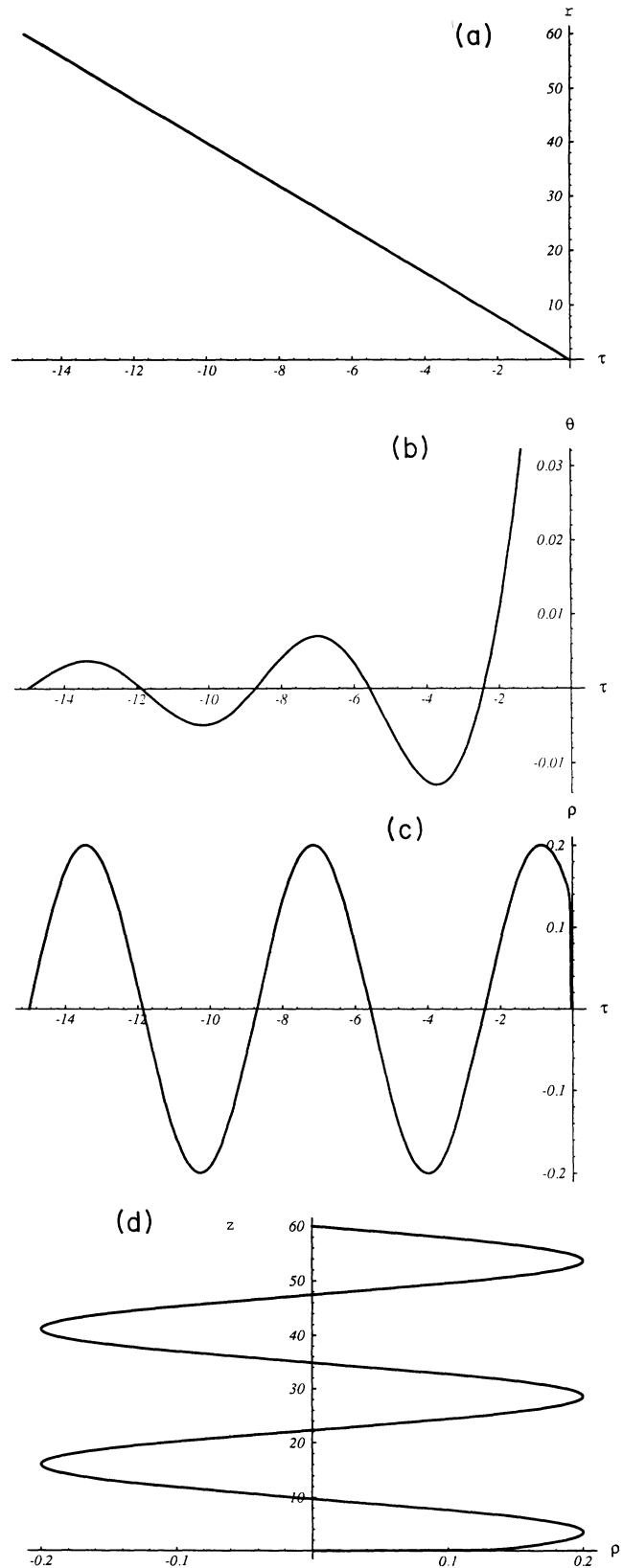


FIG. 12. Numerical solution for the equations of motion of a string in a Schwarzschild black hole background: The string falls into the black hole.

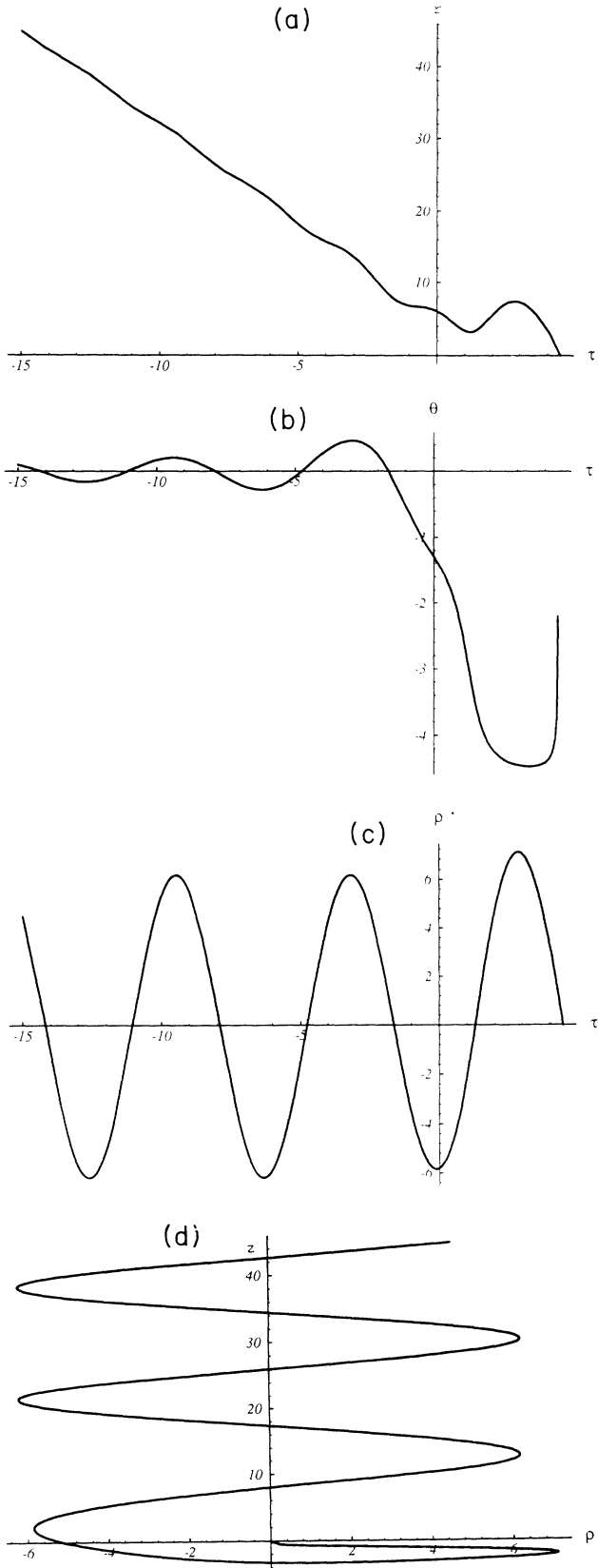


FIG. 13. Numerical solution for the equations of motion of a string in a Schwarzschild black hole background: The string falls into the black hole, but only after first going past it and then back into the singularity.

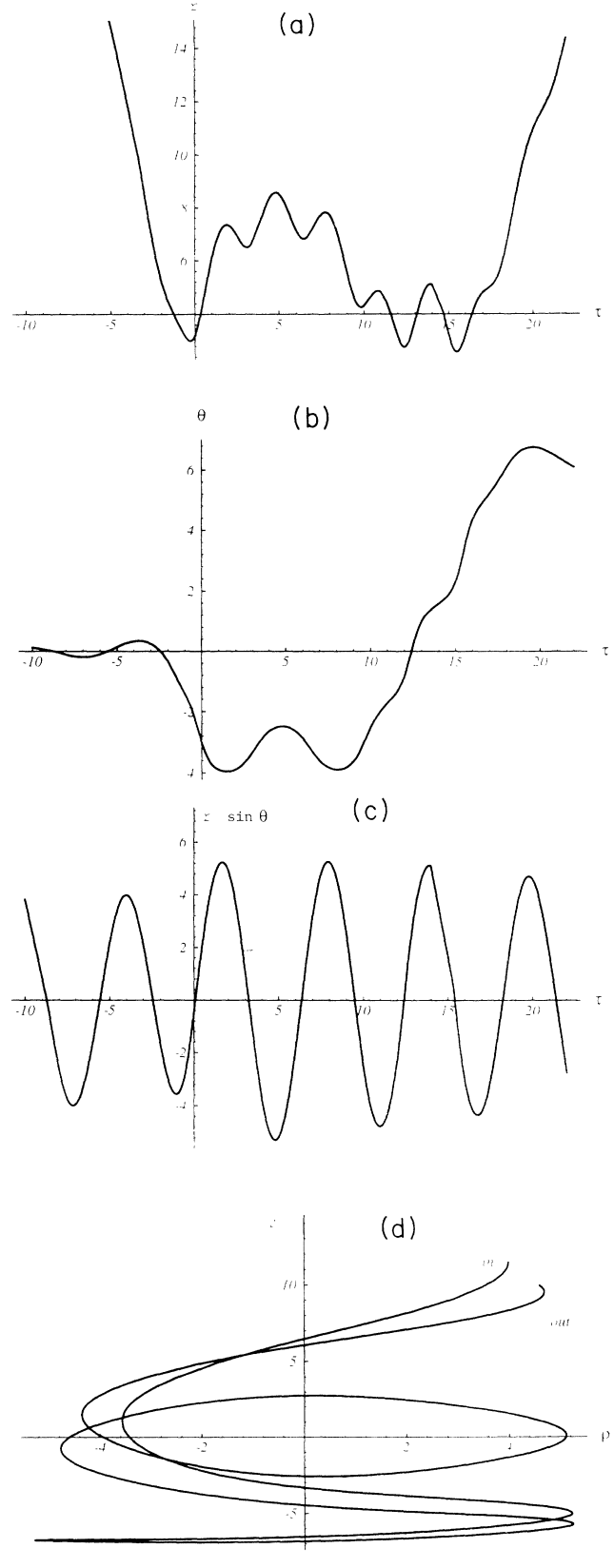


FIG. 14. Numerical solution for the equations of motion of a string in a Schwarzschild black hole background: The string goes past the black hole, circles round it, and then bounces back with a change in its amplitude and momentum.

V. STRINGS PROPAGATING IN STRINGY BLACK HOLES

Among the tree-level string vacua of this type we shall only consider nonrotating, (3+1)-dimensional black hole spacetimes of the type first presented in [30].

Before coming to those, though, let us write the equations of motion corresponding to a ring configuration of a classical string in a metric of the form (in Einstein's frame)

$$ds_E^2 = -A(r)dt^2 + \frac{1}{B(r)}dr^2 + C(r)(d\theta^2 + \sin^2\theta d\phi^2). \quad (5.1)$$

Our ansatz (4.15) can be used for this general metric. The equations of motion and constraint are

$$\ddot{r} = \frac{1}{2} \left[\frac{AB' - BA'}{AB} \dot{r}^2 + BC \left[\frac{C'}{C} - \frac{A'}{A} \right] \dot{\theta}^2 - n^2 BC \sin^2\theta \left[\frac{C'}{C} + \frac{A'}{A} \right] \right], \quad (5.2)$$

$$\ddot{\theta} = -\frac{C'}{C} \dot{r} \dot{\theta} - n^2 \sin\theta \cos\theta,$$

$$\dot{t} = \frac{e}{A},$$

$$e^2 = \frac{A}{B} \dot{r}^2 + AC \dot{\theta}^2 + ACn^2 \sin^2\theta.$$

Here e is a constant of motion and the primes denote derivatives with respect to r for A , B , and C , which are functions of r . The string size squared is in this case

$$S(\tau)^2 = C(r)n^2 \sin^2\theta. \quad (5.3)$$

As in Sec. IV [Eq. (4.19)], the string energy is given here by $E(t) = e/\alpha'$.

More specifically, let us consider the following set of metrics [30]:

$$A(r) = B(r) = \frac{(r-r_+)(r-r_-)}{r^2 - r_0^2}, \quad (5.4)$$

$$C(r) = r^2 - r_0^2.$$

This corresponds to the presence of an electric and a magnetic charge, to which the string does not couple save for their effect on the metric. The parameters of the metric are derived from the mass M and the charges Q_E (electric) and Q_M (magnetic) as follows:

$$r_0 = \frac{Q_M^2 - Q_E^2}{2M}, \quad r_{\pm} = M \pm (M^2 + r_0^2 - Q_E^2 - Q_M^2)^{1/2}. \quad (5.5)$$

If $Q_E^2 > Q_M^2$, the string coupling constant goes to zero close to the singularity $r = |r_0|$, and we would thus be justified in considering this a good approximation to an exact string vacuum. Remember that all these metrics are solutions to the tree-level effective action.

These metrics are globally very similar to a Reissner-

Nordström (RN) metric when Q_E and Q_M are both different from zero. There are two horizons: an event horizon at r_+ and an inner horizon at r_- ; and a singularity at $r = |r_0|$. As a matter of fact, RN appears as the special case $Q_E = Q_M$.

A very interesting phenomenon appears for these spacetimes for the equatorial motion of a ringlike string. We obtain the following exact solution to the equations of motion:

$$\begin{aligned} \phi &= n\sigma, \quad \theta = \pi/2, \quad \dot{t} = e/A(r(\tau)), \\ r(\tau) &= M + a \cos(n\tau + \varphi), \end{aligned} \quad (5.6)$$

with the following constraints on a , coming from the positivity of $e^2 = a^2 + Q_E^2 + Q_M^2 - M^2 - r_0^2$, and from imposing that $r(\tau) \geq |r_0|$:

$$M - |r_0| \geq |a| \geq (M^2 + r_0^2 - Q_E^2 - Q_M^2)^{1/2}.$$

From these constraints it follows that (a) the periastron lies *inside* the inner horizon: $r_{\min} \leq r_-$; (b) the apoastron lies *outside* the event horizon: $r_{\max} \geq r_+$. (c) these last two formulas are only saturated when $a = (M^2 + r_0^2 - Q_E^2 - Q_M^2)^{1/2}$. We are thus presented with the situation that a classical string enters the nonstatic region, goes on to face the singularity, and without falling into it bounces back out again. This looks clearly analogous to a timelike curve that crosses from one asymptotically flat region to another, of those present in the maximal analytic extension of this spacetime. In other words, the string travels to other universes through the wormholes, and continues to do so indefinitely. The extreme case $a = (M^2 + r_0^2 - Q_E^2 - Q_M^2)^{1/2}$ corresponds to $e^2 = 0$, from which it follows that in this case the constant t trajectories within the region $r_- \leq r \leq r_+$ are the ones the string takes, never entering neither the asymptotically flat regions, nor the regions connected with the singularity. The other extreme case, $a = M - |r_0|$, corresponds to a string that reaches the singularity.

Let us now examine the asymptotic behavior of ringlike configurations of classical strings far away from the black hole and very close to it. Far away from it, the metric is Minkowski's up to $1/r$ terms. Therefore the asymptotic behavior of ring solutions is the same as for Schwarzschild's background, formula (4.24). More interesting is the possibility of collapse onto the singularity. Whenever r_+ and r_- are real, as we are supposing all along, it will always be true that $(|r_0| - r_+)(|r_0| - r_-) \geq 0$. This inequality will only be saturated when $Q_E = Q_M = 0$, which would place us in the already analyzed Schwarzschild case. Therefore, consider $(|r_0| - r_+)(|r_0| - r_-) > 0$. We find that the string generically cannot reach the singularity $r = |r_0|$. Assume that it does reach it, say, for $\tau \rightarrow 0$. It follows from the equations of motion

$$\begin{aligned} \dot{r} &= \left[\frac{2}{r}(r-r_+)(r-r_-) - r + \frac{1}{2}(r_+ + r_-) \right] \dot{\theta}^2 \\ &\quad - \left[r - \frac{1}{2}(r_+ + r_-) \right] n^2 \sin^2 \theta, \\ \ddot{\theta} &= -\frac{2r}{r^2 - r_0^2} \dot{r} \dot{\theta} - n^2 \sin \theta \cos \theta, \end{aligned} \quad (5.7)$$

that the generic behavior is

$$\begin{aligned} r &\sim |r_0| + c\tau^\alpha, \\ \theta &\sim \theta_0 + b\tau^\beta, \end{aligned} \quad (5.8)$$

where $\beta = 1 - \alpha$ (or $\beta = 1 - 2\alpha$ for RN). There is another possibility, with $\alpha = \beta = 2$. Now, from the energy squared,

$$e^2 = \dot{r}^2 + (r - r_+)(r - r_-)(\dot{\theta}^2 + \sin^2 \theta), \quad (5.9)$$

and the positivity of $(|r_0| - r_+)(|r_0| - r_-)$, we see that α and β are necessarily greater or equal than 1, thus excluding the generic behavior $\alpha + \beta = 1$ (or $2\alpha + \beta = 1$ for RN). We stress that collapse is indeed possible, as has been seen in the exact solution (5.6) presented above for $a = M - |\eta_0|$, but a very fine tuning of the parameters of the infalling string is required to avoid the repulsion of the singularity.

It is important to observe that the most interesting effects observed in the propagation of classical strings within our approach appear already for solutions of Einstein-Maxwell theory, without having to investigate tree-level string vacua.

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