Trapped surfaces and the Penrose inequality in spherically symmetric geometries

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We demonstrate that the Penrose inequality is valid for spherically symmetric geometries even when the horizon is immersed in matter. The matter field need not be at rest. The only restriction is that the source satisfies an energy condition outside the horizon. No restrictions are placed on the matter inside the horizon. The proof of the Penrose inequality gives a new necessary condition for the formation of trapped surfaces. This formulation may also be adapted to give a sufficient condition. We show that a modification of the Penrose inequality proposed by Gibbons for charged black holes can be broken in early stages of gravitational collapse. This investigation is based exclusively on the initial data formulation of general relativity.

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In our early analyses [1] of the formation of trapped surfaces in spherically symmetric self-gravitating systems we found several criteria that determine the formation of trapped surfaces. These were all expressed in terms of quasilocal quantities. We found both necessary and sufficient conditions when the matter was at rest (moment of time symmetry data) and a sufficient condition when the matter was moving. We failed, however, to discover a necessary condition in the case of nonsymmetric in time initial data. In this paper we fill in that gap. We do this by first proving the Penrose inequality in the situation when the apparent horizon is inside matter. The resulting equation can be manipulated to give both necessary and sufficient conditions for the formation of trapped surfaces. These inequalities are interesting in that they use a global quantity, the Arnowitt-Deser-Misner (ADM) mass. The only assumptions we make are that a $1+3$ splitting of the spacetime by maximal hypersurfaces exists and that the matter density is non-negative *outside* any horizon. Only a part of the Einstein equations, the Hamiltonian and momentum constraints, are employed.

The apparent horizon in the Schwarzschild geometry is a surface which satisfies $2m/R=1$, where m is the total (ADM) mass of the spacetime and R is the Schwarzschild (areal) radius of the surface. If the matter in a spherically symmetric spacetime has compact support then the metric in the exterior region can be written in the Schwarzschild form and the condition $2m/R=1$ is both a necessary and sufficient condition for the appearance of trapped surfaces outside matter even in the case where the matter is moving. In this note we derive inequalities which are direct generalizations of the $2m/R=1$ condition for the appearance of a horizon which are valid even when the surface in question is inside the support of the matter as a follow-on from a proof of the Penrose inequality in such circumstances.

Penrose proposed [2] an inequality which he hoped would be satisfied by black holes. This inequality reduces to the condition $2m/R=1$ for the horizon of a Schwarzschild black hole. The Penrose inequality and its generalization describing charged matter proposed by Gibbons [3] was formulated in order to clarify circumstances in which the cosmic censorship hypothesis [4] can be broken. In a realistic collapse to a black hole one expects that (i) the area of an event horizon must increase, (ii) apparent horizons may not have an area greater than event horizons, and (iii) in a final stage black holes coincide with one of the known simple solutions (having no hair)—Schwarzschild, Reissner-Nordström, Kerr, or Kerr-Newman black holes.

When the total angular momentum vanishes and the global charge is 0, the final state should be the Schwarzschild solution having an area $4\pi(2m)^2$. Therefore, if there is known an initial value configuration breaking the Penrose inequality, then at least one of the arguments (i)–(iii) would not be true. The first two statements rely on the validity of the cosmic censorship hypothesis [5]; because of that, the failure of the Penrose inequality may mean that the cosmic censorship is broken.

Ludvigsen and Vickers [6] have proven that the Penrose inequality holds for a class of (possibly nonspherical) geometries, assuming a global condition on the past history of a collapsing system. There are also several partial proofs [7,8] or numerical analyses [9,10] in the framework of the initial value formalism.

The general spherically symmetric line element can be written as

$$ds^2=-\alpha^2dt^2+adr^2+br^2[d\theta^2+\sin^2(\theta)d\phi^2]\ , \qquad (1)$$

where $0<\phi<2\pi$ and $0<\theta<\pi$ are the standard angle variables.

The initial data are prescribed by giving the spatial geometry at $t=0$, i.e., by specifying the functions a and b and by giving the extrinsic curvature (which we assume to be traceless, $K_i^i=0$)

$$K_r^r=-2K_\theta^\theta=-2K_\phi^\phi=-2\partial_t R/\alpha R\ , \qquad (2)$$

where the areal radius R is defined as

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$$R = r\sqrt{b} \ . \quad (3)$$

A useful concept is that of the mean curvature of a centered sphere in a Cauchy slice:

$$p = 2\partial_r R / \sqrt{a} R \ . \quad (4)$$

It is important to note that each of these three quantities, K_r^r, R, and p are geometric three-scalars and as such do not depend on the coordinates chosen on the three-slice.

The spherically symmetric Einstein equations can be written as two initial constraints, one evolution equation and the lapse equation. We will only consider the constraints in this paper.

The Hamiltonian constraint can be written as

$$\partial_r p = -\sqrt{a}\,\{8\pi\rho_0 + \tfrac{3}{4}(K_r^r)^2 + \tfrac{3}{4}p^2 - R^{-2}\} \ , \quad (5)$$

while the momentum constraint reads

$$\partial_r K_r^r = -\tfrac{3}{2} p\sqrt{a} K_r^r - 8\pi j_r \ . \quad (6)$$

If T_μ^ν is the energy-momentum tensor of the matter field which generates the spherical solution, then

$$\rho_0 = -T_0^0 \ , \quad j_r = \alpha T_r^0 \ . \quad (7)$$

The constraints (5) and (6) possess a "conserved" quantity:

$$E = \frac{p^2 R^3}{8} - \frac{R}{2} - \frac{R^3 (K_r^r)^2}{8} - 2\pi \int_r^\infty \sqrt{a}\,\rho_0 R^3 p\,d\tilde{r} - \tfrac{1}{4}\int_r^\infty K_r^r \partial_{\tilde{r}}(K_r^r R^3)\,d\tilde{r} \ . \quad (8)$$

E is r independent, $\partial_r E = 0$. This can be proved directly by differentiating (8) and using the constraints (5) and (6).

Assume that the initial data are asymptotically flat. Assuming a suitable integrability of the current and matter densities, asymptotically the mean curvature p approaches 0 as $(2-4m/r)/r$ and K_r^r decreases like C/r^3 while the areal radius R behaves like $r(1+m/r)$, where m is the ADM mass of the system. One then obtains

$$E = -m \ . \quad (9)$$

After a little algebra one can rewrite (8) as

$$m - \left[\frac{S}{16\pi}\right]^{1/2} = -\frac{R^3}{8}\theta(S)\theta'(S) + 2\pi\int_r^\infty \sqrt{a}\,\rho_0 R^3 p\,d\tilde{r} + \tfrac{1}{4}\int_r^\infty K_r^r \partial_{\tilde{r}}(K_r^r R^3)\,d\tilde{r} \ , \quad (10)$$

where $S = 4\pi R^2$, $\theta(S)$ is the divergence of future-directed light rays outgoing from a sphere S ($\theta = \alpha^{-1} d/dt|_{\text{out}} \ln S = p - K_r^r$) and $\theta'(S)$ is the divergence of past-directed light rays outgoing from S (the convergence of future-directed light rays ingoing from S) ($\theta' = -\alpha^{-1} d/dt|_{\text{in}} \ln S = p + K_r^r$). Future trapped surfaces are those on which $\theta(S)$ is negative, past trapped surfaces are those on which θ' is negative. (Let us remark that our convention differs from that of [5].)

The Penrose inequality is the statement that the left-hand side of (10) is positive on any apparent horizon, where future horizons are defined by $\theta(S)$ vanishing and past horizons by $\theta'(S)$ being zero. Equation (10), therefore, would prove this inequality for spherically symmetric black holes provided that the combination of the two integrals on the right-hand side is nonnegative. We show that this is so if we impose very weak conditions on the initial data.

Let us notice that, using the momentum constraint (6), the last integrand can be written as

$$-8\pi R^3 K_r^r j_r \ . \quad (11)$$

The two integrands appearing in (10) can now be written as

$$2\pi\int_r^\infty \sqrt{a} R^3 \left[\rho_0 p - K_r^r \frac{j_r}{\sqrt{a}}\right] d\tilde{r} \ , \quad (12)$$

and Eq. (10) becomes

$$m - \left[\frac{S}{16\pi}\right]^{1/2} = -\frac{R^3}{8}\theta(S)\theta'(S) + 2\pi\int_r^\infty \sqrt{a} R^3 \left[\rho_0 p - K_r^r \frac{j_r}{\sqrt{a}}\right] d\tilde{r} \ . \quad (13)$$

We assume the dominant energy condition [5], which means that

$$\rho_0 \geq |j_r| / \sqrt{a} \quad (14)$$

holds *outside* of the trapped surfaces in the data. We do not place any restriction on the matter in the interior.

The expression $p\rho_0 - K_r^r j_r / \sqrt{a}$ can be written as

$$\tfrac{1}{2}(p + K_r^r)(\rho + j_r/\sqrt{a}) + \tfrac{1}{2}(p - K_r^r)(\rho - j_r/\sqrt{a}) \ .$$

This tells us that the integrand in (13) is bounded from below by

$$\sqrt{a} R^3 \inf\left[\left[\rho_0 - \frac{j_r}{\sqrt{a}}\right](p - K_r^r) \ , \left[\rho_0 + \frac{j_r}{\sqrt{a}}\right](p + K_r^r)\right] = \sqrt{a} R^3 \inf\left[\left[\rho_0 - \frac{j_r}{\sqrt{a}}\right]\theta \ , \left[\rho_0 + \frac{j_r}{\sqrt{a}}\right]\theta'\right] \ .$$

Consider the outermost future trapped surface, the (future) apparent horizon, call it S. Let us assume that S is outside the outermost past trapped surface. In other words, we assume $\theta(S) = 0$ and that both θ and θ' are positive outside S. From (14) we also have that $\rho_0 \geq |j_r| / \sqrt{a}$ in the same region. Then (13) implies that

$$m - \left[\frac{S}{16\pi}\right]^{1/2} \geq 2\pi\int_r^\infty R^3 d\tilde{r}\sqrt{a}\, \inf\left[\left[\rho_0 - \frac{j_r}{\sqrt{a}}\right]\theta, \left[\rho_0 + \frac{j_r}{\sqrt{a}}\right]\theta'\right] \geq 0 \ . \quad (15)$$

Of course, an identical argument works if the outermost trapped surface is a past apparent horizon. Thus we have demonstrated that the Penrose inequality holds for the outermost horizon. Let us stress that the only assumption we make is that the matter satisfies the dominant energy condition in the *exterior* region. This may be important if one wishes to consider quantum effects which may lead to violations of the energy conditions. These effects will tend to be significant only in the strong field regions, i.e., inside the horizon.

In summary, we have proven the following.

Theorem. Let Σ_t be a partial maximal Cauchy hypersurface that extends outwards from an outermost future (past) apparent horizon situated at a sphere S. Assume the dominant energy condition in Σ_t and let the ADM mass be m. Then the areal radius $R=\sqrt{S/4\pi}$ of the future (past) apparent horizon must be less than the Schwarzschild radius $2m$.

Remark. The same result holds true also for any nonmaximal Cauchy hypersurface, under the remaining conditions as above.

Putting the content of the theorem in yet another way, we can say that (15) constitutes a necessary condition for the formulation of an outer trapped surface and (since a region filled with trapped surfaces must be surrounded by an outermost trapped surface) also for the formation of a trapped surface within a Cauchy slice. It complements previously obtained results: we failed in [1] to find a necessary condition for nonsymmetric in time initial data [11].

We can also derive a sufficient condition using this same analysis, but under stronger conditions. Assume that the maximal initial Cauchy slice is either regular at the origin or has at most a conical singularity there. One can show [12] (assuming that the dominant weak energy condition holds everywhere on the Cauchy slice)

$$R\theta \le 2 \ , \quad R\theta' \le 2 \ .$$

This allows one to get an estimate, using the same technique as before, that

$$2\pi \int_r^\infty \sqrt{a}\, R^3 \left[\rho_0 p - K_r^r \frac{j^r}{\sqrt{a}} \right] d\tilde{r} \le M_e$$

where $M_e(S)$ is the external rest mass $\int_{\text{out } S} \rho_0 dV$ outside a surface S. Thus, a sufficient condition is

$$\textit{If } m \ge \frac{R(S)}{2} + M_e(S) \textit{ then } S \textit{ is a trapped surface} \ .$$

Unfortunately, the analysis of moment of time symmetry data suggests that the above condition may, perhaps, never be satisfied.

At moment of time symmetry, both j^r and K_r^r are identically zero and Eq. (13) reduces to

$$m - \left[\frac{S}{16\pi} \right]^{1/2} = -\frac{R^3}{8} p^2 + 2\pi \int_r^\infty \sqrt{a}\, R^3 \rho_0 p\, d\tilde{r} \ . \tag{16}$$

As before, we can show that the volume integral is less than M_e.

Hence, in the case of symmetric in time data, we always have that $m \le \frac{1}{2} R + M_e$, independent of the existence or not of trapped surfaces. We do not know whether this condition can be violated in the case of nonsymmetric in time data.

Gibbons [3] proposed a modified form of (15) for charged matter:

$$m - \left[\left[\frac{S}{16\pi} \right]^{1/2} + q^2 \left[\frac{\pi}{S} \right]^{1/2} \right] \ge 0 \ , \tag{17}$$

where q is a global charge. (17) can be proved assuming that all charged matter is enclosed within the outermost trapped surface. In this case the energy density can be written as $\rho = \rho_e + \rho_m$; that is, it splits into a purely electrostatic (monopole) part

$$\rho_e = \frac{q^2}{8\pi R^4} \tag{18}$$

and the remaining matter density, represented by ρ_m, which we assume satisfies the energy condition (14). The integral of the electrostatic part can be performed explicitly, to give

$$\int_r^\infty \frac{q^2 \partial_r R}{2R^2} d\tilde{r} = q^2 \sqrt{\pi/S} \ . \tag{19}$$

This gives the equation

$$m - \left[\left[\frac{S}{16\pi} \right]^{1/2} + q^2 \left[\frac{\pi}{S} \right]^{1/2} \right] = -\frac{R^3}{8} \theta(S) \theta'(S) + 2\pi \int_r^\infty \sqrt{a}\, \rho_m R^3 p\, d\tilde{r} + \tfrac{1}{4} \int_r^\infty K_r^r \partial_r (K_r^r R^3) d\tilde{r} \ . \tag{20}$$

The rest of the reasoning is exactly the same as before, so that finally one obtains (17).

We would like to point out that the Penrose inequality [(15) with $q=0$] is always true, irrespective of whether the matter is charged or not and independent of the detailed distribution of charged matter. The modification proposed by Gibbons for charged systems, however, requires a closer examination. We assumed above that the charged matter is contained entirely inside the outermost trapped surface. Can the above result be true without this condition? Equation (20) strongly suggests a negative answer. It is clear that Eq. (20) is valid even when the charge extends outside the horizon if we understand ρ_m to represent $\rho_m = \rho - \rho_e$, where ρ_e is defined by (18). In this case, however, ρ_e is an overestimate of the electrostatic energy and ρ_m underestimates the matter energy. Even if the matter were to satisfy the dominant energy condition we have no reason to assume that the unphysical ρ_m would do so. It is easy to imagine an initial geometry with an apparent horizon, where a large change is carried by matter outside the region with trapped surfaces. It should be simple to arrange things so that the integrals (20) are negative thus violating the inequality (17).

On the other hand, (17) will be true in the later stages of gravitational collapse. One can show that when S is chosen to be the outermost trapped surface then it must move outward faster than light rays, thus swallowing more and more of the matter; ultimately the right-hand side of (20) becomes negligible and therefore the left-hand side of (20) should equal zero. This in turn suggests that the areal radius of trapped surfaces should grow (since both global charge q and mass m are conserved) during an evolution. That is indeed true; the directional derivative $\partial_t + V\partial_r$ of the area of the outermost trapped surface S is equal to

$$R^2 K_r^r \sqrt{a}\,(V-\alpha/\sqrt{a})\ ,$$

where V is velocity of S and $\alpha/\sqrt{a}$ the velocity of outgoing light rays. At the future horizon K_r^r is strictly positive: $K_r^r = \frac{1}{2}(-\theta+\theta') = \theta'/2$.

Let us comment on our assumptions concerning the focusing properties of the spacetime geometry. The condition that θ' is strictly positive outside a sphere of vanishing $\theta(S)$ means that white holes (if there are any) are hidden inside S. Ingoing light rays are always convergent (i.e., $\theta' > 0$) in any realistic gravitational collapse that develops from smooth initial data. Or, in other words, if at some time t the geometry of a collapsing system contains a surface with vanishing θ', then the past history of that system must contain a singularity. In the case of spherical symmetry the existence of a singularity follows directly from the Raychaudhuri equations [5], while in the general nonspherical case one can invoke the Penrose-Hawking singularity theorems. And conversely, one can prove that if smooth initial data with strictly positive θ' give rise to a smooth evolution then ingoing light rays are always convergent, $\theta' > 0$ on all future Cauchy slices. It is reasonable, therefore, to assume that θ' is always positive.

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