The inflationary energy scale

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The energy scale of inflation is of much interest, as it suggests the scale of grand unified physics, governs whether cosmological events such as topological defect formation can occur after inflation, and also determines the amplitude of gravitational waves which may be detectable using interferometers. The COBE results are used to limit the energy scale of inflation at the time large scale perturbations were imprinted. An exact dynamical treatment based on the Hamilton-Jacobi equations is then used to translate this into limits on the energy scale at the end of inflation. General constraints are given, and then tighter constraints based on physically motivated assumptions regarding the allowed forms of density perturbation and gravitational wave spectra. These are also compared with the values of familiar models.

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I. INTRODUCTION

Limits on the energy density at which cosmological inflation [1] takes place are of great interest, being a prime example of a situation where cosmological observations might provide information regarding the correct physics at energies completely inaccessible by terrestrial means. An accurate estimate of the inflationary energy scale may provide vital information concerning the scale of unification for gauge interactions, for example. The energy scale is also of interest for cosmological reasons; for instance, one is interested to know whether or not the inflationary energy scale is so low as to forbid the formation after inflation of topological defects that might be of interest for structure formation [2]. The inflationary scale also determines whether or not topological defects can quantum mechanically nucleate during inflation [3]. The amplitude of gravitational waves produced during inflation [4], which may be detectable in our own solar system using interferometers, is also given directly by the evolution of the energy scale during the last stages of inflation.

The usual goal of studies such as this is to provide upper limits to the inflationary energy scale. As we shall see, lower limits are much harder to come by. Studies made in the 1980s and early 1990s [5, 6] typically made some simple modeling of inflation, and then imposed what was at that time the current upper limits on microwave fluctuations. (If further assumptions such as the existence of an axion field were made, then other constraints could be brought to bear [7].) Conceptually simpler but in general weaker limits can be obtained by considering only the effect of gravitational wave modes [4, 8]. The measurement of large scale microwave background temperature fluctuations by the Cosmic Background Explorer (COBE) Differential Microwave Radiometer (DMR) instrument [9] now allows one to be much more definite, in any given model replacing an upper limit with a definite value (and uncertainty). The understanding of the influence of gravitational wave modes excited during inflation on the microwave background [4] has also advanced considerably recently [10–12]. The detection of such modes would provide vital information as regards bounding the energy scale from below, as we shall discuss.

The discussion here is focused on inflationary models which utilize a single rolling scalar field; that is, chaotic inflation [13] in its loosest sense. In such models it is usual to assume that one can choose the potential of this field as one likes. The results derived here are rigidly true only in this case. However, they also hold in models with multiple rolling scalar fields, provided that the fluctuations in field directions orthogonal to the classical trajectory are small; indeed, as these fluctuations would inevitably reduce the allowed energy scale by soaking up some of the COBE anisotropy, any upper bounds derived here continue to be true in this case. They do not directly apply to models which rely on scalar fields trapped in metastable vacuum states, though even there one can usually, as with extended inflation [14], rephrase this situation in terms of a rolling field.

There are typically two steps in finding the inflationary energy scale. The first is to limit the energy density at the time when fluctuations observable in the microwave background were generated. This occurs as those scales crossed outside the Hubble radius during inflation, typically when the scale factor was around e^{-60} of its size at the end of inflation (normally referred to as 60 e-foldings from the end of inflation). Here we aim to provide a much more general treatment than before, utilizing results to first order in the slow-roll approximation, thus incorporating both the predicted tilt [15] in the density perturbation (scalar) spectrum from inflation and also including the effect of gravitational wave (tensor) modes with their characteristic scale dependence. This enables accurate bounds to be placed on the energy density 60 e-foldings from the end of inflation.

The second aspect of finding the inflationary energy scale is to use the limit 60 e-foldings from the end, and evolve the system so as to provide a limit on the energy scale at the end of inflation. In the past this has been

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accomplished by using the slow-roll approximation; however, inflation can only end if the slow-roll approximation breaks down, and so such approaches are necessarily inaccurate. In this paper, the equations are written in Hamilton-Jacobi form [16], which allows the inflationary dynamics to be treated *exactly*. This involves treating the Hubble parameter, which directly measures the energy density, as a function of the inflaton field ϕ.

There is a disadvantage to this procedure in that one is removing the discussion from the potential, $V(\phi)$, which is a quantity about which one believes one has an intuitive feel, and focusing it on the Hubble parameter which in specific models is supposed to be a derived quantity. However, the approach offers several compensations. One compensation is simply the ability to make analytic progress. One should also bear in mind that many of the motivations for being interested in the energy scale, such as the reheat temperature and the level of gravitational wave production, are directly connected to the total energy density as measured by H rather than to the potential alone. Further, one can utilize the results here using a potential provided one is willing to utilize the slow-roll approximation throughout, as in previous treatments, simply by neglecting the kinetic contribution to the Friedmann equation so that H^2 is just proportional to V. By contrast, a treatment based on the potential offers no opportunity to move to the more accurate results as offered here. Whether one wishes to use the potential may depend on circumstance; in particular for the normalization at 60 e-foldings, the slow-roll approximation is very accurate in most known models. However, the slow-roll approximation cannot possibly be good near the end of inflation, and so the more powerful method based on the Hubble parameter may be of much more use there.

The outline of this paper is as follows. In Sec. II, the equations are set up in Hamilton-Jacobi form. A new proposal is then implemented for the specification of inflationary models, where rather than specifying them by a potential $V(\phi)$, they are instead specified by a function $\epsilon(\phi)$ which measures how accurately the slow-roll approximation holds as a function of scalar field value. It is emphasized that this classification covers all inflationary models involving rolling fields, and that the dynamics are treated exactly, *not* subject to any form of slow-roll approximation. Section III discusses the generation of perturbation spectra by a given inflationary model, and uses this to bound from above the inflationary energy scale 60 e-foldings from the end of inflation. Section IV takes advantage of the Hamilton-Jacobi formalism to produce limits on the energy scale at the end of inflation, both under very general circumstances and more restrictively by imposing physically motivated constraints on the form of perturbation spectra produced. Section V provides the conclusions, and also discusses the possibility of producing lower bounds on the energy scale.

II. INFLATIONARY DYNAMICS

The Hamilton-Jacobi equations arise when one rewrites the equations of motion in a way that allows one to write the Hubble parameter as a function of the scalar field ϕ. The usual equations of motion are

$$H^2 = \frac{8\pi}{3m_{\rm Pl}^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \qquad (1)$$

$$\ddot{\phi} + 3H\dot{\phi} = -V'(\phi), \qquad (2)$$

with $H = \dot{a}/a$ the Hubble parameter, a the scale factor, $m_{\rm Pl}$ the Planck mass, and where as usual dots are time derivatives and primes derivatives with respect to the scalar field ϕ. Differentiating the first with respect to t and using the second gives

$$2\dot{H} = -\frac{8\pi}{m_{\rm Pl}^2}\dot{\phi}^2. \qquad (3)$$

We assume that $\dot{\phi}$ never passes through zero during inflation, allowing us to use ϕ as a time variable.[1] We may therefore divide each side by it and eliminate the time dependence in the Friedmann equation, obtaining the Hamilton-Jacobi equations [16]

$$(H')^2 - \frac{12\pi}{m_{\rm Pl}^2}H^2 = -\frac{32\pi^2}{m_{\rm Pl}^4}V(\phi), \qquad (4)$$

$$\dot{\phi} = -\frac{m_{\rm Pl}^2}{4\pi}H'. \qquad (5)$$

Without loss of generality, we shall throughout make the choice that $\dot{\phi} > 0$. With the equations in this form, it is natural to think of specifying inflationary models by a choice of $H(\phi)$ rather than $V(\phi)$ [17]; one can then easily generate a large set of exact inflationary solutions simply by differentiation, whereas a choice of $V(\phi)$ requires the normally impossible task of analytically solving the nonlinear equation (4).

We can now define what we shall refer to as slow-roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ by [11]

$$\epsilon(\phi) = \frac{m_{\rm Pl}^2}{4\pi}\left(\frac{H'}{H}\right)^2, \qquad (6)$$

$$\eta(\phi) = \frac{m_{\rm Pl}^2}{4\pi}\frac{H''}{H} = \epsilon(\phi) - \sqrt{\frac{m_{\rm Pl}^2}{16\pi}}\,\frac{\epsilon'(\phi)}{\sqrt{\epsilon(\phi)}}. \qquad (7)$$

The sign of the last term, like the signs of other equations featuring $\sqrt{\epsilon(\phi)}$ later, is determined from the choice $\dot{\phi} > 0$. Wherever square roots are utilized, it is the positive root that is to be taken, with the overall sign incorporated in the prefactor. These parameters measure how accurate the slow-roll approximation would be at a given value of ϕ; their smallness corresponds, respectively, to

[1] In rolling models, this is always a good assumption while inflation occurs. It can only be violated while inflation is still occurring if the potential has a local minimum with nonzero potential energy, in which case the field will become a trapped one. It will generally be violated *after* inflation ends, with the field oscillating about a minimum with zero potential energy, but our intention here is only to reach to the end of inflation. Assuming inflation ends with the field approaching a minimum with zero potential, inflation always ends before field oscillations begin.

the validity of neglecting the first term in Eq. (4) and the first term of its ϕ derivative. Let us emphasize again though that we will not make a slow-roll approximation in considering the dynamics. Further, $\epsilon(\phi)$ possesses the extremely useful property that the condition for inflation, $\ddot{a} > 0$, is precisely equivalent to $\epsilon(\phi) < 1$.

In this paper, it is convenient to go one small step further than specifying models by $H(\phi)$; instead we shall specify models by choosing $\epsilon(\phi)$. By allowing arbitrary forms of this function, we can specify arbitrary inflationary models just as well as if we were to use $V(\phi)$. Our choice though allows analytic progress without the slow-roll approximation.

The number of e-foldings N between scalar field values ϕ and $\phi_{\rm end}$ (the latter being the scalar field value when inflation ends) is given by

$$N(\phi,\phi_{\rm end}) \equiv \ln\frac{a(\phi_{\rm end})}{a(\phi)} = \sqrt{\frac{4\pi}{m_{\rm Pl}^2}}\int_{\phi}^{\phi_{\rm end}}\frac{1}{\sqrt{\epsilon(\phi)}}\,d\phi\,. \tag{8}$$

Unlike the similar equation often seen featuring V/V', this expression is exact. The end of inflation, when the scale factor stops accelerating, is given precisely by $\epsilon(\phi) = 1$, which determines $\phi_{\rm end}$.[2]

One computes $H(\phi)$ by quadrature from

$$\frac{d\ln H}{d\phi} = -\sqrt{\frac{4\pi\epsilon(\phi)}{m_{\rm Pl}^2}}\,, \tag{9}$$

to get

$$H(\phi) = H_{\rm end}\exp\left(\int_{\phi}^{\phi_{\rm end}}\sqrt{\frac{4\pi\epsilon(\phi)}{m_{\rm Pl}^2}}\,d\phi\right)\,, \tag{10}$$

where $H_{\rm end}$ is of course $H(\phi_{\rm end})$, the Hubble parameter at the end of inflation. The Hubble parameter is a direct measure of the energy scale, and so bounding the energy scale simply amounts to bounding H at different epochs.

The potential which generates the solutions is then

$$V(\phi) = \frac{3m_{\rm Pl}^2}{8\pi}H^2(\phi)\left(1-\frac{\epsilon(\phi)}{3}\right)\,. \tag{11}$$

Whenever slow roll is good (small ϵ) one has $V(\phi) \propto H^2(\phi)$. One can thus generate an endless set of exact solutions from choices of $H(\phi)$, or from $\epsilon(\phi)$ in those cases where the integration giving $H(\phi)$ can be done analytically.

One can use these equations to calculate the density perturbation amplitude $\delta_H(k)$, as formally defined in [18], which to lowest order in slow roll is

$$\delta_H(k) \equiv \left.\frac{H^2(\phi)}{5\pi|\dot{\phi}|}\right|_{aH=k} \tag{12}$$

$$= \left.\frac{2}{5\sqrt{\pi}}\frac{H(\phi)}{m_{\rm Pl}\sqrt{\epsilon(\phi)}}\right|_{aH=k}\,, \tag{13}$$

the right-hand sides being evaluated when the comoving scale k equals the (inverse) Hubble radius during inflation. One can then satisfy the COBE result [9], most conveniently taken to be evaluated 60 e-foldings from the end of inflation. We shall henceforth take δ_H to indicate the amplitude of the spectrum at this time. This fixes $H_{\rm end}$, provided one knows how to incorporate tilt and gravitational wave corrections into the correct normalization of δ_H.

Provided inflation ends at $\epsilon(\phi) = 1$, one then has

$$V_{\rm end} = \frac{m_{\rm Pl}^2}{4\pi}H_{\rm end}^2\,, \tag{14}$$

though to estimate the energy density one should include the kinetic contribution, writing

$$\rho_{\rm end} = \frac{3m_{\rm Pl}^2}{8\pi}H_{\rm end}^2\,. \tag{15}$$

It is best to illustrate this formalism via an example, which corresponds rather closely to the usual polynomial chaotic inflation models [13] with potentials $V(\phi) \propto \phi^{\alpha}$. Let us choose

$$\epsilon(\phi) = \frac{m_{\rm Pl}^2\alpha^2}{16\pi\phi^2} \tag{16}$$

with negative ϕ and α a constant. Inflation ends at $\epsilon(\phi) = 1$, giving $\phi_{\rm end}^2 = \alpha^2 m_{\rm Pl}^2/16\pi$, and we have

$$N(\phi,\phi_{\rm end}) = \frac{4\pi}{\alpha}\frac{\phi^2}{m_{\rm Pl}^2} - \frac{\alpha}{4}\,. \tag{17}$$

Solving, we get

$$H(\phi) = H_{\rm end}\left(\frac{\phi}{\phi_{\rm end}}\right)^{\alpha/2}\,, \tag{18}$$

and so

$$H_{60} = H_{\rm end}\left(1+\frac{240}{\alpha}\right)^{\alpha/4}\,, \tag{19}$$

where H_{60} is the Hubble parameter 60 e-foldings from the end of inflation. Thus

$$\delta_H = \frac{2}{5\sqrt{\pi}}\frac{H_{\rm end}}{m_{\rm Pl}\sqrt{\epsilon_{60}}}\left(1+\frac{240}{\alpha}\right)^{\alpha/4}\,, \tag{20}$$

with

$$\epsilon_{60} = \frac{\alpha}{240+\alpha}\quad,\quad \eta_{60} = \frac{\alpha-2}{240+\alpha}\,. \tag{21}$$

In fact $\alpha = 2$ corresponds to the special case where $\eta(\phi)$ is identically zero for all ϕ.

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[2]This statement is true for all rolling models. If one is considering models which end inflation by an unusual means such as bubble nucleation in a field other than the rolling one (e.g., extended inflation [14]), this provides an exception and $\phi_{\rm end}$ must be determined via the physics of the nucleation process. In such cases ϵ may be less than unity at the end of inflation, though one could imagine that it had increased extremely rapidly to unity.

In the following section, we shall see that for models with small ϵ_{60} and η_{60} such as these, the appropriate δ_H to explain the COBE result is 1.7×10^{-5} [18]. Consequently, one has

$$\frac{H_{\rm end}}{m_{\rm Pl}} = 7.5\times10^{-5}\left(1+\frac{240}{\alpha}\right)^{-\frac{2+\alpha}{4}} \quad (22)$$

$$= \begin{cases} 6.2\times10^{-7} & \text{for } \alpha=2\,, \\ 1.6\times10^{-7} & \text{for } \alpha=4\,. \end{cases} \quad (23)$$

The potential supplying this $\epsilon(\phi)$ is

$$V(\phi) = \frac{3m_{\rm Pl}^2}{8\pi}H_{\rm end}^2\left(1-\frac{m_{\rm Pl}^2\alpha^2}{48\pi\phi^2}\right)\left(\frac{\phi}{\phi_{\rm end}}\right)^{\alpha}, \quad (24)$$

confirming that in the slow-roll limit we just get the polynomial potentials of the simplest chaotic inflation models. The analytic solution requires that the potential has the extra ϕ-dependent correction term which makes the solutions exact. A suitable adjustment of the original $\epsilon(\phi)$ can be used to give exactly $V(\phi)\propto\phi^\alpha$, though it cannot be written analytically.

III. THE PERTURBATION SPECTRA, AND LIMITING H_{60}

Returning to the general case, we now need to examine in detail what the COBE normalization means. In the last section, we mentioned the fiducial normalization $\delta_H = 1.7\times10^{-5}$ which is correct only for sufficiently flat scalar spectra with negligible gravitational waves. This is appropriate only if the slow-roll parameters $\epsilon(\phi)$ and $\eta(\phi)$ are small at the time the relevant scales leave the horizon. By utilizing standard results [18], we can improve this to incorporate the first level of slow-roll corrections, a treatment which is adequate for all models which appear viable when confronted with the full range of large scale structure observations [18]. To this order one can also use the quantities V_{60}, V'_{60}, and V''_{60} in the analysis, but we shall retain our focus on the Hubble parameter.

In the spirit of the above, we shall assume that the scales corresponding to quadrupole anisotropies passed out of the horizon 60 e-foldings from the end of inflation,[3] and that across the scales which contribute significantly to the COBE observation (which are only a few e-foldings) the spectral indices of the scalar and tensor modes can be treated as scale independent (that is, the spectra are approximated by power laws). It is then easy to show [11] that the spectral indices are given from the slow-roll parameters at that time as[4]

$$n_{60}^S = 1-4\epsilon_{60}+2\eta_{60} \quad (25)$$

$$= 1-2\epsilon_{60}+\sqrt{\frac{m_{\rm Pl}^2}{4\pi}}\frac{\epsilon'_{60}}{\sqrt{\epsilon_{60}}}\,, \quad (26)$$

$$n_{60}^T = -2\epsilon_{60}\,. \quad (27)$$

In addition to this, we need to know the contributions of the scalars and tensors to the microwave anisotropies. As usual, the fractional temperature anisotropy is split into multipoles (with the monopole and dipole removed)

$$\frac{\Delta T}{T}(\theta,\phi) = \sum_{l,m} a_{lm}Y_m^l(\theta,\phi)\,. \quad (28)$$

Inflation predicts the (rotationally invariant) expectation of these multipoles, $\Sigma_l^2 = \langle|a_{lm}|^2\rangle$, where the average is over all possible observer points.

The scalar amplitude from an inflation model can be calculated analytically for power-law spectra, giving [18]

$$\Sigma_l^2({\rm scalar}) = \frac{\pi}{2}\left[\frac{\sqrt{\pi}}{2}l(l+1)\frac{\Gamma(1+2\epsilon-\eta)\,\Gamma(l-2\epsilon+\eta)}{\Gamma(3/2+2\epsilon-\eta)\,\Gamma(l+2+2\epsilon-\eta)}\right]\frac{\delta_H^2}{l(l+1)}\,, \quad (29)$$

where, for *every* multipole, δ_H is evaluated at the scale $H_0/2$ corresponding to the quadrupole. As we are assuming this scale leaves the horizon 60 e-foldings from the end, we have

$$\delta_H^2 = \frac{4}{25\pi}\frac{H_{60}^2}{m_{\rm Pl}^2\epsilon_{60}}\,. \quad (30)$$

For the gravitational wave spectrum the general case involves a messy double integration which must be carried out numerically. To first order in slow roll we can evade this by using an approximation due to Lucchin, Matarrese, and Mollerach [12], which shows that if the scalar and tensor power-law indices satisfy $n_{60}^T = n_{60}^S - 1$ (equivalently $\epsilon'_{60}=0$), thus giving power-law inflation, then to a good approximation the contributions of scalars and tensors to the microwave multipoles remain in fixed proportion, that proportion being given by $25\epsilon_{60}/2$. The gravitational wave multipoles can therefore be generated using the scalar result, but with the spectral index $1-2\epsilon_{60}$ rather than the true scalar index. The expectations then add in quadrature to give the total Σ_l^2.

Finally, one must calculate the prediction for the COBE 10° result. The 10° variance σ_{10}^2 is given by a weighted sum over the multipole expectations, where the weighting function F_l corresponds to the beam profile of the experiment. That is, one writes

[3]Note that the normalization of the energy scale at the time the quadrupole scale crossed the Hubble radius does *not* depend on the choice for this number; the normalization depends only on the local physics at the time the anisotropies were generated. However, this number (which depends weakly on the physics of reheating) does crop up in the extrapolation to the end of inflation, as it must. Even there though, its effect is small, as we shall see in the next section.

[4]These are correct to first order in the slow-roll parameters. Stewart and Lyth [19] have provided expressions correct to second order, these corrections normally being small. We shall not utilize these here. Note that the numerical factors are different from those in [11], due to a slightly different definition of the slow-roll parameters.

$$\sigma_{10}^2 = \frac{1}{2\pi}\sum_l (2l+1)\Sigma_l^2 F_l \,, \tag{31}$$

and the COBE weight function is

$$F_l = \frac{1}{2}\exp\left[-\left(\frac{l+1/2}{13.5}\right)^2\right] . \tag{32}$$

The procedure is now clear. A given model makes a prediction for ϵ_{60} and η_{60}. In all the above expressions, we can pull out the dependence on H_{60} as a prefactor, and so obtaining the correct normalization determines H_{60} as a function of ϵ_{60} and η_{60}, and hence directly gives the energy scale at that stage of inflation.

Throughout we quote figures based on the COBE result $\sigma_{10} = 1.1 \times 10^{-5}$ [9]. One has that $H_{60} \propto \sigma_{10}$, and so if this result is revised one can just scale the results; $H_{60} \to H_{60}(\sigma_{10}/1.1 \times 10^{-5})$ and $V_{60}^{1/4} \to V_{60}^{1/4}\sqrt{(\sigma_{10}/1.1 \times 10^{-5})}$. This simple scaling also applies to the values at the end of inflation. If one is merely interested in upper limits, then one chooses one's preferred upper limit on σ_{10}; at present one might advocate the COBE 2σ upper limit of 1.5×10^{-5}, and thus say that the inflation scales $V^{1/4}$ are constrained to be no more than $\sqrt{15/11}$ of the values quoted here.

One should also remember the possibility of cosmic variance — that a single observer may see a different 10° variance than the ensemble average as calculated above. For the COBE observation, the cosmic variance introduces an uncertainty of about 10% [18] (weakly dependent on the spectral indices) which is negligible as compared to the present observational errors when added in quadrature.

It is unfortunate, but perhaps unsurprising, that the largest values of H_{60} correspond to large departures from the slow-roll regime, and hence stretch the validity of approximations used. (We shall see later though that this is considerably less of a problem with $H_{\rm end}$.) It is therefore necessary in bounding H_{60} to impose some constraints of physical reasonableness. The choice made here is to assume that the scalar spectral index lies between about 0.5 and 1.5, as indicated at the 1σ level by COBE [9] (though cases with large gravitational wave contributions will invalidate their analysis on the scalar index). This therefore requires $-0.5 \le 4\epsilon_{60} - 2\eta_{60} \le 0.5$. Note that although this is in principle only a 1σ bound, all inflation-based models with any chance of satisfying large scale structure data are well within this band [18], so in fact the limits derived here are almost certainly rather conservative.

We also impose the additional restriction that ϵ_{60} and $|\eta_{60}|$ do not exceed 0.25. The largest values of H_{60} do occur outside this regime, but some limit must be imposed to assure that the approximations used do not lose their validity. As we shall see, large values of the slow-roll parameters are normally not compatible with the requirement that there be 60 e-foldings of inflation to follow, and also these restrictions are not important in bounding $H_{\rm end}$.

The general trends are illustrated in Fig. 1. In particular one notices the following properties.

(1) If one fixes a small ϵ_{60}, to exclude gravitational waves, and varies the tilt using η_{60}, then one finds a larger δ_H required as η_{60} is made negative, tilting the spectrum to remove short scale power. The effect on H_{60} is rather modest, however.

(2) At fixed tilt ($2\epsilon_{60} - \eta_{60} = \text{const}$), δ_H of course gets smaller as ϵ_{60} is increased introducing gravitational waves. However, in determining H_{60} the increasing ϵ_{60} is a more important effect (recall $H_{60} \propto \delta_H\sqrt{\epsilon_{60}}$) and the energy scale H_{60} is increased as ϵ_{60} and η_{60} are increased in concert.

(3) The standard normalization $\delta_H = 1.7 \times 10^{-5}$ is accurately achieved only in a small region about $\epsilon_{60}, |\eta_{60}| \simeq 0$. Increasing ϵ_{60} at fixed η_{60} decreases it, as does increasing η_{60} at fixed ϵ_{60}.

(4) The energy scale drops dramatically as $\epsilon_{60} \to 0$, in accord with Eq. (30). Physically, small ϵ_{60} corresponds to a very flat potential and hence very slow classical evolution of ϕ. To keep quantum effects small relative to the

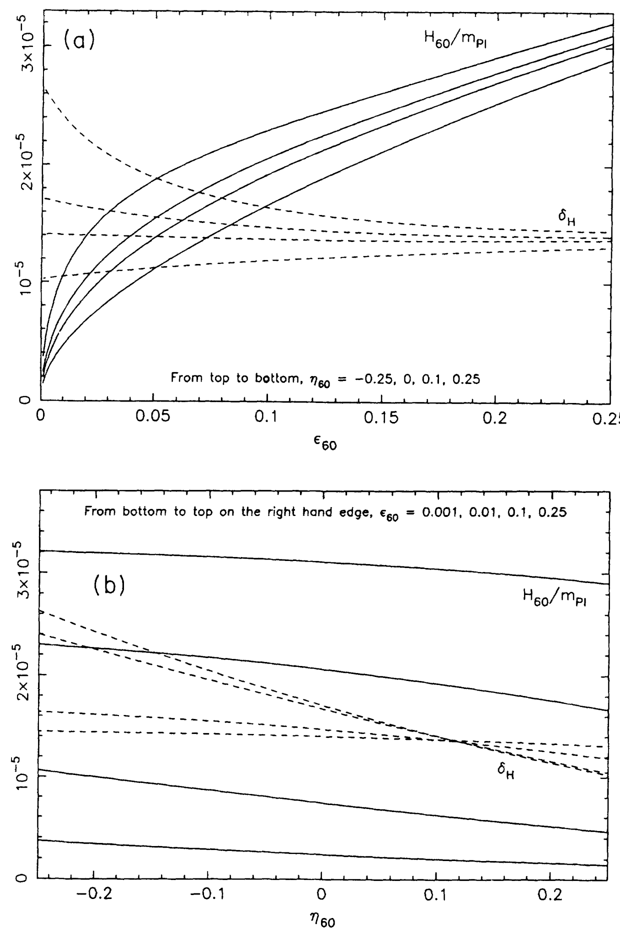

FIG. 1. (a) and (b) These figures give the COBE normalization of H_{60} (solid lines) and the corresponding scalar spectrum normalization δ_H (dashed lines), as a function of ϵ_{60} and η_{60} (the numerical values being conveniently close to each other). These include both tilted spectrum and gravitational wave corrections, as described in text. (a) plots them vs ϵ_{60} for various fixed η_{60}, and (b) vs η_{60} at various fixed ϵ_{60}. The trends are summarized in the text.

classical behavior therefore requires a much lower energy scale.

In the light of this, the largest values of H_{60} come from choosing the largest ϵ_{60} and η_{60} consistent with the assumptions being made (trying a large negative η_{60} falls foul of the tilt bound). The maximum value found is $H_{60} = 2.9\times 10^{-5} m_{\rm Pl}$, corresponding to a potential energy at that time of $V_{60}^{1/4} = 3.8 \times 10^{16}$ GeV, for the values $\epsilon_{60} = \eta_{60} = 0.25$.

IV. FROM H_{60} TO $H_{\rm end}$

The COBE normalization gives us specific information about H_{60}, dependent only on ϵ_{60} and its derivative. To limit the energy at the end of inflation requires one to evolve the system to $\phi_{\rm end}$. At this point, we remind the reader that inflation can end in two distinct ways.

(1) In most slow-rolling models, inflation ends because $\epsilon(\phi)$ grows to equal unity.

(2) In certain models such as power-law [20] and intermediate [21] inflation, $\epsilon(\phi)$ never reaches unity in the basic models, threatening eternal inflation. One escape route often postulated is that the form of the potential is modified to allow $\epsilon(\phi)$ to increase, which brings us back to case (1). However, an alternative is that a new mechanism intervenes to end inflation. The key example is extended inflation [14], which looks like power-law inflation in the Einstein conformal frame, but is brought to an end by the tunneling of another field with $\epsilon(\phi)$ still small.

We shall largely be concerned with the first, more common case. However, the results are typically also applicable in the second, as noted below.

The key constraint is that 60 e-foldings remain, which means that $\epsilon(\phi)$ must satisfy the integral constraint

$$\frac{60}{\sqrt{4\pi}} = \int_{\phi_{60}}^{\phi_{\rm end}} \frac{1}{\sqrt{\epsilon(\phi)}} \frac{d\phi}{m_{\rm Pl}} . \tag{33}$$

At the same time, we can write

$$\frac{H_{\rm end}}{H_{60}} = \exp\left(-\sqrt{4\pi} \int_{\phi_{60}}^{\phi_{\rm end}} \sqrt{\epsilon(\phi)} \frac{d\phi}{m_{\rm Pl}} \right) , \tag{34}$$

where $\epsilon(\phi_{\rm end}) = 1$ and H_{60} is determined from the COBE normalization for the given ϵ_{60} and η_{60}.

This is most conveniently represented graphically, as in Fig. 2, by plotting $1/\sqrt{\epsilon(\phi)}$ against $\phi/m_{\rm Pl}$. Equation (33) then gives the required area under the curve between the initial value and $\epsilon(\phi)$ reaching unity. The area under the curve of $\sqrt{\epsilon(\phi)}$ subject to this constraint measures the reduction of $H_{\rm end}$ relative to H_{60}; in bounding $H_{\rm end}$ from above one's aim is to minimize this reduction.

We can see that again constraints of physical reasonableness must be applied in order to gain worthwhile results. This is because one can always choose $\epsilon(\phi)$ so as to bring $H_{\rm end}$ close to H_{60}. This is done as follows; very rapidly *decrease* $\epsilon(\phi)$ until it is arbitrarily close to zero, keep it there until 60 e-foldings have passed, and then immediately increase it to unity. This corresponds to the bizarre situation of the field being fired *up* the potential, as we are keeping the total energy density fixed while converting the initial kinetic component into potential energy. Of course, this choice is dubious on physical grounds, and will also violate the initial assumption that the spectra are power laws on which the calculation of H_{60} was based, so this should only be taken as illustrating a general point that by sufficient contrivance $H_{\rm end}$ can always be placed near H_{60}.

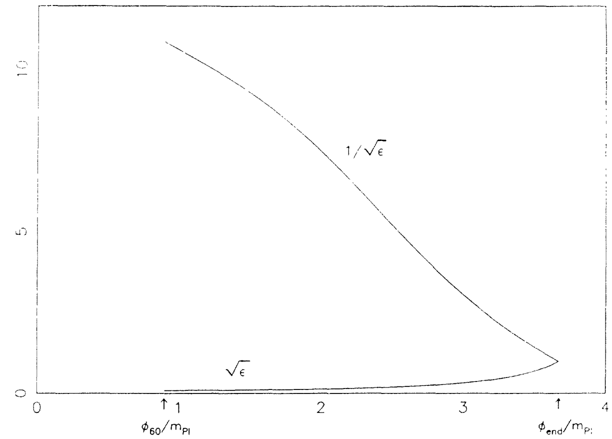

FIG. 2. A graphical illustration of Eqs. (33) and (34). The value of ϕ can be shifted by the addition of an arbitrary constant. At ϕ_{60} one has some particular values for ϵ_{60} and ϵ'_{60}. As ϕ increases, $1/\sqrt{\epsilon(\phi)}$ must vary in such a way that the area under it reaches $60/\sqrt{4\pi}$ just as $\epsilon(\phi)$ reaches unity to end inflation (in general it need not do so monotonically as illustrated here). At the same time, the area under the curve $\sqrt{\epsilon(\phi)}$ measures the decrease of H relative to H_{60} in accord with Eq. (34). It is clear that the smallest decrease in H during the last 60 e-foldings is achieved by keeping $\epsilon(\phi)$ as small as possible for as long as possible.

[Perhaps more reasonable is to imagine the potential suddenly flattening out, so that the kinetic energy term is left simply to redshift to zero. This would simply reduce the Hubble parameter by a factor $(1 - \epsilon_{60}/3)^{1/2}$. However such circumstances are not favorable for ending inflation, and indeed one may not even be able to redshift the kinetic term before 60 e-foldings have passed.]

Let us therefore impose constraints intended to be physically "reasonable" on the form of $\epsilon(\phi)$. The motivation here lies in assuming that the form is functionally simple, motivated by the notion that were it not, then the inferred potential would also be functionally complex, undermining one's prejudice that it is the potential which belongs to a simple underlying theory.

Case A: $\epsilon(\phi)$ is monotonic. This follows a suggestion by Lyth [6], though he employed a slow-roll approximation. As ϵ must ultimately increase to unity, and given that the last 60 e-foldings sample only a limited part of the overall potential, this appears physically well motivated.[5] Without paying too much attention at this point to ϵ'_{60} beyond noting that $\epsilon' > 0 \Leftrightarrow \epsilon > \eta$, we can see that the largest final energy density will be generated if one keeps $\epsilon(\phi)$ at the constant value ϵ_{60} until 60 e-foldings pass, and then as before increase it suddenly up to unity. Note that although this would again require an unusual potential in the single-field case, featuring a very flat plateau followed by a sharp drop, this is in fact exactly what happens in models based on two fields [22, 18], where inflation driven by the first field ends when the second field becomes dynamically unstable. This scenario should therefore certainly not be considered unreasonable.

With these assumptions, it is easy to show that

$$\frac{H_{\rm end}}{H_{60}} = \exp\left(-60\epsilon_{60}\right) . \tag{35}$$

This is a very interesting result, because we recall that it was large values of ϵ_{60} which gave the largest H_{60}, but now we see that such large values have a detrimental effect on the size of $H_{\rm end}$. In fact, as far as large $H_{\rm end}$ is concerned one needs small ϵ_{60}. The largest $H_{\rm end}$ we can achieve is $6.0\times10^{-6}m_{\rm Pl}$ for $\epsilon_{60} \simeq 0.007$, a significant reduction on H_{60}. Further, this is for $\eta_{60} = -0.25$, which is not really consistent with our notion of ϵ' being small. For more realistic values of $\eta_{60} \simeq 0$, the limit strengthens yet further to $H_{\rm end} < 4.1\times10^{-6}m_{\rm Pl}$, with the maximum at $\epsilon_{60} \simeq 0.008$. Indeed, in the small ϵ, η limit this is an analytic result utilizing the fiducial COBE normalization, from the maximization of

$$H_{\rm end}^{\rm max} \simeq 7.5\times10^{-5}\sqrt{\epsilon_{60}}\exp\left(-60\epsilon_{60}\right)m_{\rm Pl} , \tag{36}$$

where "max" indicates that this is the maximum possible $H_{\rm end}$ for a given ϵ_{60}. For the maximizing ϵ_{60}, $H_{\rm end}$ is within a factor 2 of H_{60}.

This is a good point to return to our assumption that the quadrupole scale and the end of inflation are separated by 60 e-foldings. As stated above, the normalization at the quadrupole scale depends only on the physics at that time; the choice of 60 only manifests itself in the scaling to the end of inflation. It is easy to see that if one were to make the choice 50 instead in the above equation, the effect on $H_{\rm end}^{\rm max}$ would be tiny, as the exponential plays a very insignificant role.

Note that because there is no reduction in H during the rapid growth of $\epsilon(\phi)$ to unity once 60 e-foldings have passed, these limits also hold in the case where inflation ends with $\epsilon(\phi)$ still less than one through some additional mechanism, again subject only to the assumption that $\epsilon'(\phi) \geq 0$. It is interesting to note that extended inflation features exactly a constant $\epsilon(\phi)$, and hence amongst the models permitted by the monotonicity assumption it minimizes the reduction of H during the last 60 e-foldings for a given ϵ_{60}.

Recall that this is only subject to the constraint of a monotonic $\epsilon(\phi)$, making no further assumptions as to the form of the inflationary potential or approximations to the inflationary dynamics, and represents a dramatic tightening of the constraints. There is only an extremely weak dependence, contained in the exponential which is of order unity, on the assumption that there are 60 e-foldings between the quadrupole scale leaving the horizon and the end of inflation. Different reheating mechanisms have the power to shift this number by say 10, but this has a negligible impact on the conclusions.

The results illustrate a fundamental point; maximizing H_{60} is not in general the best way to go about maximizing $H_{\rm end}$. Our exact treatment of the last stages of inflation also indicates that the highest-energy densities available at the end of inflation are much lower than the highest-energy densities available at 60 e-foldings.

Case B: $\epsilon(\phi)$ and $\epsilon'(\phi)$ are monotonic. This assumption poses yet tighter constraints. In accord with it, the slowest that $\epsilon(\phi)$ can rise is linearly (from its initial conditions at ϕ_{60}), and it is easy to see that linear growth gives the maximum number of e-foldings that could occur. By solving the appropriate equations, we can get an upper limit on the number of e-foldings such a linear extrapolation would give, as

$$N_{\rm linear} < \frac{1}{(\epsilon_{60}-\eta_{60})\sqrt{\epsilon_{60}}} . \tag{37}$$

If $N_{\rm linear}$ is less than 60, then the construction would be inconsistent; that is, if one were to keep $\epsilon(\phi)$ and its derivative monotonic then it would be impossible to achieve 60 e-foldings before inflation ends. This imposes a restriction on the values of ϵ_{60} and η_{60} that are allowed within this assumption. Having satisfied that, then in accord with the above the smallest reduction in H during the last 60 e-foldings is achieved if $\epsilon(\phi)$ behaves linearly until 60 e-foldings have passed, and then increases rapidly to unity. A tedious but straightforward calculation shows that the reduction factor in this case is

$$\frac{H_{\rm end}}{H_{60}} = \exp\left\{-\frac{\epsilon_{60}}{3\left(\epsilon_{60}-\eta_{60}\right)}\left\{\left[1+60\left(\epsilon_{60}-\eta_{60}\right)\right]^3-1\right\}\right\} , \tag{38}$$

which in the limit $\epsilon'_{60} \to 0$ (equivalent to $\epsilon_{60} \to \eta_{60}$) recovers the result of case A.

Case B tightens the constraint from case A by enforcing that η_{60} be close to ϵ_{60}, in order to minimize the reduction factor. However, it does not offer significantly stronger limits than the analytic result mentioned there for small $|\eta_{60}|$, because the reduction factor is the same if one chooses $\epsilon_{60} = \eta_{60}$, and it happens that the COBE normalization does not change much for $\epsilon_{60} = 0.008$ if η_{60} is increased from zero to equal ϵ_{60}. With $\epsilon_{60} = \eta_{60} \simeq 0.008$, we get the maximum value consistent with the case B assumptions; $H_{\rm end} < 4.1\times10^{-6}m_{\rm Pl}$. However, in more

[5]It is violated weakly by some models based on two fields [22, 18], though in any case they tend to give very low-energy scales. The only known example where it is violated strongly is intermediate inflation [21], which can be implemented in an extended inflation framework [23] and features $\epsilon(\phi)$ decreasing as ϕ^2 until tunneling brings inflation to an end.

specific circumstances the case B assumptions do lead to a tightening of the limits; for instance if $\epsilon_{60} = 0.008$ and $\eta_{60} = 0$, then the limit advertised in case A is tightened to 75% of its case A limit.

V. DISCUSSION

In conclusion, limits have been provided on the inflationary energy scale both 60 e-foldings from the end of inflation and at the end of inflation. As noted in the previous section, the results have negligible dependence on the specific choice of 60 for the number of e-foldings between the quadrupole scale leaving the horizon and the end of inflation, so its dependence on the details of reheating can be ignored.

Throughout these conclusions, numbers are quoted based on the central COBE normalization for the 10° variance; to convert to an upper limit, one multiplies by the factor by which one is willing to let the true 10° variance go up. At present, we recommend using the 2σ upper limit, thus multiplying the numbers for H by 15/11, though it is worth recalling that for structure formation models based on inflation the intermediate angle microwave experiments probably leave little room for the true 10° variance to be above the COBE result at all [24].

When one incorporates tilt and gravitational wave corrections to the COBE normalization, one finds that the largest values of H_{60} occur in regions far from the slow-roll limit, where the validity of the calculations is breaking down. Nevertheless, by imposing physically motivated constraints based on prejudice regarding structure formation, it is reasonable to say that the largest value of H_{60} which can generate the central COBE value is $H_{60} = 2.9 \times 10^{-5} m_{\rm Pl}$.

The Hamilton-Jacobi equations are used to provide an exact analytic treatment of the translation of limits on H_{60} into limits on $H_{\rm end}$. Again it is possible by sufficient contrivance in the choice of $\epsilon(\phi)$ to put $H_{\rm end}$ close to H_{60}. However, by imposing very reasonable physically motivated constraints the situation changes dramatically. Here the properties of the maximization are much nicer, for the maximum values of $H_{\rm end}$ occur in situations where slow roll was accurately obeyed 60 e-foldings from the end. This fits in with the picture that if slow roll is not accurate, then the expansion is far from de Sitter and hence the energy scale must be decreasing rapidly. The best motivated assumption is that $\epsilon(\phi)$ monotonically increases with scale, which in general leads to a maximum $H_{\rm end}$ of $6.0 \times 10^{-6} m_{\rm Pl}$. With further reasonable assumptions this is tightened further to a maximum $H_{\rm end}$ of $4.1 \times 10^{-6} m_{\rm Pl}$.

Let us compare these rather abstractly generated limits with the sorts of values arising in polynomial chaotic inflation models, taking as illustration $V(\phi) \propto \phi^2$, which coincidentally gives values of ϵ_{60} and η_{60} very similar to those we have advocated as helping to maximize $H_{\rm end}$, though the general $\epsilon(\phi)$ behavior is of course different. In Sec. II, we provided an exact inflationary solution based on choosing a polynomial $H(\phi)$, which yielded $H_{60} = 6.8 \times 10^{-6} m_{\rm Pl}$ and $H_{\rm end} = 6.2 \times 10^{-7} m_{\rm Pl}$. This solution is a good approximation to that of a quadratic potential whenever the slow-roll parameters are small, so the estimate of H_{60} is a good approximation to that appropriate to $V(\phi) \propto \phi^2$. As slow roll is a poor approximation at the end of inflation, the value for $H_{\rm end}$ is not as accurate. Using the normalization at H_{60}, but using exact numerical simulation to evolve to $H_{\rm end}$ with the polynomial potential yields $H_{\rm end} = 5.4 \times 10^{-7} m_{\rm Pl}$.

Throughout, we have been providing what amounts to upper limits on the energy scale, by finding the largest values of the energy scale consistent with the COBE normalization and various dynamical constraints. No mention has yet been made of lower limits, for the reason that the energy scale can be made as low as one likes while still satisfying COBE, provided one is willing to accept very small values of ϵ_{60}. As models do exist where ϵ_{60} can be tiny (such a class are the "hybrid" models featuring one inflaton field and a trigger field to end inflation [22], and natural inflation [25] provides a further example), lower limits cannot be derived using COBE alone. However, there is one very promising route by which a lower limit could be placed, which would be if it were to be demonstrated that some sizable component of the COBE signal were due to gravitational waves [26]. Such a discovery effectively places a lower limit on ϵ_{60}, and hence on the inflationary energy scale. It has already been noted that some knowledge of tensor modes is essential if one hopes to determine the detailed form of the inflaton potential [27].

There is also considerable interest in the region between H_{60} and $H_{\rm end}$, as the Hubble parameter governs the amplitude of gravitational wave production. Around 20–30 e-foldings from the end is the time when wavelengths which can be seen by Earth and space-based interferometers are generated, and so the COBE normalization allows one to make rather specific statements on the expected amplitude from COBE normalized inflation models [28].

Limits on $\rho_{\rm end}$ can be converted to limits on the reheat temperature $T_{\rm reh}$, given two uncertainties. The first is that the energy is to be distributed evenly amongst some unknown number g_* of particle degrees of freedom available at that energy; g_* is assumed to be at least the standard model value of 106.75 but could be much larger. Secondly, there is a parameter $\alpha < 1$ which measures the efficiency of reheating, $\rho_{\rm reh} = \alpha \rho_{\rm end}$, where $\rho_{\rm reh}$ is the energy density when the post-inflationary thermalization can first be said to have completed. In weakly coupled theories α is expected to be rather small, though in theories where inflation ends violently, such as through bubble collisions, it may not be too far from unity. Putting all this together gives

$$\frac{T_{\rm reh}}{m_{\rm Pl}} \simeq 0.78 g_*^{-1/4} \alpha^{1/4} \left(\frac{H_{\rm end}}{m_{\rm Pl}} \right)^{1/2} . \tag{39}$$

Making the weak, but not essential, assumption that $\epsilon'(\phi) \geq 0$ during the last 60 e-foldings of inflation, and using the standard model degrees of freedom, it is reasonable to expect that the reheat temperature after inflation will not exceed

$$T_{\rm reh} = 7.2 \alpha^{1/4} \times 10^{15}\ \text{GeV} . \tag{40}$$

We end with some brief comments concerning topological defects. It has been shown that typically one needs a defect scale of slightly over 10^{16} GeV if defects are to explain large scale structure [29]. It is clear that forming such defects after reheating will be very difficult here, because as a first step one must reduce the inflationary contribution to COBE to almost negligible size (as topological defect theories are already likely to produce excessive distortions if normalized to other large scale structure data). The reheat temperature comes down as the square root of the fractional lowering of the COBE signal, so to remove the inflationary density perturbations will bring down the reheat temperature by another factor of at least 3. There is however another possibility which can be realized with particular ease in hybrid models [22], which is to form defects as inflation ends in the field which is triggering the end of inflation. In that case typically all the inflationary energy density is available to go into defects, evading both the g_* and α suppression factors, and removing the need to restore the symmetry. Given the tight constraints illustrated above, this seems the most promising route to salvaging compatability of defect theories with the inflationary cosmology.

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