Real-polynomial formulation of general relativity in terms of connections

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I show in this Brief Report that it is possible to construct a Hamiltonian description for Lorentzian general relativity in terms of two real SO(3) connections. The constraints are simple polynomials in the basic variables. The present framework gives us a new formulation of general relativity that keeps some of the interesting features of the Ashtekar formulation without the complications associated with the complex character of the latter.

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Using a somewhat conservative approach in which both general relativity and the general quantization program are taken as the starting point, Ashtekar developed a new description for gravity that has some features that may, at the end, lead to a successful quantization of the theory [1,2]. A key element of his approach is changing the emphasis from geometrodynamics to connection dynamics. In fact, it is not only the simplicity of the constraints in Ashtekar's Hamiltonian formulation that makes it possible to advance in the quantization program but also the availability of geometrical objects that are absent in the geometrodynamical description. This fact is at the root of the successful introduction of the loop variables by Rovelli and Smolin [3]. They have provided a very appealing picture of the structure of space at the Planck scale and made it possible to find solutions to all the constraints in the Hamiltonian formulation of the theory.

In spite of all the deep new insight that has been gained, it must be said that the program is not complete in its present form and we cannot claim yet that gravity has been successfully quantized. There are many technical issues that must be addressed (and many conceptual questions too). One of them, the intrinsically complex nature of the Ashtekar variables, will be the subject of this Brief Report. The fact that the Ashtekar connection must be genuinely complex is something that can be seen at the levels of both the Hamiltonian and the Lagrangian descriptions. In the Hamiltonian formalism an imaginary unit must be introduced in the canonical transformation that leads from the usual geometrodynamical phase-space coordinates to the Ashtekar variables. It is necessary if one wants to eliminate some terms involving derivatives of the triads that would complicate the final form of the Hamiltonian constraint. In the Lagrangian description (using the Samuel-Jacobson-Smolin action [4,5]), use is made of self-dual connections that are, again, complex in the Lorentzian case. The real theory is recovered by imposing "reality conditions" on the fields. Although these conditions may prove to be useful in order to obtain the inner product of the theory, in practice they are difficult to implement. It is, in my opinion, desirable to have manifestly real formulations of general relativity that avoid these problems while keeping the simplicity of the Ashtekar approach (or, at least, a significant part of it).

The main result presented in this Brief Report is to show that it is possible to describe $(3+1)$-dimensional gravity in a phase space spanned by two real SO(3) connections (much in the spirit of [6]) with constraints that are low-order polynomials in the phase-space variables both for Euclidean and for Lorentzian general relativity. Although this approach is close to the Ashtekar point of view of using connections as the basic objects to describe the gravitational field, the geometric nature of the fields involved is different, and thus it may allow us to find new sets of elementary variables for the quantization of gravity that are not obvious in the previous formulations.

A point that I want to discuss before proceeding further is the meaning of polynomiality and its relationship with "reality" of a formulation. As has been stressed by several authors (see [7] and references therein), the geometrodynamical constraints can always be cast in a polynomial form by introducing powers of the determinant of the three-metric as global factors. The issue is not really whether the constraints are polynomial but how simple their polynomial expressions are. If in the Ashtekar formulation we do not introduce an imaginary unit in the canonical transformation that brings us from the Lorentzian Arnowitt-Deser-Misner (ADM) phase space to the new one but work, instead, with real fields, we find a real formulation in terms of "Ashtekar-like" variables. The problem is that, proceeding in this way, the final Hamiltonian constraint has a complicated expression and a high density weight (if one wants it to be polynomial). This makes very difficult the passage to the quantum theory in which we must impose the quantum version of the constraints as conditions of the wave functionals. Some of the advantages of working with the Ashtekar variables are then lost. However, the fact that the basic variables are still the connection-densitized triad pair may still allow us to use, for example, the loop variable approach when attempting the quantization of the theory and get some interesting results. It must be emphasized, also, that the really important issue seems to be finding a simple way to write the *quantum* constraints, and so it is conceivable that a somewhat complicated set of elementary variables could do the trick and give a simple quantum theory. Even if a formulation does not have simple constraints, the geometrical nature of the basic variables may suggest a set of elementary variables that simplifies the quantum theory. This, in itself, is a motivation to describe general relativity using different sets of basic variables.

In the following I will further exploit the ideas introduced in [6] to describe 3+1 complex general relativity in a phase space coordinatized by two complex SO(3) connections. I will start by giving an argument that shows that it may be possible to find an appealing polynomial formulation for Lorentzian general relativity and then give the full construction.

The phase space introduced in [6] to describe complex general relativity is coordinatized by two complex SO(3) connections ${}^1A_a^i$ and ${}^2A_a^i$. I introduce here the notation that will be used throughout this Brief Report. Tangent space indices and SO(3) indices are represented by latin letters from the beginning and the end of the alphabet, respectively, and run from 1 to 3. The space-time manifold is restricted to have the form $\mathscr{M}=\mathbb{R}\times\Sigma$ where Σ is a compact three-manifold with no boundary. I introduce also the objects $e_{ai}\equiv{}^2A_{ai}-{}^1A_{ai}$, ${}^1\tilde{B}_i^a\equiv\tilde{\eta}^{abc}\,{}^1F_{bci}$, ${}^2\tilde{B}_i^a\equiv\tilde{\eta}^{abc}\,{}^2F_{bci}$, $\det e\equiv\tilde{e}$, and $\tilde{b}_i^a\equiv{}^2\tilde{B}_i^a-{}^1\tilde{B}_i^a$, where $\tilde{\eta}^{abc}$ is the three-dimensional Levi-Civita tensor density, ϵ_{ijk} is the internal Levi-Civita tensor, and ${}^1F_{ab}^i, {}^2F_{ab}^i$ are the curvatures of ${}^1A_a^i$ and ${}^2A_a^i$ ($F_{ab}^i\equiv 2\partial_{[a}A_{b]}^i+\epsilon^i_{jk}A_a^jA_b^k$). I represent the density weights by the usual convention of using tildes above and below the fields.

The symplectic structure in this model is given by [6]

$$\Omega=2\kappa\int_\Sigma d^3x\,\tilde{\eta}^{abc}\epsilon_{ijk}[{}^1A_a^i(x)-{}^2A_a^i(x)] \times d\,{}^1A_b^j(x)\wedge d\,{}^2A_c^k(x)\ , \tag{1}$$

where $\kappa=i$ and 1 for Lorentzian and Euclidean gravity, respectively. The constraints are [6]

$$\begin{aligned} &\epsilon_i^{jk}e_{aj}\,{}^1\tilde{B}_k^a=0\ ,\\ &\epsilon_i^{jk}e_{aj}\,{}^2\tilde{B}_k^a=0\ ,\\ &e_a^k\,{}^1\tilde{B}_k^a=0 \end{aligned} \tag{2}$$

(the Lagrange multipliers for the Gauss law and the diffeomorphism constraint must be taken as purely imaginary in the Lorentzian case). The scalar constraint has now the density weight +1 because this formulation is well defined only when the e_a^i are nondegenerate (a condition that may be traced back to the nondegeneracy of the symplectic two-form [6]). We can then drop a factor $\tilde{e}$ that appears in the Hamiltonian constraint.

Following the arguments given by Hojman, Kuchař, and Teitelboim [8], it is straightforward to see that the three-metric is just $q_{ab}=e_a^ie_{bi}$. Introducing now the Hamiltonian constraint functional

$$H(\underset{\sim}{N})=\int_\Sigma d^2x\,\underset{\sim}{N}(x)\tilde{e}e_a^i\,{}^1\tilde{B}_i^a \tag{3}$$

it is possible to compute the Poisson brackets of $H(\underset{\sim}{N})$ and q_{ab} to get the extrinsic curvature in terms of the two connections:

$$\begin{aligned}\{H(\underset{\sim}{N}),q_{ab}\}&=\tfrac{1}{2}\kappa[-\underset{\sim}{N}\tilde{e}q_{ab}+\tfrac{1}{2}\underset{\sim}{N}(q_{ab}e_c^k\tilde{b}_k^c-2q_{e(a}e_{b)}^k\tilde{b}_k^e)]\\ &\equiv-2NK_{ab}\ ,\end{aligned} \tag{4}$$

where I define $N\equiv\underset{\sim}{N}\tilde{e}$. One possibility now is to impose reality conditions. One must demand that both the three-metric q_{ab} and the extrinsic curvature K_{ab} given by (4) are real (the presence of κ will give nontrivial conditions for the Lorentzian case). We can, however, realize that the expression $\tilde{q}[K^{ab}K_{ab}-K^2]$ ($\tilde{q}\equiv\det q_{ab}$) that appears in the usual Hamiltonian constraint of geometrodynamics is now a simple polynomial in e_a^i and $\tilde{b}_i^a$. This, together with the fact that e_a^i and $\tilde{b}_i^a$ are canonically conjugate [as can be seen from the Poisson brackets derived from (1)], suggests that it may be possible to find a change of coordinates from the geometrodynamical phase space (or rather the triad-extrinsic curvature version of it) to a phase space coordinatized by two real connections in which the constraints are simple polynomials. The reason one expects this to happen is because, even in the Lorentzian case, the dangerous terms quadratic in the extrinsic curvatures have simple polynomial expressions as deduced from (4).

I give now the complete construction of the new formulation. The starting point will be the geometrodynamical description of Lorentzian general relativity in terms of the triad and the extrinsic curvature. The phase space is $\Gamma(\tilde{E},K)$ coordinatized by $\tilde{E}_i^a$ and K_{ai} with the symplectic structure

$$\Omega=\int_\Sigma d^3x\,dK_a^i\wedge d\tilde{E}_i^a \tag{5}$$

and the constraints

$$\begin{aligned} &2q^{-1/2}\tilde{E}_{[i}^b\tilde{E}_{j]}^aK_a^iK_b^j+\zeta q^{1/2}R=0\ ,\\ &\mathscr{D}_b(K_a^i\tilde{E}_i^b-K_c^i\tilde{E}_i^c\delta_a^b)=0\ ,\\ &\epsilon_{ijk}K_a^j\tilde{E}^{ak}=0\ . \end{aligned} \tag{6}$$

Here, $\mathscr{D}_a$ is the torsion-free covariant derivative, compatible with e_{ai} that acts both on internal and spatial indices ($\mathscr{D}_ae_{bi}\equiv\partial_ae_{bi}+\epsilon_i^{jk}\Gamma_{aj}e_{bk}-\Gamma_{ab}^ce_{ci}=0$) and $\zeta=-1$ or 1 for Lorentzian and Euclidean generate relativity, respectively. The constraints (6) generate time evolution, spatial diffeomorphisms, and SO(3) gauge transformations [9]. The previous symplectic structure gives the Poisson brackets

$$\begin{aligned} &\{K_a^i(x),K_b^j(y)\}=0\ ,\\ &\{\tilde{E}_i^a(x),\tilde{E}_j^b(y)\}=0\ ,\\ &\{\tilde{E}_i^a(x),K_b^j(y)\}=\delta_b^a\delta_i^j\delta^3(x,y)\ . \end{aligned} \tag{7}$$

Let us introduce now a change of coordinates from the geometrodynamical phase space to a new one coordinatized by two *real* SO(3) connections ${}^1A_a^i, {}^2A_a^i$:

$$\tilde{E}_i^a=\tilde{\eta}^{abc}\epsilon_{ijk}e_b^je_c^k\ , \tag{8}$$

$$K_a^i=\frac{1}{4\tilde{e}}[e_a^ie_b^j-2e_a^je_b^i]\tilde{b}_j^b\ . \tag{9}$$

It is straightforward to check that the Jacobian of the previous transformation is well defined and different from zero if and only if $\tilde{e}\neq0$. Substituting now (8) and (9) in (5), we conclude that the symplectic structure can be written in terms of the two connections as in (1) with $\kappa=1$. The Poisson brackets in terms of these new variables are

$$\{{}^1A_a^i(x),{}^1A_b^j(y)\}=0\ ,$$
$$\{{}^2A_a^i(x),{}^2A_b^j(y)\}=0\ , \qquad (10)$$
$$\{{}^1A_a^i(x),{}^2A_b^j(y)\}=\frac{1}{4\tilde{e}}[e_a^ie_b^j-2e_a^je_b^i]\delta^3(x,y)\ .$$

They coincide with the result obtained in [6] for the Husain-Kuchař model. We must see now how the constraints (6) are written in terms of the two connections. As far as the Gauss law and vector constraints are concerned, we expect to find the result already obtained in [6] and given by the first two expressions in (2). Indeed these constraints give us just the kinematical symmetries of the theory. For the Gauss law we find that it is translated into the condition

$$\epsilon_{ijk}e_a^j\tilde{b}^{ak}=0\ , \qquad (11)$$

which is equivalent to

$${}^1\nabla_b\,{}^2B_i^b+{}^2\nabla_b\,{}^1\tilde{B}_i^b=0\ . \qquad (12)$$

The diffeomorphism constraint

$$\mathcal{D}_b(K_a^i\tilde{E}_i^b-K_c^i\tilde{E}_i^c\delta_a^b)+\epsilon_{ijk}\Gamma_a^iK_b^j\tilde{E}^{bk}=0$$

becomes now

$${}^1A_a^i\,{}^1\nabla_b\,{}^2\tilde{B}_i^b+{}^2A_a^i\,{}^2\nabla_b\,{}^1\tilde{B}_i^b=0\ . \qquad (13)$$

In the last expressions, ${}^1\nabla$ and ${}^2\nabla$ denote the covariant derivatives built out of the connections ${}^1A_a^i$ and ${}^2A_a^i$ and defined by

$${}^1\nabla_a\lambda_i=\partial_a\lambda_i+\epsilon_i^{jk}\,{}^1A_{aj}\lambda_k$$

(and the analogous expression for ${}^2\nabla$). Using the Bianchi identities, it is easy to show now that, when $\tilde{e}\neq 0$, Eqs. (11) and (13) are equivalent to

$$\epsilon_i^{jk}e_{aj}\,{}^1\tilde{B}_k^a=0\ ,$$
$$\epsilon_i^{jk}e_{aj}\,{}^2\tilde{B}_k^a=0\ . \qquad (14)$$

The generating functionals of SO(3) gauge transformations and diffeomorphisms are given by (see [6])

$$G(N)=-\int_\Sigma d^3x\ N^i[{}^1\nabla_b\,{}^2\tilde{B}_i^b+{}^2\nabla_b\,{}^1\tilde{B}_i^b]\ ,$$
$$D(\mathbf{N})=\int_\Sigma d^3x\ N^a[{}^2A_a^i\,{}^2\nabla_b\,{}^1\tilde{B}_i^b+{}^1A_a^i\,{}^1\nabla_b\,{}^2\tilde{B}_i^b]\ . \qquad (15)$$

Let us concentrate now on writing the Hamiltonian constraint in terms of two connections. As a preliminary step, it is useful to point out that the general solution to the equation

$$2\tilde{\eta}^{abc}\epsilon_{ijk}e_b^jA_c^k=\tilde{H}_i^a \qquad (16)$$

(where A_c^k is the unknown) in terms of e_a^i and $\tilde{H}_i^a$ is equal to

$$A_a^i=\frac{1}{4\tilde{e}}[e_a^ie_b^j-2e_a^je_b^i]\tilde{H}_j^b\ . \qquad (17)$$

The SO(3) connection Γ_a^i compatible with e_{ai} is given by a particular case of (17):

$$\Gamma_a^i=-\frac{1}{2\tilde{e}}[e_a^ie_b^j-2e_a^je_b^i]\tilde{\eta}^{bcd}\partial_ce_{dj}\ . \qquad (18)$$

Taking into account that $e_a^i\equiv{}^2A_a^i-{}^1A_a^i$, we see that

$$\tilde{b}_i^a-2\tilde{\eta}^{abc}\partial_be_{ci}=\tilde{\eta}^{abc}\epsilon_{ijk}e_b^j({}^1A_c^k+{}^2A_c^k)\ , \qquad (19)$$

and solving (19) for the real SO(3) connection $A_a^i=\frac{1}{2}({}^1A_a^i+{}^2A_a^i)$, we find, using Eqs. (9) and (18), that

$$A_a^i=\frac{1}{4\tilde{e}}[e_a^ie_b^j-2e_a^je_b^i][\tilde{b}_j^b-2\tilde{\eta}^{bcd}\partial_ce_{dj}]=K_a^i+\Gamma_a^i\ . \qquad (20)$$

The previous expression gives

$$\Gamma_a^i({}^1A,{}^2A)=A_a^i({}^1A,{}^2A)-K_a^i({}^1A,{}^2A)\ .$$

We can use it to simplify the computation of the scalar curvature term that appears in the geometrodynamical scalar constraint in terms of the two curvatures. In fact, we can follow the procedure used by Ashtekar to write the Hamiltonian constraint for both Euclidean and Lorentzian general relativity. We must keep in mind that the final expressions should be written in terms of ${}^1A_a^i$ and ${}^2A_a^i$.

In the Euclidean case we have

$$\tilde{e}^2\epsilon^{ijk}e_i^ae_j^b[F_{abk}({}^1A,{}^2A)-2\mathcal{D}_aK_{bk}({}^1A,{}^2A)]=0\ , \qquad (21)$$

where

$$F_{abk}({}^1A,{}^2A)=\tfrac{1}{2}\,{}^1F_{abk}+\tfrac{1}{2}\,{}^2F_{abk}-\tfrac{1}{4}\epsilon_k^{lm}e_{al}e_{bm}\ .$$

For Lorentzian general relativity we find

$$\tilde{e}^2\epsilon^{ijk}e_i^ae_j^b[F_{abk}({}^1A,{}^2A)+2\epsilon_{klm}K_a^l({}^1A,{}^2A)K_b^m({}^1A,{}^2A)-2\mathcal{D}_aK_{bk}({}^1A,{}^2A)]=0\ . \qquad (22)$$

Taking into account that $\mathcal{D}_a$ is compatible with e_{ai}, we see that the term involving $\mathcal{D}_aK_{bk}$ is proportional to the Gauss law and hence we can remove it. In terms of ${}^1A_a^i$ and ${}^2A_a^i$, the Hamiltonian constraints for Euclidean and Lorentzian general relativity become, respectively,

$$\tilde{e}[e_c^k\,{}^1\tilde{B}_k^c+e_c^k\,{}^2\tilde{B}_k^c-3\tilde{e}]=0\ , \qquad (23)$$

$$\tilde{e}[e_c^k\,{}^1\tilde{B}_k^c+e_c^k\,{}^2\tilde{B}_k^c-3\tilde{e}]-\tfrac{1}{2}[e_b^ke_c^l-2e_b^le_c^k]\tilde{b}_k^b\tilde{b}_l^c=0\ . \qquad (24)$$

Although the term quadratic in $\tilde{b}_i^a$ in (24) makes this expression more complicated than its Euclidean counterpart, it is still a low-order polynomial in the basic variables and has density weight $+2$, just as the Hamiltonian constraint in the Ashtekar formulation. It is possible to further simplify (23) and (24) by using the canonical transformation

$${}^1A_a^i\to{}^1A_a^i+\alpha e_a^i\ , \qquad (25)$$
$${}^2A_a^i\to{}^2A_a^i+\alpha e_a^i\ , \qquad (26)$$

where α is a real constant. Under the action of (25) and (26), ${}^1\tilde{B}_i^a$, ${}^2\tilde{B}_i^a$, and $\tilde{b}_i^a$ transform as

$${}^1\tilde{B}_i^a\to(1-\alpha){}^1\tilde{B}_i^a+\alpha\,{}^2\tilde{B}_i^a+\alpha(\alpha-1)\tilde{\eta}^{abc}\epsilon_{ijk}e_b^je_c^k\ , \qquad (27)$$
$${}^2\tilde{B}_i^a\to(1+\alpha){}^1\tilde{B}_i^a-\alpha\,{}^2\tilde{B}_i^a+\alpha(\alpha+1)\tilde{\eta}^{abc}\epsilon_{ijk}e_b^je_c^k\ , \qquad (28)$$
$$\tilde{b}_i^a\to\tilde{b}_i^a+2\alpha\tilde{\eta}^{abc}\epsilon_{ijk}e_b^je_c^k\ . \qquad (29)$$

The Gauss law and the diffeomorphism constraint are invariant under the action of the previous canonical transformation, whereas the Hamiltonian constraints (23) and (24) transform, respectively, into (remember that $\tilde{e}\neq 0$)

$$\tilde{e}[(1-2\alpha)e_c^k\,{}^1\tilde{B}_k^c+(1+2\alpha)e_c^k\,{}^2\tilde{B}_k^c+3(4\alpha^2-1)\tilde{e}]=0\ , \qquad (30)$$

$$\tilde{e}[(1+2\alpha)e_c^k\,{}^1\tilde{B}_k^c+(1-2\alpha)e_c^k\,{}^2\tilde{B}_k^c+3(12\alpha^2-1)\tilde{e}]-\tfrac{1}{2}(e_b^ke_c^l-2e_b^le_c^k)\tilde{b}_k^b\tilde{b}_l^c=0\ . \qquad (31)$$

By choosing now some α, we can simplify the previous expressions. If we take, for example, $\alpha=-\frac{1}{2}$, we get

$$e_c^k\,{}^1\tilde{B}_k^c=0\ , \qquad (32)$$

$$\tilde{e}[e_c^k\,{}^2\tilde{B}_k^c+3\tilde{e}]-\tfrac{1}{4}[e_b^ke_c^l-2e_b^le_c^k]\tilde{b}_k^b\tilde{b}_l^c=0 \qquad (33)$$

for Euclidean and Lorentzian general relativity, respectively. The first expression reproduces the result found in [6] for complex general relativity, as one would expect, because of the triviality of the reality conditions in the Euclidean case. The second one gives a polynomial Hamiltonian constraint for Lorentzian general relativity.

Several comments are in order at this point. The formulation described above in terms of SO(3) connections is polynomial of low order in the basic variables and real by construction. It avoids, then, the introduction of reality conditions. This may prove to be an advantage of this formalism. In fact, reality conditions are difficult to deal with even in the pure gravity case. It is not known, for example, how to implement them if one uses the loop variables to quantize the theory. It must be said, nevertheless, that reality conditions may also be a useful tool. They can be used, for example, to select the scalar product of the theory [2], as can be shown in several nontrivial examples such as electromagnetism or linearized gravity.

Of course, we must decide if the simplification brought about by the real character of the theory compensates for the complications associated with both the nontrivial symplectic structure and the presence of terms quadratic in the curvatures in the Hamiltonian constraint. As far as the symplectic structure is concerned, it must be said that, although it is not trivial, it can be found in some familiar examples such as the two-connection formulation of the Husain-Kuchař model [10,6]. This kind of symplectic structure may be actually a common feature of theories formulated in terms of two connections. The structure of (33) is not as simple as in Euclidean formulation but has some nice features. In a sense, it is somewhat in between the Hamiltonian constraints in the Arnowitt-Deser-Misner (ADM) and the Ashtekar formulations; actually the last term in (33) exactly corresponds to the term quadratic in the extrinsic curvatures that appears in the ADM scalar constraint, whereas the first term is close in form to the Euclidean Hamiltonian constraint and, thus, to the direct translation of the Hamiltonian constraint of the Ashtekar formulation to the two-connection phase space.

The fact that ${}^1A_a^i$ and ${}^2A_a^i$ are not canonically conjugate makes it difficult to talk about a canonical quantization of the theory. This means that the passage to the quantum theory is not straightforward in this formulation. One really needs to find a set of elementary variables suitable for the task. Although this may not be easy, it must be said that, in spite of the complications associated with the symplectic structure, it is possible to find pairs of canonically conjugate variables that are not obvious in the Ashtekar formalism. One example of this is given by e_a^i and $\tilde{b}_i^a$.

Another feature that makes the previous framework appealing is the way the degenerate metrics can be taken care of. They are simply excluded by the requirement that the symplectic structure be nondegenerate (or, equivalently, by the condition that the coordinate transformation introduced above is well defined). It may be that degenerate metrics convey interesting physical information and it may even be possible to deal satisfactorily with them (as it happens in $2+1$ dimensions when one uses Witten's formulation [11]). Nevertheless, they are known to be a possible source of trouble as has been emphasized by Smolin [12] and clearly shown by Varadarajan [13] with his example of a spherically symmetric solution to all the constraints of general relativity in $3+1$ dimensions that is regular everywhere, degenerate in some regions, and has arbitrary negative energy. My opinion is that it is certainly reassuring to have a consistent way of dealing with degenerate metrics.

The ideas presented above strongly suggest that it may be possible to find a real action for Lorentzian general relativity in terms of two real connections. The inclusion of matter in this action could provide an explicit realization of the idea suggested by Ashtekar [2] of unifying all the interactions as a consequence of the fact that gravity can be described in terms of connections. Although some work in this direction has already been done [14,15], it may be worth looking at this problem from the perspective of the real, two-connection formulation presented above.

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