Multipole radiation from massive fields: Application to binary pulsar systems

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A general multipole expansion for radiation from massive vector and scalar fields is developed for periodic sources. This formalism is then combined with data on the binary pulsar PSR 1913+16 to set limits on the electric charge of astrophysical bodies, and on the coupling strengths of new weak forces.

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I. INTRODUCTION

There is at present renewed interest in the possibility of detecting new intermediate-ranged forces arising from the exchange of ultralight quanta [1–5]. This interest has been stimulated by both theoretical and experimental considerations. Theoretically an extensive body of literature now exists which suggests that ultralight scalar and vector bosonic fields arise naturally in many extensions of the standard model [1–5]. The exchange of such quanta typically leads to a weak force which acts even over macroscopic distances, and which can thus simulate in some ways gravitational [6] or electromagnetic [7,8] interactions. The coexistence of such a force and gravitation leads to apparent deviations from the predictions of Newtonian gravity, and similarly for electromagnetism.

Stimulated by these theoretical ideas and also by the suggestion of a possible "fifth force" [9], experimentalists have searched for new forces by looking for deviations from the inverse-square law of Newtonian gravity, and also for violations of the weak equivalence principle. Using a variety of innovative techniques, these experiments have set stringent constraints on the coupling strength f and the Compton wavelength $1/\mu$ which characterize the interactions of such new fields [see, for example, Eq. (2.1) below]. Along with these results has come the recognition that further significant improvements in sensitivity are possible, and this has served to motivate continued interest in such gravitational experiments. To date, the possibility of new forces coexisting with electromagnetism has received much less experimental attention [7,8], largely because of the widespread (if inaccurate) belief that electromagnetism has been tested sufficiently to exclude such forces.

In searching for systems where effects of putative new forces might show up most prominently, one is led naturally to examine those where the background gravitational or electromagnetic effects are suppressed for some reason. In modern-day Eötvös experiments, for example, such a suppression can be achieved by arranging for the two test masses to have similar shapes. Another example is the detection of new forces by searching for the energy that the corresponding fields carry off in radiation. It is well known that the leading multipole for gravitational radiation is quadrupole, whereas it is monopole or dipole for scalar or vector fields, respectively [10]. Since each successive multipole order is suppressed by a factor of order d/λ, where d is the characteristic dimension of the system and λ is the wavelength of the emitted radiation, emission of scalar or vector radiation may compete favorably with gravitational radiation in some systems. One such system may be the Hulse-Taylor pulsar PSR 1913+16, which provides the strongest evidence to date for the existence of gravitational radiation at the level predicted by general relativity [11]. Since any discrepancy between theory and experiment could be attributed to the radiation of some hitherto unknown field, this system can provide constraints on new fields which increase in sensitivity with time as data continue to accumulate.

In this paper we generalize earlier work by various authors dealing with the radiation of scalar and vector fields of small nonzero mass. The focus of our paper is the formulation of a multipole expansion for the radiation of massive vector and scalar fields, which is then applied to the binary pulsar in order to extract the experimental limits presented in Sec. IV. Our formulation thus makes it easy to directly compare the constraints on the coupling strength of new fields that arise from the binary pulsar to those that emerge from laboratory experiments. In addition it allows for a systematic refinement in various limits as the pulsar data improve.

Previous work on radiation of massive fields has focused primarily on massive electrodynamics [12–16], and has dealt with such questions as how Maxwell's equations are recovered in the limit as the quantum mass goes to zero. Radiation of massive quanta at a single frequency has been studied by Crandall and Wheeler [14], van Nieuwenhuizen [15], and by Crone and Sher [16]. For massive scalar fields, radiation from a point source has been considered by Cawley and Marx [17]. In the wake of the "fifth force" hypothesis [9], a number of authors raised the possibility of setting limits on new fields by studying the radiation they carry off. Li and Ruffini [18] argued that the gravitational radiation from binaries would dominate over that from massive vector fields. Bertotti and Sivaram [19] have considered the possibility of detecting the radiation of quanta from a composition-dependent fifth force by means of an interferometer using test masses of different composition. They also discuss, as we do below, the dependence of the binary radiation

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on the compositions of the pulsar and its companion. Fujii [5] has studied the radiation of a light scalar field in the context of the "fifth force" hypothesis. He has observed, as we noted above, that scalar radiation could be important compared to gravitational radiation, because it enters at a lower multipole.

The outline of our paper is as follows: in Sec. II we review the formalism for massive vector fields, which we then apply to develop the multipole expansion. The explicit expressions for the leading multipole contributions to the time-averaged radiated power are given by Eqs. (2.71) and (2.72). The analogous formalism for scalar fields is presented in Sec. III, and the corresponding formulas for the radiated power are given in Eqs. (3.18) and (3.19). In Sec. IV the results of the preceding sections are applied to the binary pulsar to extract experimental limits on the radiation of vector and scalar fields, for various models of the pulsar and its companion. A summary of our conclusions is presented in Sec. V.

II. MASSIVE VECTOR FIELDS

A. Background and overview

To establish our notation and conventions we begin by deriving the field equations and energy-momentum tensor for the massive field from the appropriate Lagrangian. We will later follow a similar approach to develop the formalism for the case of a massive scalar field where the results are less familiar. Following Li and Ruffini [18] we restrict our attention to sources $J^\alpha(\mathbf{x},t)$ that are periodic in time, noting that our results can be easily extended to more general situations. After expanding the sources and fields in Fourier series, we use Green's function techniques to obtain a general solution for the vector field $A^\alpha(\mathbf{x},t)$. From this result we identify the radiation fields by examining the asymptotic behavior of $A^\alpha(\mathbf{x},t)$ for large $|\mathbf{x}|$. We find that only modes above a certain threshold frequency can contribute to radiation, and show how this can be easily understood in the language of photons. We then calculate a general formula for the time-averaged energy flux per unit solid angle. Although this formula is too cumbersome for most applications, it can be developed in a multipole expansion for radiation as in conventional massless electromagnetism. We then explicitly evaluate the time-averaged power radiated for a system as electric dipole, magnetic dipole, and electric quadrupole radiation, and show that these results reduce to their familiar forms when the quanta become massless.

B. Field equations and energy-momentum tensor

In this section we establish our notation and review the conventional formalism for describing a massive vector field $A^\alpha(\mathbf{x},t)$ whose quanta have a mass m_γ. We assume that $A^\alpha(\mathbf{x},t)$ is described by the usual Proca Lagrangian density [20]:

$$\mathcal{L} = -\frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} - \frac{f_V}{c}J_\alpha A^\alpha + \frac{\mu^2}{8\pi}A_\alpha A^\alpha, \qquad (2.1)$$

where $F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$, $\mu = m_\gamma c/\hbar$, and $J^\alpha(\mathbf{x},t)$ is the source current in units of the fundamental charge f_V. For massive electrodynamics, $f_V = e$. The inhomogeneous field equations for $A^\alpha(\mathbf{x},t)$, which are obtained from Eq. (2.1) by use of the Euler-Lagrange equations, are given by

$$\partial_\beta F^{\beta\alpha} + \mu^2 A^\alpha = \frac{4\pi f_V}{c}J^\alpha. \qquad (2.2)$$

The homogeneous equations

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0 \qquad (2.3)$$

follow as an immediate consequence of the definition of $F^{\alpha\beta}$, and have the same form as in the massless case. If the electric and magnetic fields are defined as in massless electrodynamics,

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0, \qquad (2.4a)$$

$$\mathbf{B} = \nabla \times \mathbf{A}, \qquad (2.4b)$$

then the Maxwell equations for a massive electromagnetic field become [8,12,13]

$$\nabla \cdot \mathbf{E} = 4\pi f_V \rho - \mu^2 A^0, \qquad (2.5a)$$

$$\nabla \cdot \mathbf{B} = 0, \qquad (2.5b)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad (2.5c)$$

$$\nabla \times \mathbf{B} = \frac{4\pi f_V}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} - \mu^2 \mathbf{A}. \qquad (2.5d)$$

Note that the source-independent equations are identical to those in the massless case. For $\mu \neq 0$, the potentials $\mathbf{A}$ and A^0 are not arbitrary, in contrast with the massless case. This can be seen from Eq. (2.5a) by noting that A^0 can be expressed in terms of $\nabla \cdot \mathbf{E}$ and ρ, both of which are gauge independent. A similar argument follows for $\mathbf{A}$ starting from Eq. (2.5d). Returning to Eq. (2.2) we find

$$f_V \partial_\alpha J^\alpha = \left(\frac{c\mu^2}{4\pi}\right)\partial_\alpha A^\alpha, \qquad (2.6)$$

so that if $\mu \neq 0$ and J^α is conserved, then $A^\alpha(\mathbf{x},t)$ must respect the Lorentz condition

$$\partial_\alpha A^\alpha = 0. \qquad (2.7)$$

Using this condition, the field equations Eq. (2.2) may be rewritten as

$$(\Box + \mu^2)A^\alpha = \frac{4\pi f_V}{c}J^\alpha, \qquad (2.8)$$

where

$$\Box \equiv \partial_\alpha \partial^\alpha = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2. \qquad (2.9)$$

We note for later purposes that if the source $J^\alpha(\mathbf{x},t)$ is assumed to be conserved, then it follows from the Lorentz gauge condition (2.7) that only three of the four components of $A^\alpha(\mathbf{x},t)$ are independent.

To determine the energy and momentum radiated in the form of massive quanta, we construct the energy-momentum tensor $T^{\alpha\beta}$ for the free vector fields, from which the energy density u and energy flux $\mathbf{S}$ (the analog of the Poynting vector in electromagnetism) can be identified. $T^{\alpha\beta}$ can be obtained from [21]

$$T^{\alpha\beta} \equiv 2\frac{\delta\mathcal{L}_{\rm free}}{\delta g_{\alpha\beta}} - g^{\alpha\beta}\mathcal{L}_{\rm free}, \tag{2.10}$$

where the explicit expression for the free field Lagrangian $\mathcal{L}_{\rm free}$ in terms of the metric tensor $g_{\alpha\beta} = {\rm diag}(1,-1,-1,-1)$ is

$$\mathcal{L}_{\rm free} = -\frac{1}{16\pi}g_{\alpha\gamma}g_{\beta\delta}F^{\alpha\beta}F^{\gamma\delta} + \frac{\mu^2}{8\pi}g_{\alpha\beta}A^\alpha A^\beta. \tag{2.11}$$

Combining Eqs. (2.10) and (2.11) we find

$$T^{\alpha\beta} = \frac{1}{16\pi}g^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} + \frac{1}{4\pi}F^{\alpha}{}_{\gamma}F^{\gamma\beta} + \frac{\mu^2}{4\pi}(A^\alpha A^\beta - \tfrac{1}{2}g^{\alpha\beta}A_\gamma A^\gamma). \tag{2.12}$$

From Eq. (2.12) the energy density u and the ith component S^i of the energy flux $\mathbf{S}$ for the free vector field are given by [12,13]

$$u \equiv T^{00} = \frac{1}{8\pi}\{|\mathbf{E}|^2 + |\mathbf{B}|^2 + \mu^2[(A^0)^2 + |\mathbf{A}|^2]\}, \tag{2.13a}$$

$$S^i \equiv cT^{0i} = \frac{c}{4\pi}[\mathbf{E}\times\mathbf{B} + \mu^2 A^0\mathbf{A}]^i. \tag{2.13b}$$

As a check, we see that the expressions for u and $\mathbf{S}$ reduce to the usual electromagnetic results in the limit $\mu \to 0$ [22].

C. General solution for periodic sources

Given a source $J^\alpha(\mathbf{x},t)$, the resulting fields $A^\alpha(\mathbf{x},t)$ are found by solving the inhomogeneous wave equation, Eq. (2.2). In the limit $\mu \to 0$, the general solution to Eq. (2.2) is easily found, since its retarded Green's function $G^+(\mathbf{x},t;\mathbf{x}',t')$ has the simple form [23]

$$\lim_{\mu\to 0} G^+(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|}. \tag{2.14}$$

When $\mu \neq 0$ the retarded Green's function is given by [24]

$$G^+(\mathbf{x},t;\mathbf{x}',t') = \frac{\delta\left[t' - \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{|\mathbf{x}-\mathbf{x}'|} - \frac{\theta\left[-t' + \left(t - \frac{|\mathbf{x}-\mathbf{x}'|}{c}\right)\right]}{s}\mu c\mathcal{J}_1(\mu s), \tag{2.15}$$

where

$$s^2 \equiv c^2(t-t')^2 - (\mathbf{x}-\mathbf{x}')^2 \tag{2.16}$$

is the spacetime interval, $\theta(x)$ is the step function

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{2.17}$$

and $\mathcal{J}_1(\mu s)$ is the Bessel function of the first kind. Rather than solving the full inhomogeneous equation Eq. (2.2), the problem simplifies greatly if we follow the method of Li and Ruffini [18] and consider localized sources $J^\alpha(\mathbf{x},t)$ that are periodic in time with period T. The sources and fields can be expanded in a Fourier series:

$$J^\alpha(\mathbf{x},t) = \sum_{n=-\infty}^{\infty} J_n^\alpha(\mathbf{x})e^{-in\omega_0 t}, \tag{2.18}$$

$$A^\alpha(\mathbf{x},t) = \sum_{n=-\infty}^{\infty} A_n^\alpha(\mathbf{x})e^{-in\omega_0 t}, \tag{2.19}$$

where

$$J_n^\alpha(\mathbf{x}) = \frac{1}{T}\int_0^T dt\, J^\alpha(\mathbf{x},t)e^{in\omega_0 t}, \tag{2.20}$$

and

$$\omega_0 \equiv \frac{2\pi}{T} \tag{2.21}$$

is the characteristic frequency of the system. Since the source $J^\alpha(\mathbf{x},t)$ is real, it follows that

$$J_n^{\alpha *}(\mathbf{x}) = J_{-n}^\alpha(\mathbf{x}). \tag{2.22}$$

If the sources are not periodic, one simply replaces the sums with integrals.

Substituting Eqs. (2.18) and (2.19) into the wave equation (2.2) yields an equation for each Fourier component:

$$(\nabla^2 + k_n^2)A_n^\alpha(\mathbf{x}) = -\frac{4\pi f_V}{c}J_n^\alpha(\mathbf{x}), \tag{2.23}$$

where

$$k_n \equiv \frac{n\omega_0}{c}\sqrt{1 - \left(\frac{n_0}{n}\right)^2} \tag{2.24}$$

and

$$n_0 \equiv \frac{\mu c}{\omega_0}. \tag{2.25}$$

Assuming no bounding surfaces except at infinity, the inhomogeneous Helmholtz equation (2.23) can be solved by noting that the Green's function $G_n(\mathbf{x},\mathbf{x}')$ for an outgoing wave is given by

$$G_n(\mathbf{x},\mathbf{x}') = \frac{e^{ik_n|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}, \tag{2.26}$$

where

$$(\nabla^2 + k_n^2)G_n(\mathbf{x},\mathbf{x}') = -4\pi\delta(\mathbf{x}-\mathbf{x}'). \tag{2.27}$$

The solution to Eq. (2.23) is then

$$A_n^\alpha(\mathbf{x}) = \frac{f_V}{c}\int d^3x' G_n(\mathbf{x},\mathbf{x}')J_n^\alpha(\mathbf{x}') = \frac{f_V}{c}\int d^3x' \frac{e^{ik_n|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}J_n^\alpha(\mathbf{x}'). \tag{2.28}$$

Combining Eqs. (2.19) and (2.28), the general solution for $A^\alpha(\mathbf{x},t)$ for a localized periodic source $J^\alpha(\mathbf{x},t)$ is given by

$$A^\alpha(\mathbf{x},t) = \frac{f_V}{c}\sum_{n=-\infty}^{\infty}\int d^3x' \frac{\exp\{in\omega_0[\frac{|\mathbf{x}-\mathbf{x}'|}{c}\sqrt{1-(\frac{n_0}{n})^2}-t]\}}{|\mathbf{x}-\mathbf{x}'|}J_n^\alpha(\mathbf{x}'), \tag{2.29}$$

where the Fourier component $J_n^\alpha(\mathbf{x})$ is obtained from Eq. (2.20). As a check on Eq. (2.29), we note that, for a point charge f_V located at the origin,

$$J^0(\mathbf{x},t) = c\, f_V\, \delta(\mathbf{x}) \tag{2.30}$$

and hence from Eq. (2.20),

$$J_n^0(\mathbf{x}) = c\, f_V\, \delta(\mathbf{x})\, \delta_{n0}. \tag{2.31}$$

Combining Eqs. (2.31) and (2.29) then leads to the familiar result

$$A^0(\mathbf{x},t) = f_V \frac{e^{-\mu r}}{r}, \tag{2.32}$$

where $r \equiv |\mathbf{x}|$.

D. Radiation fields

From the general solution for $A^\alpha(\mathbf{x},t)$ in Eq. (2.29) we wish to identify the radiation fields $A_{\text{rad}}^\alpha(\mathbf{x},t)$ associated with freely propagating waves by examining the behavior of the fields far from the source. If the radiation fields are traveling freely, then their energy flux per unit solid angle given by

$$\frac{d\dot{E}}{d\Omega} = r^2(\hat{\mathbf{r}}\cdot\mathbf{S}), \tag{2.33}$$

is nonzero infinitely far from the source:

$$\lim_{r\to\infty} r^2(\hat{\mathbf{r}}\cdot\mathbf{S}_{\text{rad}}) \neq 0. \tag{2.34}$$

From Eq. (2.13b) we note that since $\mathbf{S}$ is proportional to $(A^\alpha)^2$, Eq. (2.34) is satisfied if

$$\lim_{r\to\infty} A_{\text{rad}}^\alpha \propto \frac{1}{r}. \tag{2.35}$$

The radiation fields can then be obtained by examining the general solution for $A^\alpha(\mathbf{x},t)$ in the limit $r\to\infty$, and then identifying the fields which fall of as $1/r$. Returning to Eq. (2.28), we note that if d is the characteristic dimension of the system and $r \gg d$, then

$$|\mathbf{x}-\mathbf{x}'| \simeq r - \hat{\mathbf{r}}\cdot\mathbf{x}', \tag{2.36}$$

where $\hat{\mathbf{r}} \equiv \mathbf{x}/r$. It follows that far from the source $A_n^\alpha(\mathbf{x})$ can be written in the form

$$A_n^\alpha(\mathbf{x}) = \frac{e^{ik_n r}}{cr}I_n^\alpha(\theta,\phi) + O\left(\frac{1}{r^2}\right), \tag{2.37}$$

where [18]

$$I_n^\alpha(\theta,\phi) \equiv f_V\int d^3x' e^{-ik_n\hat{\mathbf{r}}\cdot\mathbf{x}'}J_n^\alpha(\mathbf{x}'). \tag{2.38}$$

We note that $I_n^\alpha(\theta,\phi)$ depends only upon the direction of the observation point and not on r. Also, for real sources,

$$I_n^{\alpha *} = I_{-n}^\alpha. \tag{2.39}$$

Summing over all modes, we find, from Eq. (2.37),

$$A^\alpha(\mathbf{x},t) = \sum_{n=-\infty}^{\infty} I_n^\alpha(\theta,\phi)\frac{e^{i(k_n r - n\omega_0 t)}}{cr} + O\left(\frac{1}{r^2}\right). \tag{2.40}$$

Although the sum over n (which can be chosen to be positive) includes all harmonics, there are no contributions from those with $n < n_0$ in Eq. (2.24). This follows by noting from Eq. (2.24) that when $n < n_0$, $k_n = i|k_n|$ so that

$$\frac{e^{ik_n r}}{r} = \frac{e^{-|k_n|r}}{r} \tag{2.41}$$

which vanishes in the limit $r\to\infty$. The radiation fields $A_{\text{rad}}^\alpha(\mathbf{x},t)$ are thus determined by the nonzero modes in the limit $r\to\infty$:

$$A_{\text{rad}}^\alpha(\mathbf{x},t) = \sum_{|n|>n_0} I_n^\alpha(\theta,\phi)\frac{e^{i(k_n r - n\omega_0 t)}}{cr}, \tag{2.42}$$

which have the form of spherical waves propagating outward from the source. It is interesting to note that the group velocity $v_n = d\omega_n/dk_n$ is different for each mode. Combining Eqs. (2.25) and (2.24) we have

$$\omega_n = c\sqrt{k_n^2+\mu^2} \tag{2.43}$$

which yields [16]

$$v_n = ck_n\left(k_n^2+\mu^2\right)^{-1/2} = c\sqrt{1-\left(\frac{n_0}{n}\right)^2}. \tag{2.44}$$

We see that the large n (high energy) modes travel at nearly the speed of light, but the small n (low energy) modes may travel much more slowly. This dispersion helps to explain why the radiation problem for massive fields simplifies by Fourier decomposing the fields and sources. While each mode behaves simply, the dispersion described by Eq. (2.44) makes the superposition of all harmonics complicated. For massless fields (in vacuum), all harmonics travel at the speed of light so the Fourier decomposition is unnecessary.

The fact that not all modes contribute in the massive case can be understood by noting that the energy $E_n = n\hbar\omega_0$ of the nth mode must be greater than the rest energy $m_\gamma c^2$ of the quantum for a massive photon to be emitted. This implies the condition

$$n > \frac{m_\gamma c^2}{\hbar\omega_0} = \frac{\mu c}{\omega_0} \equiv n_0, \tag{2.45}$$

so that for a given photon mass m_γ there is a minimum frequency

$$\omega_{\text{min}} \equiv \mu c = \frac{m_\gamma c^2}{\hbar} \tag{2.46}$$

of radiation. A source will radiate only weakly if its characteristic frequency ω_0 is such that $\omega_0 \ll \omega_{\text{min}}$, since the dominant (small n) modes will not appear. The finite mass of the photon also leads to the dispersion of the group velocities found in Eq. (2.44).

Returning to Eq. (2.42) we note that by virtue of the Lorentz condition only three of the components of A^α are independent. If we separate the time and space components of the radiation fields in Eq. (2.42) by writing

$$A^0_{\text{rad}}(\mathbf{x},t) = \sum_{|n|>n_0} I^0_n(\theta,\phi)\frac{e^{i(k_n r - n\omega_0 t)}}{cr}, \tag{2.47a}$$

$$\mathbf{A}_{\text{rad}}(\mathbf{x},t) = \sum_{|n|>n_0} \mathbf{I}_n(\theta,\phi)\frac{e^{i(k_n r - n\omega_0 t)}}{cr}, \tag{2.47b}$$

and apply the Lorentz condition Eq. (2.7) in the form

$$-\frac{1}{c}\frac{\partial A^0}{\partial t} = \nabla\cdot\mathbf{A}, \tag{2.48}$$

we find

$$\frac{n\omega_0}{c}I^0_n = k_n(\hat{\mathbf{r}}\cdot\mathbf{I}_n) + O\left(\frac{1}{r^2}\right). \tag{2.49}$$

Hence to $O(1/r)$,

$$I^0_n = \sqrt{1-\left(\frac{n_0}{n}\right)^2}(\hat{\mathbf{r}}\cdot\mathbf{I}_n), \tag{2.50}$$

from which if follows that all results for the radiation fields can be expressed in terms of $\mathbf{I}_n$.

Returning to the expressions for A^0_{rad} and $\mathbf{A}_{\text{rad}}$ given by Eqs. (2.47a) and (2.47b) we can calculate $\mathbf{E}_{\text{rad}}$ and $\mathbf{B}_{\text{rad}}$ for the radiation fields using Eqs. (2.4a) and (2.4b). Retaining only terms $O(1/r)$ we find

$$\mathbf{E}_{\text{rad}} = \sum_{|n|>n_0}\left\{[(\hat{\mathbf{r}}\times\mathbf{I}_n)\times\hat{\mathbf{r}}] + \left(\frac{n_0}{n}\right)^2(\hat{\mathbf{r}}\cdot\mathbf{I}_n)\hat{\mathbf{r}}\right\} \times\frac{in\omega_0}{c^2 r}e^{i(k_n r - n\omega_0 t)}, \tag{2.51a}$$

$$\mathbf{B}_{\text{rad}} = \sum_{|n|>n_0}(\hat{\mathbf{r}}\times\mathbf{I}_n)\frac{ik_n}{cr}e^{i(k_n r - n\omega_0 t)}. \tag{2.51b}$$

Note the appearance in $\mathbf{E}_{\text{rad}}$ of a term proportional to $\hat{\mathbf{r}}$ (i.e., a longitudinal component), which does not occur in electromagnetism. If the expressions for $\mathbf{A}_{\text{rad}}$, A^0_{rad}, $\mathbf{E}_{\text{rad}}$, and $\mathbf{B}_{\text{rad}}$ are substituted into the energy flux $\mathbf{S}$ given by Eq. (2.13b), we find, for the instantaneous energy flux carried by the radiation fields,

$$\mathbf{S}_{\text{rad}} = \frac{1}{4\pi c r^2}\sum_{|n|,|m|>n_0}\left(\frac{n\omega_0}{c}\right)e^{i(k_n r - n\omega_0 t)}e^{i(k_m r - m\omega_0 t)}\left[-k_m\left\{[(\hat{\mathbf{r}}\times\mathbf{I}_n)\cdot(\hat{\mathbf{r}}\times\mathbf{I}_m)]\hat{\mathbf{r}} + \left(\frac{n_0}{n}\right)^2(\hat{\mathbf{r}}\cdot\mathbf{I}_n)[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{I}_m)]\right\} + k_n\left(\frac{n_0}{n}\right)^2(\hat{\mathbf{r}}\cdot\mathbf{I}_n)\mathbf{I}_m\right]. \tag{2.52}$$

Using Eq. (2.33), the instantaneous energy flux per unit solid angle is then

$$\frac{d\dot{E}}{d\Omega} = \frac{1}{4\pi c}\sum_{|n|,|m|>n_0}\left(\frac{n\omega_0}{c}\right)e^{i(k_n r - n\omega_0 t)}e^{i(k_m r - m\omega_0 t)}\left[-k_m(\hat{\mathbf{r}}\times\mathbf{I}_n)\cdot(\hat{\mathbf{r}}\times\mathbf{I}_m) + k_n\left(\frac{n_0}{n}\right)^2(\hat{\mathbf{r}}\cdot\mathbf{I}_n)(\hat{\mathbf{r}}\cdot\mathbf{I}_m)\right]. \tag{2.53}$$

The quantity of physical interest is the time-averaged energy flux defined by

$$\left\langle\frac{d\dot{E}}{d\Omega}\right\rangle \equiv \frac{1}{T}\int_0^T\frac{d\dot{E}}{d\Omega}dt. \tag{2.54}$$

Inserting Eq. (2.53) into Eq. (2.54) and using

$$\frac{1}{T}\int_0^T dt\, e^{i(k_n r - n\omega_0 t)}e^{i(k_m r - m\omega_0 t)} = \delta_{m,-n}, \tag{2.55}$$

we find, for the time-averaged energy flow per unit solid angle,

$$\left\langle\frac{d\dot{E}}{d\Omega}\right\rangle = \frac{1}{2\pi c}\sum_{n>n_0}\left\{\left(\frac{n^2\omega_0^2}{c^2}\right)\sqrt{1-\left(\frac{n_0}{n}\right)^2} \times\left[|\hat{\mathbf{r}}\times\mathbf{I}_n|^2 + \left(\frac{n_0}{n}\right)^2|\hat{\mathbf{r}}\cdot\mathbf{I}_n|^2\right]\right\}. \tag{2.56}$$

The result given by Eq. (2.56) is exact, but is too cumbersome for all but the simplest sources. In the next section we develop a multipole expansion valid in the long wavelength limit from which a general formula for the total radiated power can be derived.

E. Multipole expansion of radiation fields

The radiation problem in conventional electrodynamics is greatly simplified by using a multipole expansion, since typically only the lowest multipoles contribute significantly to the total radiated energy. In this section we derive the analogous multipole expansion for the massive case, and examine the differences from the usual massless electrodynamics. We then explicitly evaluate the power radiated as electric dipole, magnetic dipole, and quadrupole radiation in the long wavelength limit, and show that the results reduce to those for the electromagnetic case in the limit $\mu \to 0$.

As we have seen in the previous section, the Lorentz condition implies that the radiation fields and total radiation flux can be written in terms of the quantity $\mathbf{I}_n(\theta,\phi)$ defined in Eq. (2.38). Although $\mathbf{I}_n(\theta,\phi)$ is difficult to evaluate in general, it may be simplified by expanding the exponential in powers of $k_n d$ where d is the characteristic size of the source:

$$\begin{aligned}\mathbf{I}_n(\theta,\phi) &= f_V \int d^3x'(1 - ik_n\hat{\mathbf{r}}\cdot\mathbf{x}')\mathbf{J}_n(\mathbf{x}') + O[(k_n d)^2] \\ &= \mathbf{I}_n^{(0)} + \mathbf{I}_n^{(1)} + O[(k_n d)^2], \end{aligned} \qquad (2.57)$$

where

$$\mathbf{I}_n^{(0)}(\theta,\phi) \equiv f_V \int d^3x' \mathbf{J}_n(\mathbf{x}'), \qquad (2.58a)$$

$$\mathbf{I}_n^{(1)}(\theta,\phi) \equiv -ik_n f_V \int d^3x'(\hat{\mathbf{r}}\cdot\mathbf{x}')\mathbf{J}_n(\mathbf{x}'). \qquad (2.58b)$$

In the long wavelength limit, $k_n d \ll 1$ and hence to good approximation we need retain only $\mathbf{I}_n^{(0)}$ and $\mathbf{I}_n^{(1)}$ in Eq. (2.57). This approximation is useful in describing nonrelativistic systems. If $\omega_0 d \simeq v$ where v is a characteristic velocity of the radiating system, then when $v/c \ll 1$, from Eq. (2.24) $k_n d \lesssim (n\omega_0 d)/c \sim (nv)/c \ll 1$ for sufficiently small n. As in ordinary electrodynamics, the long wavelength approximation is valid only when the harmonics close to the fundamental frequency ω_0 are dominant.

It is then straightforward to show that, as in the usual massless electrodynamics, $\mathbf{I}_n^{(0)}$ is proportional to the Fourier component $\mathbf{p}_n$ of the dipole moment:

$$\mathbf{I}_n^{(0)}(\theta,\phi) = -in\omega_0 f_V \int d^3x'\, \mathbf{x}'\rho_n(\mathbf{x}) \equiv -in\omega_0\mathbf{p}_n. \qquad (2.59)$$

Hence in the dipole approximation,

$$|\hat{\mathbf{r}}\times\mathbf{I}_n(\theta,\phi)|^2 \simeq |\hat{\mathbf{r}}\times\mathbf{I}_n^{(0)}|^2 = n^2\omega_0^2|\mathbf{p}_n|^2\sin^2\Theta, \qquad (2.60a)$$

$$|\hat{\mathbf{r}}\cdot\mathbf{I}_n(\theta,\phi)|^2 \simeq |\hat{\mathbf{r}}\cdot\mathbf{I}_n^{(0)}|^2 = n^2\omega_0^2|\mathbf{p}_n|^2\cos^2\Theta, \qquad (2.60b)$$

where $\sin\Theta = \hat{\mathbf{r}}\times\hat{\mathbf{p}}$. Combining Eqs. (2.60a), (2.60b), and (2.56) the dipole approximation for the time-averaged energy flux per unit solid angle is given by

$$\left\langle \frac{d\dot{E}}{d\Omega}\right\rangle = \frac{1}{2\pi c}\sum_{n>n_0}\left\{\frac{n^4\omega_0^4}{c^2}|\mathbf{p}_n|^2\sqrt{1-\left(\frac{n_0}{n}\right)^2} \times\left[\sin^2\Theta + \left(\frac{n_0}{n}\right)^2\cos^2\Theta\right]\right\}. \qquad (2.61)$$

We see from Eq. (2.61) that for massive electrodynamics ($\mu \neq 0$), both the magnitude and angular distribution of $\langle d\dot{E}/d\Omega\rangle$ are different from what they are in the massless case. The contribution to $\langle d\dot{E}/d\Omega\rangle$ from $\mathbf{I}_n^{(1)}(\theta,\phi)$ in Eq. (2.57) can be obtained in the usual way [25] by writing

$$\begin{aligned}\mathbf{I}_n^{(1)}(\theta,\phi) &= -\frac{ik_n f_V}{2}\int d^3x'\{(\hat{\mathbf{r}}\cdot\mathbf{x}')\mathbf{J}_n(\mathbf{x}') + [\hat{\mathbf{r}}\cdot\mathbf{J}_n(\mathbf{x}')]\mathbf{x}' + [\mathbf{x}'\times\mathbf{J}_n(\mathbf{x}')]\times\hat{\mathbf{r}}\} \\ &= -ick_n\left[\mathbf{m}_n\times\hat{\mathbf{r}} - \frac{in\omega_0}{2c}\int d^3x'(\hat{\mathbf{r}}\cdot\mathbf{x}')\,\mathbf{x}'\,\rho_n(\mathbf{x}')\right], \end{aligned} \qquad (2.62)$$

where $\mathbf{m}_n$ is the Fourier component of the magnetic dipole moment $\mathbf{m}$ defined by

$$\mathbf{m}(t) = \frac{f_V}{2c}\int d^3x'\,\mathbf{x}'\times\mathbf{J}(\mathbf{x}',t). \qquad (2.63)$$

The expression for $\mathbf{I}_n^{(1)}$ in Eq. (2.62) can be expressed in terms of the (traceless) quadrupole moment tensor $\mathcal{Q}_{ij}$,

$$\mathcal{Q}_{ij}(t) = f_V\int d^3x'(3x_i'x_j' - r'^2\delta_{ij})\rho(\mathbf{x}',t), \qquad (2.64)$$

and the vector $\mathcal{Q}_i$ defined by contracting one of its indices with x^i/r:

$$\mathcal{Q}_i \equiv \mathcal{Q}_{ij}x^j/r. \qquad (2.65)$$

For our purposes, it is also convenient to define the mean-square-charge radius $\langle R^2\rangle$:

$$\langle R^2\rangle \equiv \frac{f_V}{Q}\int d^3x' r'^2\rho(\mathbf{x}'), \qquad (2.66)$$

where Q is the total charge

$$Q \equiv f_V\int d^3x'\rho(\mathbf{x}',t). \qquad (2.67)$$

With these definitions, $\mathbf{I}_n^{(1)}(\theta,\phi)$ may be written as

$$\mathbf{I}_n^{(1)}(\theta,\phi) = -ick_n\left[\mathbf{m}_n\times\hat{\mathbf{r}} - \frac{n\omega_0 k_n}{6c}\left(\boldsymbol{\mathcal{Q}}_n + Q\langle R^2\rangle_n\hat{\mathbf{r}}\right)\right], \qquad (2.68)$$

where $\mathcal{Q}_n$ and $\langle R^2\rangle_n$ are the Fourier components of $\mathcal{Q}$ and $\langle R^2\rangle$.

The time-averaged energy flux per unit solid angle can be obtained by combining Eq. (2.56) with the expressions for $\mathbf{I}_n^{(0)}$ in Eq. (2.59) and $\mathbf{I}_n^{(1)}(\theta,\phi)$ in Eq. (2.68). We find, to order $(k_n d)^2$,

$$\left\langle \frac{d\dot{E}}{d\Omega}\right\rangle = \frac{1}{2\pi c}\sum_{n>n_0}\left(\frac{n^2\omega_0^2}{c^2}\right)\sqrt{1-\left(\frac{n_0}{n}\right)^2}\left[\left|-in\omega_0\mathbf{p}_n\times\hat{\mathbf{r}} - ick_n(\mathbf{m}_n\times\hat{\mathbf{r}})\times\hat{\mathbf{r}} - \frac{n\omega_0 k_n}{6}\mathcal{Q}_n\times\hat{\mathbf{r}}\right|^2 + \left(\frac{n_0}{n}\right)^2\left|-in\omega_0\mathbf{p}_n\cdot\hat{\mathbf{r}} - \frac{n\omega_0 k_n}{6}\left(\mathcal{Q}_n\cdot\hat{\mathbf{r}} + Q\langle R^2\rangle_n\right)\right|^2\right]. \quad (2.69)$$

The total time-averaged power radiated $\langle \dot{E}\rangle = \int\langle d\dot{E}/d\Omega\rangle d\Omega$ can be obtained from Eq. (2.69) by using the identities

$$\frac{1}{r^2}\int x_i x_j d\Omega = \frac{4\pi}{3}\delta_{ij}, \quad (2.70a)$$

$$\frac{1}{r^4}\int x_i x_j x_k x_l d\Omega = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}). \quad (2.70b)$$

We can express $\langle \dot{E}\rangle$ in terms of the dominant electric and magnetic multipoles $\mathcal{E}1$, $\mathcal{M}1$, and $\mathcal{E}2$ as

$$\langle \dot{E}\rangle = \langle \dot{E}\rangle_{\mathcal{E}1} + \langle \dot{E}\rangle_{\mathcal{M}1} + \langle \dot{E}\rangle_{\mathcal{E}2} + O[(k_n d)^2], \quad (2.71)$$

where

$$\langle \dot{E}\rangle_{\mathcal{E}1} = \frac{4}{3c^3}\sum_{n>n_0}\left\{n^4\omega_0^4|\mathbf{p}_n|^2\sqrt{1-\left(\frac{n_0}{n}\right)^2}\left[1+\frac{1}{2}\left(\frac{n_0}{n}\right)^2\right]\right\}, \quad (2.72a)$$

$$\langle \dot{E}\rangle_{\mathcal{M}1} = \frac{4}{3c^3}\sum_{n>n_0} n^4\omega_0^4|\mathbf{m}_n|^2\left[1-\left(\frac{n_0}{n}\right)^2\right]^{3/2}, \quad (2.72b)$$

$$\langle \dot{E}\rangle_{\mathcal{E}2} = \frac{1}{90c^5}\sum_{n>n_0} n^6\omega_0^6\left[1-\left(\frac{n_0}{n}\right)^2\right]^{3/2}\left\{|(\mathcal{Q}_{ij})_n|^2 + \left(\frac{n_0}{n}\right)^2\left[\frac{4}{3}|(\mathcal{Q}_{ij})_n|^2 + \frac{2}{3}Q^2|\langle R^2\rangle_n|^2\right]\right\}. \quad (2.72c)$$

We thus see that for massive vector fields the $\mathcal{E}2$ contribution to $\langle \dot{E}\rangle$ cannot be calculated from a knowledge of the traceless quadrupole moment tensor $\mathcal{Q}_{ij}$ alone as in the massless case.

We conclude this section by verifying that the preceding results reduce to their electromagnetic counterparts when the photon is massless. Setting $n_0 = 0$, the power radiated for each multipole becomes

$$\langle \dot{E}\rangle_{\mathcal{E}1} = \frac{4}{3}\frac{1}{c^3}\sum_{n=1}^{\infty} n^4\omega_0^4|\mathbf{p}_n|^2, \quad (2.73a)$$

$$\langle \dot{E}\rangle_{\mathcal{M}1} = \frac{4}{3}\frac{1}{c^3}\sum_{n=1}^{\infty} n^4\omega_0^4|\mathbf{m}_n|^2, \quad (2.73b)$$

$$\langle \dot{E}\rangle_{\mathcal{E}2} = \frac{1}{90c^5}\sum_{n=1}^{\infty} n^6\omega_0^6|(\mathcal{Q}_{ij})_n|^2. \quad (2.73c)$$

One can then show that

$$\langle|\ddot{\mathbf{p}}|^2\rangle = 2\sum_{n=1}^{\infty} n^4\omega_0^4|\mathbf{p}_n|^2, \quad (2.74a)$$

$$\langle|\ddot{\mathbf{m}}|^2\rangle = 2\sum_{n=1}^{\infty} n^4\omega_0^4|\mathbf{m}_n|^2, \quad (2.74b)$$

$$\langle|\dddot{\mathcal{Q}}_{ij}|^2\rangle = 2\sum_{n=1}^{\infty} n^6\omega_0^6|(\mathcal{Q}_{ij})_n|^2. \quad (2.74c)$$

Hence the total time-averaged power radiated in the limit $\mu\to 0$ is

$$\langle \dot{E}\rangle_{\mu=0} = \frac{2}{3}\frac{\langle|\ddot{\mathbf{p}}|^2\rangle}{c^3} + \frac{2}{3}\frac{\langle|\ddot{\mathbf{m}}|^2\rangle}{c^3} + \frac{1}{180}\frac{\langle|\dddot{\mathcal{Q}}_{ij}|^2\rangle}{c^5}, \quad (2.75)$$

which agrees with the usual electromagnetic results [26].

III. MULTIPOLE RADIATION FOR MASSIVE SCALAR FIELDS

In this section we develop the formalism to describe radiation of massive scalar fields in a manner analogous to that for the vector field case considered previously. An elementary account of the properties of classical scalar fields has been given by Kahana and Coish [27], and the radiation problem for a scalar point source has been investigated previously by Cawley and Marx [17]. However, our approach will closely follow the method previously applied to the massive vector field.

We begin with the Lagrangian density $\mathcal{L}$ for a scalar field $\Phi(\mathbf{x},t)$ of mass m_γ interacting with a source ρ (in units of f_S),

$$\mathcal{L} = \frac{1}{8\pi}\left(\partial^\alpha\Phi\partial_\alpha\Phi - \mu^2\Phi^2\right) + f_S\rho\Phi \equiv \mathcal{L}_{\text{free}} + f_S\rho\Phi, \quad (3.1)$$

where $\mu = m_\gamma c/\hbar$ as before. Our scalar Lagrangian density Eq. (3.1), which differs from conventional treatments by a factor of $1/4\pi$, facilitates a comparison with the massive vector field results derived in the previous sections. From Eq. (3.1) the field equation for Φ is

$$\Box\Phi + \mu^2\Phi = 4\pi f_S \rho, \tag{3.2}$$

and the energy-momentum tensor $T^{\alpha\beta}$ obtained from Eq. (2.10) is

$$T^{\alpha\beta} = \frac{1}{4\pi}\partial^\alpha\Phi\partial^\beta\Phi - \frac{1}{8\pi}g^{\alpha\beta}(\partial_\gamma\Phi\partial^\gamma\Phi - \mu^2\Phi^2). \tag{3.3}$$

The energy density u and the ith component S^i of the energy flux $\mathbf{S}$ are then given by

$$u \equiv T^{00} = \frac{1}{8\pi}\left[\frac{1}{c^2}\left(\frac{\partial\Phi}{\partial t}\right)^2 + (\boldsymbol{\nabla}\Phi)^2 + \mu^2\Phi^2\right], \tag{3.4a}$$

$$S^i \equiv cT^{0i} = -\frac{1}{4\pi}\left(\frac{\partial\Phi}{\partial t}\right)\frac{\partial\Phi}{\partial x^i}. \tag{3.4b}$$

We note that the expression for S^i in Eq. (3.4b) is independent of μ so it has the same form as the massless case. However, S^i depends on μ implicitly through $\Phi(\mathbf{x},t)$. This contrasts with the case of the massive vector field where S^i given by Eq. (2.13b) depends explicitly on μ.

Returning to the field equation for $\Phi(\mathbf{x},t)$ in Eq. (3.2), we expand both the field $\Phi(\mathbf{x},t)$ and the source $\rho(\mathbf{x},t)$ in a Fourier series:

$$\Phi(\mathbf{x},t) = \sum_{n=-\infty}^{\infty}\Phi_n(\mathbf{x})e^{-in\omega_0 t}, \tag{3.5a}$$

$$\rho(\mathbf{x},t) = \sum_{n=-\infty}^{\infty}\rho_n(\mathbf{x})e^{-in\omega_0 t}, \tag{3.5b}$$

where $\omega_0 = 2\pi/T$ is the characteristic frequency of the system, and

$$\rho_n(\mathbf{x}) = \frac{1}{T}\int_0^T dt\,\rho(\mathbf{x},t)e^{in\omega_0 t}. \tag{3.6}$$

Combining Eqs. (3.2), (3.5a), and (3.5b) we see that each Fourier component satisfies

$$(\nabla^2 + k_n^2)\Phi_n(\mathbf{x}) = -4\pi f_S\rho_n(\mathbf{x}), \tag{3.7}$$

where k_n and n_0 are given by Eqs. (2.24) and (2.25). Equation (3.7) is identical in form to Eq. (2.23), and hence its general solution can be written as

$$\Phi_n(\mathbf{x}) = f_S\int d^3x'\frac{e^{ik_n|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\rho_n(\mathbf{x}') \equiv f_S\int d^3x' G_n(\mathbf{x},\mathbf{x}')\rho_n(\mathbf{x}'). \tag{3.8}$$

Combining Eqs. (3.8) and Eq. (3.5a), the solution for $\Phi(\mathbf{x},t)$ can be written as

$$\Phi(\mathbf{x},t) = f_S\sum_{n=-\infty}^{\infty}\int d^3x'\frac{\exp\left\{in\omega_0\left[\frac{|\mathbf{x}-\mathbf{x}'|}{c}\sqrt{1-\left(\frac{n_0}{n}\right)^2} - t\right]\right\}}{|\mathbf{x}-\mathbf{x}'|}\rho_n(\mathbf{x}'). \tag{3.9}$$

We can use Eq. (3.9) to identify the radiation fields by examining the behavior at distances far from the source. Combining Eqs. (2.36) and (3.9) we find that, when r is much greater than the characteristic size of the radiating system,

$$\Phi(\mathbf{x},t) = \sum_{n=-\infty}^{\infty} I_n(\theta,\phi)\frac{e^{i(k_n r - n\omega_0 t)}}{r} + O\left(\frac{1}{r^2}\right), \tag{3.10}$$

where

$$I_n(\theta,\phi) \equiv f_S\int d^3x' e^{-ik_n\hat{\mathbf{r}}\cdot\mathbf{x}'}\rho_n(\mathbf{x}'). \tag{3.11}$$

Since k_n in Eqs. (3.10) and (3.11) has the same form as in the massive case, it follows that, as before, only modes with $n > n_0$ will contribute to radiation. We then identify the radiation field $\Phi_{\rm rad}(\mathbf{x},t)$ as

$$\Phi_{\rm rad}(\mathbf{x},t) = \sum_{|n|>n_0} I_n(\theta,\phi)\frac{e^{i(k_n r - n\omega_0 t)}}{r}, \tag{3.12}$$

which has the form of an outwardly propagating spherical wave.

To calculate the instantaneous energy flux $\mathbf{S}$ we combine Eqs. (3.12) and (3.4b) and retain only terms $O(1/r^2)$:

$$\mathbf{S}_{\rm rad} = -\frac{1}{4\pi r^2}\sum_{|n|,|m|>n_0}(n\omega_0 k_m I_n I_m \hat{\mathbf{r}})e^{i(k_n r - n\omega_0 t)}e^{i(k_m r - m\omega_0 t)}. \tag{3.13}$$

From Eq. (3.13), the instantaneous energy flux per unit solid angle $d\dot{E}/d\Omega$ is given by

$$\frac{d\dot{E}}{d\Omega} = r^2(\hat{\mathbf{r}}\cdot\mathbf{S}_{\rm rad}) = -\frac{1}{4\pi}\sum_{|n|,|m|>n_0}(n\omega_0 k_m I_n I_m)e^{i(k_n r - n\omega_0 t)}e^{i(k_m r - m\omega_0 t)}. \tag{3.14}$$

Averaging the expression in Eq. (3.14) over time we find

$$\left\langle \frac{d\dot{E}}{d\Omega} \right\rangle \equiv \frac{1}{T}\int_0^T \frac{d\dot{E}}{d\Omega}dt = \sum_{n>n_0} \frac{n^2\omega_0^2}{2\pi c}\sqrt{1-\left(\frac{n_0}{n}\right)^2}|I_n|^2. \tag{3.15}$$

As in the case of the massive vector field, we can carry out a multipole expansion of $I_n(\theta,\phi)$ in the long wavelength limit. A new feature of scalar fields is the appearance of monopole radiation which arises because the radiating charge is not necessarily conserved; for a conserved charge the leading multipole is the dipole. Expanding the exponent in the integrand of Eq. (3.11) in powers of $k_n d$, where d is the size of the source, we find

$$\begin{aligned} I_n(\theta,\phi) &= f_S \int d^3x' \left[1 - ik_n\hat{\mathbf{r}}\cdot\mathbf{x}' - \frac{k_n^2}{2}(\hat{\mathbf{r}}\cdot\mathbf{x}')^2\right]\rho_n(\mathbf{x}') \\ &\quad + O[(k_n d)^3] \\ &= Q_n - ik_n\mathbf{p}_n\cdot\hat{\mathbf{r}} - \frac{k_n^2}{6}\left[\boldsymbol{\mathcal{Q}}_n\cdot\hat{\mathbf{r}} + \left(q\langle R^2\rangle\right)_n\right] \\ &\quad + O[(k_n d)^3]. \end{aligned} \tag{3.16}$$

In Eq. (3.16), $\mathbf{p}_n$ and $\boldsymbol{\mathcal{Q}}_n$ are the Fourier components of the electric dipole moment and the electric quadrupole moment vector defined in Eqs. (2.59) and (2.65), respectively and Q_n is Fourier component of the total charge. Since Q may also be a function of time, the Fourier component of $Q(t)\langle R^2\rangle$ has been written as $(Q\langle R^2\rangle)_n$. The time-averaged energy flux per unit solid angle is obtained by combining Eqs. (3.15) and (3.16) to yield

$$\begin{aligned} \left\langle \frac{d\dot{E}}{d\Omega} \right\rangle &= \sum_{n>n_0} \frac{n^2\omega_0^2}{2\pi c}\sqrt{1-\left(\frac{n_0}{n}\right)^2} \\ &\quad \times \left| Q_n - ik_n\mathbf{p}_n\cdot\hat{\mathbf{r}} - \frac{k_n^2}{6}\left[\boldsymbol{\mathcal{Q}}_n\cdot\hat{\mathbf{r}} + \left(Q\langle R^2\rangle\right)_n\right]\right|^2. \end{aligned} \tag{3.17}$$

This may be integrated in the same manner as for the massive vector field giving for the total-averaged power radiated:

$$\langle\dot{E}\rangle = \langle\dot{E}\rangle_{\mathcal{E}0} + \langle\dot{E}\rangle_{\mathcal{E}1} + \langle\dot{E}\rangle_{\mathcal{E}2}, \tag{3.18}$$

where

$$\langle\dot{E}\rangle_{\mathcal{E}0} = \frac{2}{c}\sum_{n>n_0} n^2\omega_0^2\sqrt{1-\left(\frac{n_0}{n}\right)^2}\,|Q_n|^2, \tag{3.19a}$$

$$\langle\dot{E}\rangle_{\mathcal{E}1} = \frac{2}{3c^3}\sum_{n>n_0} n^4\omega_0^4\left[1-\left(\frac{n_0}{n}\right)^2\right]^{3/2}\left\{|\mathbf{p}_n|^2 - \mathrm{Re}\left[Q_n^*(Q\langle R^2\rangle)_n\right]\right\}, \tag{3.19b}$$

$$\langle\dot{E}\rangle_{\mathcal{E}2} = \frac{1}{135c^5}\sum_{n>n_0} n^6\omega_0^6\left[1-\left(\frac{n_0}{n}\right)^2\right]^{5/2}\left[|(\mathcal{Q}_{ij})_n|^2 + \frac{15}{2}|(Q\langle R^2\rangle)_n|^2\right]. \tag{3.19c}$$

Since the total charge Q may be time dependent, there is a new contribution to the $\mathcal{E}1$ power in addition to the usual dipole term.

It is instructive to consider the power radiated by each multipole in the limit $\mu \to 0$. We have

$$\langle\dot{E}\rangle_{\mathcal{E}0} = \frac{2}{c}\sum_{n>n_0} n^2\omega_0^2|Q_n|^2, \tag{3.20a}$$

$$\langle\dot{E}\rangle_{\mathcal{E}1} = \frac{2}{3c^3}\sum_{n>n_0} n^4\omega_0^4\left\{|\mathbf{p}_n|^2 - \mathrm{Re}\left[Q_n^*(Q\langle R^2\rangle)_n\right]\right\}, \tag{3.20b}$$

$$\langle\dot{E}\rangle_{\mathcal{E}2} = \frac{1}{135c^5}\sum_{n>n_0} n^6\omega_0^6\left[|(\mathcal{Q}_{ij})_n|^2 + \tfrac{15}{2}|(Q\langle R^2\rangle)_n|^2\right]. \tag{3.20c}$$

One can show that

$$\langle|\dot{Q}|^2\rangle = 2\sum_{n=1}^{\infty} n^2\omega_0^2|Q_n|^2, \tag{3.21a}$$

$$\left\langle \dot{Q}\frac{d}{dt}\left[Q(t)\langle R^2\rangle\right]\right\rangle = 2\sum_{n=1}^{\infty} n^2\omega_0^2 Re\left[Q_n^*(Q\langle R^2\rangle)_n\right], \tag{3.21b}$$

$$\left\langle \left|\frac{d^3}{dt^3}\left[Q(t)\langle R^2\rangle\right]\right|^2\right\rangle = 2\sum_{n=1}^{\infty} n^6\omega_0^6\left|(Q\langle R^2\rangle)_n\right|^2, \tag{3.21c}$$

which along with Eqs. (2.74a) and (2.74c) leads to

$$\langle\dot{E}\rangle_{\mu=0} = \frac{\langle|\dot{Q}(t)|^2\rangle}{c} + \frac{1}{3c^3}\left[\langle|\ddot{\mathbf{p}}(t)|^2\rangle - \left\langle \dot{Q}\frac{d}{dt}\left[Q(t)\langle R^2\rangle\right]\right\rangle\right] + \frac{1}{270c^5}\left[\langle|\dddot{\mathcal{Q}}_{ij}|^2\rangle + \frac{15}{2}\left\langle\left|\frac{d^3}{dt^3}\left[Q(t)\langle R^2\rangle\right]\right|^2\right\rangle\right]. \tag{3.22}$$

As noted earlier, if the total charge Q of a system is conserved, and hence is time independent, then no monopole radiation will be emitted and the dipole term reduces to its familiar form. In addition, even for massless scalar radiation, the quadrupole radiation may not be completely described by a traceless quadrupole moment tensor as in electromagnetism.

IV. APPLICATION TO BINARY PULSAR SYSTEMS

A. Overview

In the preceding sections we have developed a formalism for describing the radiation of massive vector

TABLE I. Some of the measured orbital parameters of PSR 1913+16. The uncertainties in the last digits are quoted in parentheses.

Parameter	Value	Reference
Orbital period P_b (s)	27906.9807804(6)	[33]
Eccentricity ϵ	0.6171308(4)	[33]
Advance of periastron $\dot{\omega}$ (degrees/yr)	4.226621(11)	[33]
Time dilation γ (ms)	4.295(2)	[33]
Orbital period derivative $\dot{P}_b$ (10^{-12} s s^{-1})	−2.422(6)	[33]
Pulsar mass m_p (solar masses)	1.442(3)	[11]
Companion mass m_c (solar masses)	1.386(3)	[11]

and scalar fields which parallels that for electromagnetism. In principle, any system carrying charges associated with these fields would radiate when accelerated, just as in electromagnetism. However, present limits on new intermediate-range interactions indicate that should they exist, their coupling strength will be much smaller than for gravity. Hence, the only existing systems in which observable effects of radiation damping due to new forces might be detected are binary pulsar systems, where evidence for gravitational radiation has been seen [5,18,19]. Although the new forces may be inherently weaker than gravity, the leading multipole radiated is monopole or dipole, whereas the lowest multipole for gravitational radiation is quadrupole. Hence the weakness of the new interaction may be at least partially compensated by the fact that the radiation is dominated by a lower multipole than gravity. It is this circumstance which allows us to set interesting limits on possible new fields by studying the radiation from macroscopic bodies.

Presently, the binary pulsar system PSR 1913+16 has become the premier astrophysical laboratory for testing relativistic gravity [11,28]. Since its discovery nearly 30 years ago, timing models fitting the observed pulses have allowed the accurate measurement of most of the system's parameters (see Table I). These measurements have provided a clean test of general relativity in a moderately relativistic system, and the gradual decrease in the system's orbital period is the first clear evidence for gravitational radiation. Recent measurements have become sufficiently accurate to detect the effects of galactic rotation on the time derivative of the orbital period as measured by an Earth observer (Table II) [32,33].

It has also been realized that this system might be used to constrain putative new forces [5,18,19,29,30]. We now apply the radiation formalism developed in previous sections to quantitatively determine the effects of radiation damping on the time derivative of the orbital period for the binary system. Comparison of these results with observation then leads to radiation limits on the coupling strengths of these new forces.

B. Radiation damping and the orbital period derivative

If the pulsar and its companion carry charges Q_p and Q_c which are the sources of a massive vector or scalar field, then the binary system would radiate energy in addition to the usual gravitational radiation. This radiation manifests itself as a secular decrease in the system's orbital period P_b. The rate at which the orbital period changes $\dot{P}_b$ (averaged over one orbital period) is related to the total time-averaged power $\langle \dot{E}_{\rm tot}\rangle$ radiated by the system by [31]

$$\frac{\dot{P}_b}{P_b} = -\frac{3}{2}\frac{\langle \dot{E}_{\rm tot}\rangle}{E_{\rm tot}}, \tag{4.1}$$

where $E_{\rm tot}$ is the total energy of the system. However, we noted earlier that galactic rotation will affect $\dot{P}_b^{\rm obs}$ measured by an observer on the Earth. To isolate the effects due purely to radiation damping we rewrite Eq. (4.1) as

$$\frac{\dot{P}_b^{\rm obs-gal}}{P_b} = -\frac{3}{2}\frac{\langle \dot{E}_{\rm tot}\rangle}{E_{\rm tot}}, \tag{4.2}$$

where $\dot{P}_b^{\rm obs-gal}$ is obtained by subtracting nonradiative effects due to the galactic rotation from the observed orbital period derivative.

The total time-averaged radiated power can be expressed as the sum of the power radiated by gravitational radiation $\langle \dot{E}_{\rm GR}\rangle$, and that radiated by other fields $\langle \dot{E}_X\rangle$:

$$\langle \dot{E}_{\rm tot}\rangle = \langle \dot{E}_{\rm GR}\rangle + \langle \dot{E}_X\rangle, \tag{4.3}$$

where $\langle \dot{E}_{\rm GR}\rangle$ is given by [11,36]

TABLE II. The orbital period derivative for PSR 1913+16: Observation versus theory [33]. The uncertainties in the last digits are quoted in parentheses.

Parameter	Value (10^{-12} s s^{-1})
Observed value $\dot{P}_b^{\rm obs}$	−2.4225(56)
Galactic contribution $\dot{P}_b^{\rm gal}$	−0.0124(64)
Intrinsic orbital period decay $\dot{P}_b^{\rm obs-gal} = \dot{P}_b^{\rm obs} - \dot{P}_b^{\rm gal}$	−2.4101(85)
General relativistic prediction $\dot{P}_b^{\rm GR}$	−2.4025(1)
$\dot{P}_b^{\rm obs-gal}/\dot{P}_b^{\rm GR}$	1.0032(35)

$$\langle \dot{E}_{\rm GR}\rangle = -\frac{192\pi}{5c^5}\left(\frac{2\pi G}{P_b}\right)^{5/3} m_1 m_2 (m_1+m_2)^{-1/3} g_{\rm GR}(\epsilon). \tag{4.4}$$

Here ϵ is the orbital eccentricity and

$$g_{\rm GR}(\epsilon) = \frac{1+(73/24)\epsilon^2+(37/96)\epsilon^4}{(1-\epsilon^2)^{7/2}}. \tag{4.5}$$

Using Eq. (4.2), the ratio of the change in the orbital period due to these other sources of energy loss $\dot{P}_b^X$, to the total period change $\dot{P}_b$ is (to first order in $\langle \dot{E}_X\rangle/\langle \dot{E}_{\rm GR}\rangle$)

$$\frac{\dot{P}_b^X}{\dot{P}_b} = 1 - \frac{\dot{P}_b^{\rm GR}}{\dot{P}_b^{\rm obs-gal}} = \frac{\langle \dot{E}_X\rangle}{\langle \dot{E}_{\rm GR}\rangle}, \tag{4.6}$$

where

$$\frac{\dot{P}_b^X}{P_b} \equiv -\frac{3}{2}\frac{\langle \dot{E}_X\rangle}{E_{\rm tot}}. \tag{4.7}$$

After effects due to the relative motion between the solar system and pulsar are included, present observations indicate that gravitational radiation accounts for nearly all of the effects on the orbital period derivative (Table II) [33]:

$$\frac{\dot{P}_b^{\rm GR}}{\dot{P}_b^{\rm obs-gal}} = 0.9968 \pm 0.0035. \tag{4.8}$$

Hence from Eq. (4.6) we have

$$\frac{\langle \dot{E}_X\rangle}{\langle \dot{E}_{\rm GR}\rangle} = 0.0032 \pm 0.0035. \tag{4.9}$$

C. Radiation limits on new weak forces

Using the results from the previous sections we can calculate the contribution to the total radiation of our putative massive vector and scalar fields. Since the characteristic speeds in the pulsar system are of order $v/c \sim 10^{-3}$, the long wavelength approximation is valid. If we assume that the total charge of the system is constant, so that the lowest radiated multipole is dipole, then the total power is given by Eq. (2.72a) for massive vector fields, and by Eq. (3.19b) for massive scalar fields. Both of these formulas depend on the Fourier component $\mathbf{p}_n$ of the dipole moment for a system of two bodies orbiting under the influence of an inverse square force law [34]:

$$\mathbf{p}_n = \frac{m_p m_c}{m_p+m_c}\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)\frac{a}{n} \times\left[\mathcal{J}_n'(n\epsilon)\hat{\mathbf{x}} + i\frac{\sqrt{1-\epsilon^2}}{\epsilon}\mathcal{J}_n(n\epsilon)\hat{\mathbf{y}}\right]. \tag{4.10}$$

Here m_p and m_c are the masses of the pulsar and its companion, a is the semimajor axis of the relative orbit, ϵ is the orbital eccentricity, $\mathcal{J}_n(x)$ is the nth order Bessel function, and the prime on the Bessel function denotes differentiation with respect to its argument. In deriving Eq. (4.10) it is assumed that the interaction between the two bodies via this new field is much weaker than gravity, so as to not substantially affect the purely gravitational orbital trajectory. The time-averaged power radiated can then be found by inserting Eq. (4.10) into Eqs. (2.72a) and (3.19b), respectively:

$$\langle \dot{E}_V\rangle = \frac{2}{3}\left(\frac{m_p m_c}{m_p+m_c}\right)^2\left(\frac{a^2\omega_0^4}{c^3}\right) g_V(\mu,\epsilon)\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \tag{4.11a}$$

$$\langle \dot{E}_S\rangle = \frac{1}{3}\left(\frac{m_p m_c}{m_p+m_c}\right)^2\left(\frac{a^2\omega_0^4}{c^3}\right) g_S(\mu,\epsilon)\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \tag{4.11b}$$

where

$$g_V(\mu,\epsilon) \equiv \sum_{n>n_0} 2n^2\left[\mathcal{J}_n'^2(n\epsilon)+\left(\frac{1-\epsilon^2}{\epsilon^2}\right)\mathcal{J}_n^2(n\epsilon)\right] \times\left\{\sqrt{1-\left(\frac{n_0}{n}\right)^2}\left[1+\frac{1}{2}\left(\frac{n_0}{n}\right)^2\right]\right\}, \tag{4.12a}$$

$$g_S(\mu,\epsilon) \equiv \sum_{n>n_0} 2n^2\left[\mathcal{J}_n'^2(n\epsilon)+\left(\frac{1-\epsilon^2}{\epsilon^2}\right)\mathcal{J}_n^2(n\epsilon)\right] \times\left[1-\left(\frac{n_0}{n}\right)^2\right]^{3/2}, \tag{4.12b}$$

and

$$n_0 = \frac{cP_b}{2\pi\lambda}. \tag{4.13}$$

Since the total energy $E_{\rm tot}$ is given to lowest order by it usual Newtonian form

$$E_{\rm tot} = -\frac{Gm_p m_c}{2a}, \tag{4.14}$$

then the orbital period derivative due the radiation of vector or scalar fields is found by combining Eqs. (4.14) and (4.7) with Eq. (4.11a) or Eq. (4.11b):

$$\dot{P}_b^V = -\frac{32 m_p m_c}{G(m_p+m_c)^2}\left(\frac{\pi^4 a^3}{c^3 P_b^3}\right) g_V(\mu,\epsilon)\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \tag{4.15a}$$

$$\dot{P}_b^S = -\frac{16 m_p m_c}{G(m_p+m_c)^2}\left(\frac{\pi^4 a^3}{c^3 P_b^3}\right) g_S(\mu,\epsilon)\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \tag{4.15b}$$

where we have used $\omega_0 = 2\pi/P_b$.

Before continuing, we discuss some of the general features of these results. First, for dipole radiation, the difference of the charge-to-mass ratios,

$$\Delta\left(\frac{Q}{m}\right) \equiv \frac{Q_p}{m_p}-\frac{Q_c}{m_c}, \tag{4.16}$$

appears. It follows that in highly symmetric binary systems, where $\Delta(Q/m)$ can be very small, dipole radiation would be suppressed and the dominant radiation multipole would be electric quadrupole (magnetic dipole radiation vanishes for a system containing only two bodies [26]). Secondly significant radiation will occur only if $n_0 \equiv m_\gamma c^2/\hbar\omega_0 \lesssim 1$ or, using Eq. (4.13), if the range λ

of the new interaction satisfies $\lambda = 1/\mu \lesssim cP_b/2\pi$. For the binary pulsar PSR 1913+16, $P_b = 27\,907$ s so it will radiate significantly for $\lambda \gtrsim 10^{12}$ m$\sim 10^3 a$. More generally, a binary system will produce substantial radiation for massive fields only if the range of the interaction is much larger than the characteristic size of the system.

To obtain practical expressions that can be used to constrain the parameters of new forces, we return to Eq. (4.6) where the ratio $\langle \dot{E}_X\rangle/\langle \dot{E}_{\rm GR}\rangle$ appears. By combining Eq. (4.4) with Eqs. (4.11a) and (4.11b) we obtain the ratios

$$\frac{\langle \dot{E}_V\rangle}{\langle \dot{E}_{\rm GR}\rangle} = \frac{5}{48}\frac{c^2 g_V(\mu,\epsilon)}{G g_{\rm GR}(\epsilon)}\left[\frac{P_b}{2\pi G(m_p+m_c)}\right]^{2/3}\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \qquad (4.17a)$$

$$\frac{\langle \dot{E}_S\rangle}{\langle \dot{E}_{\rm GR}\rangle} = \frac{5}{96}\frac{c^2 g_S(\mu,\epsilon)}{G g_{\rm GR}(\epsilon)}\left[\frac{P_b}{2\pi G(m_p+m_c)}\right]^{2/3}\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \qquad (4.17b)$$

where Kepler's third Law has been used to eliminate the dependence upon a. Equations (4.17a) and (4.17b) can be simplified even further by first noting that the rate of periastron advance $\dot{\omega}_{\rm GR}$ predicted by general relativity can be written as

$$\dot{\omega}_{\rm GR} = \frac{6\pi G(m_p+m_c)}{c^2 P_b a(1-\epsilon^2)} = \frac{6\pi}{c^2 P_b(1-\epsilon^2)}\left[\frac{2\pi G(m_p+m_c)}{P_b}\right]^{2/3}. \qquad (4.18)$$

Combining Eq. (4.18) with Eqs. (4.17a) and (4.17b) we find

$$\frac{\langle \dot{E}_V\rangle}{\langle \dot{E}_{\rm GR}\rangle} = \frac{5\pi}{8}\frac{g_V(\mu,\epsilon)}{G\dot{\omega}_{\rm GR}P_b g_{\rm GR}(\epsilon)(1-\epsilon^2)}\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2, \qquad (4.19a)$$

$$\frac{\langle \dot{E}_S\rangle}{\langle \dot{E}_{\rm GR}\rangle} = \frac{5\pi}{16}\frac{g_S(\mu,\epsilon)}{G\dot{\omega}_{\rm GR}P_b g_{\rm GR}(\epsilon)(1-\epsilon^2)}\left(\frac{Q_p}{m_p}-\frac{Q_c}{m_c}\right)^2. \qquad (4.19b)$$

It is convenient to rewrite the charge-to-mass ratio Q_i/m_i in terms of dimensionless quantities. Using the dimensionless interaction strength ξ, defined by

$$\xi \equiv \frac{f^2}{G m_H^2}, \qquad (4.20)$$

where f is the coupling strength of the new force and $m_H = m({}_1\mathrm{H}^1) = 1.00782519(8)u$ is the mass of hydrogen, one obtains

$$\frac{Q_i}{\sqrt{G}m_i} = \sqrt{\xi}\frac{N_i}{\mu_i}. \qquad (4.21)$$

In Eq. (4.21) $N_i = Q_i/f$ and $\mu_i = m_i/m_H$ is the mass in units of the mass of a hydrogen atom. Combining Eq. (4.21) with Eqs. (4.19a) and (4.19b) gives

$$\frac{\langle \dot{E}_V\rangle}{\langle \dot{E}_{\rm GR}\rangle} = \frac{5\pi}{8}\frac{g_V(\mu,\epsilon)}{\dot{\omega}_{\rm GR}P_b g_{\rm GR}(\epsilon)(1-\epsilon^2)}\xi_V\left(\frac{N_p}{\mu_p}-\frac{N_c}{\mu_c}\right)^2, \qquad (4.22a)$$

$$\frac{\langle \dot{E}_S\rangle}{\langle \dot{E}_{\rm GR}\rangle} = \frac{5\pi}{16}\frac{g_S(\mu,\epsilon)}{\dot{\omega}_{\rm GR}P_b g_{\rm GR}(\epsilon)(1-\epsilon^2)}\xi_S\left(\frac{N_p}{\mu_p}-\frac{N_c}{\mu_c}\right)^2, \qquad (4.22b)$$

where ξ_V and ξ_S are the dimensionless coupling strengths for vector and scalar fields. The dependence upon ξ and N_i in Eqs. (4.22a) and (4.22b) can be isolated and, using Eq. (4.6), takes the form

$$\xi_V\left(\frac{N_p}{\mu_p}-\frac{N_c}{\mu_c}\right)^2 \le \frac{8}{5\pi}\dot{\omega}_{\rm GR}P_b(1-\epsilon^2)\left[\frac{g_{\rm GR}(\epsilon)}{g_V(\mu,\epsilon)}\right]\left(1-\frac{\dot{P}_b^{\rm GR}}{\dot{P}_b^{\rm obs-gal}}\right), \qquad (4.23a)$$

$$\xi_S\left(\frac{N_p}{\mu_p}-\frac{N_c}{\mu_c}\right)^2 \le \frac{16}{5\pi}\dot{\omega}_{\rm GR}P_b(1-\epsilon^2)\left[\frac{g_{\rm GR}(\epsilon)}{g_S(\mu,\epsilon)}\right]\left(1-\frac{\dot{P}_b^{\rm GR}}{\dot{P}_b^{\rm obs-gal}}\right), \qquad (4.23b)$$

where the inequality has been introduced to accommodate the possibility that more than one force contributes to the radiated power. For infinite-ranged fields ($\mu = 0$),

$$g_V(\mu=0,\epsilon) = g_S(\mu=0,\epsilon) \equiv g_X(\epsilon), \qquad (4.24)$$

where [36]

$$g_X(\epsilon) = \sum_{n=1}^{\infty} 2n^2\left[\mathcal{J}_n'^2(n\epsilon) + \left(\frac{1-\epsilon^2}{\epsilon^2}\right)\mathcal{J}_n^2(n\epsilon)\right] = \frac{2+\epsilon^2}{2(1-\epsilon^2)^{5/2}}. \qquad (4.25)$$

These results are applicable to any binary system. If we now restrict our attention to PSR 1913+16, then using the results from Table I we have $\dot{\omega} = 8.42\times 10^{-7}$ rad/s, $P_b = 27{,}907$ s, $\epsilon = 0.6171$, $g_{\rm GR}(\epsilon) = 11.9$, $\mu_p = 1.71\times 10^{57}$, and $\mu_c = 1.65\times 10^{57}$. Since the effects of new forces on the periastron advance are expected to be small, it is sufficient for the accuracy needed to set $\dot{\omega}_{\rm GR} \simeq \dot{\omega}$. Finally, we have seen in the previous section that $\dot{P}_b^{GR}/\dot{P}_b^{\rm obs-gal}$ is roughly unity within present uncertainties, and hence, it is convenient to write

$$\frac{\dot{P}_b^{\rm GR}}{\dot{P}_b^{\rm obs-gal}} \equiv 1-\delta = 1-(0.0032\pm 0.0035). \qquad (4.26)$$

The constraints imposed on new weak forces for the binary pulsar PSR 1913+16 can be expressed in the form

$$\xi_V \left(\frac{N_p}{\mu_p} - \frac{N_c}{\mu_c}\right)^2 \leq \frac{2.45 \times 10^{-4}}{g_V(\mu, \epsilon = 0.6171)}\delta, \quad (4.27a)$$

$$\xi_S \left(\frac{N_p}{\mu_p} - \frac{N_c}{\mu_c}\right)^2 \leq \frac{4.90 \times 10^{-4}}{g_S(\mu, \epsilon = 0.6171)}\delta. \quad (4.27b)$$

For massless fields, Eq. (4.25) gives $g_X(\epsilon) = 3.95$ which leads to the constraints

$$\lambda \to \infty : \quad \xi_V \left(\frac{N_p}{\mu_p} - \frac{N_c}{\mu_c}\right)^2 \leq 6.20 \times 10^{-5}\delta, \quad (4.28a)$$

$$\lambda \to \infty : \quad \xi_S \left(\frac{N_p}{\mu_p} - \frac{N_c}{\mu_c}\right)^2 \leq 1.24 \times 10^{-4}\delta. \quad (4.28b)$$

The numerical values for μ_p and μ_c have not been inserted, since in certain applications the ratio N_i/μ_i is more easily calculated than N_i or μ_i alone. We now apply these results to several interesting cases.

D. Electromagnetic radiation from the binary pulsar

If the pulsar and/or its companion have a net electric charge, then the binary system could be a source of electromagnetic (as well as gravitational) radiation. Since the observed rate of energy loss agrees well with that predicted by general relativity, we can use the results in Eqs. (4.8) and (4.9) to set limits on the charge carried by the pulsar and its companion. In the electromagnetic case, the range of the force is infinite and the fundamental unit of charge is the electronic charge $f_V = e$ (in Gaussian units), so the dimensionless coupling constant $\xi_{\rm EM}$ is known:

$$\xi_{\rm EM} = \frac{e^2}{G m_H^2} = 1.23 \times 10^{36}. \quad (4.29)$$

The unknown quantities are the numbers of charges, $N_p = Z_p$ and $N_c = Z_c$, on the pulsar and its companion. Let us consider, for example, a scenario in which the pulsar and companion carry equal and opposite charges,

$$Z \equiv Z_p = -Z_c, \quad (4.30)$$

so that the net charge of the binary system is zero. Since electrodynamics is a vector interaction, Eqs. (4.29) and (4.30) can be combined with the infinite-range vector constraint Eq. (4.28a) and the numerical values for μ_p and μ_c to obtain the radiation constraint on Z:

$$Z \leq 5.9 \times 10^{36}\delta^{1/2}. \quad (4.31)$$

Using the present value of δ given by Eq. (4.26) we find

$$Z \lesssim 10^{36}. \quad (4.32)$$

We note that Eq. (4.32) is consistent with other constraints on the electric charge carried by astrophysical bodies. If the charge were too large, the resulting electric field could produce enough electron-positron pairs to neutralize the object even in a vacuum. Hanni [37] has found that for neutron stars, this vacuum polarization effect implies

$$Z \lesssim 10^{35} \quad (4.33)$$

which is somewhat more restrictive than our radiation constraint Eq. (4.32). There is, however, a constraint which is more stringent than either (4.32) or (4.33). This follows from the observation that a charged astrophysical body cannot possess a substantial electric charge for a long period of time since it would accrete oppositely charged matter from the surrounding plasma [38]. To obtain an estimate of the limits on Z implied by this argument, consider an object of mass M and charge $Q = eZ$ surrounded by a plasma containing positive and negative charges. If Z is too large, so that the net electrostatic force dominates over gravity, the star will selectively accrete particles of opposite charge, thereby neutralizing itself. If, on the other hand, the gravitational force is dominant, then the matter accreted would be nearly neutral since gravity attracts positive and negative charges equally. For gravity to dominate over Coulomb forces, and thus prevent the star from neutralizing itself, we must have

$$\frac{eZq}{r^2} \ll \frac{GMm}{r^2}, \quad (4.34)$$

where q is the charge of a typical particle in the interstellar medium and m is its mass. For an electron, the particle with the largest charge-to-mass ratio q/m, Eq. (4.34) leads to the constraint [39]

$$Z \ll 5.2 \times 10^{17} \left(\frac{M}{M_\odot}\right). \quad (4.35)$$

For PSR 1913+16 this gives

$$Z \lesssim 10^{17}, \quad (4.36)$$

which is substantially more stringent than the other constraints. It should be emphasized, however, that Eqs. (4.33) and (4.36) are theoretical bounds, while Eq. (4.32) results from direct observations. Finally, the constraint given by Eq. (4.32) can be shown to be consistent with the assumption that the orbital motion does not deviate significantly from that determined by gravity alone.

While the radiation limit obtained by Eq. (4.32) is not as restrictive as the others described, this value will continually improve with observation time. Furthermore, PSR 1913+16 is only the first binary pulsar system to be observed long enough to obtain useful limits. It is not difficult to envision other binary systems whose orbital parameters would allow which much tighter limits can be set.

E. Radiation from vector and scalar fields coupled to baryon number

Another interaction that can be constrained by the binary pulsar is that of a vector field coupled to baryon number B. Such a force with infinite range was first proposed by Lee and Yang [40], while a short-ranged ($\sim$ 10–

10^3 m) baryonic interaction (the "fifth force") has been considered recently by a number of authors [1–3,5,9], In Eqs. (4.27a)—(4.28b) let $N_p = B_p$ and $N_c = B_c$ be the baryon numbers of the pulsar and its companion, and we wish to constrain the dimensionless coupling constant ξ_5 associated with this interaction. Although the binary pulsar is not useful in setting limits on interactions such as the proposed "fifth force" which have short ranges, we will investigate the constraints for ranges much larger than the dimensions of the pulsar system as in the original Lee-Yang theory.

For astrophysical bodies, it is convenient to define the baryon mass M_A by

$$M_A \equiv B m_H \tag{4.37}$$

which is simply the mass of the constituent particles. M_A is related to the "gravitational mass" M of the body (the actual mass determined by a gravitational measurement) by

$$M = M_A - \frac{E_{\text{binding}}}{c^2}, \tag{4.38}$$

where E_{binding} is the absolute value of the binding energy of the object. In terms of M_A we have

$$\frac{B}{\mu} = \frac{M_A/m_H}{M/m_H} = \frac{M_A}{M}, \tag{4.39}$$

which then allows one to express the difference in charge-to-mass ratios of the pulsar and companion as

$$\Delta\left(\frac{B}{\mu}\right) \equiv \left(\frac{B_p}{\mu_p}\right) - \left(\frac{B_c}{\mu_c}\right) = \left(\frac{M_{Ap}}{m_p}\right) - \left(\frac{M_{Ac}}{m_c}\right). \tag{4.40}$$

Although one cannot determine M_A/M precisely for the pulsar or its companion, existing models yield equations of state that can give reasonable estimates. For example, interpolating the values of M_A for most of the models given by Arnett and Bowers [41], one finds for the binary pulsar system that typically $B/\mu \simeq 1.1$ (i.e., the binding energy is about 10% of the total mass), and that $\Delta(B/\mu)$ lies in the range 0.004–0.008. Lattimer and Yahil [42] have found for the neutron star models they consider, that for $M > M_\odot$, the binding energy is related to the mass by

$$E_{\text{binding}} \simeq 1.5 \times 10^{53} \left(\frac{M}{M_\odot}\right) \text{ ergs.} \tag{4.41}$$

Combining Eqs (4.38), (4.40), and (4.41) we find

$$\Delta\left(\frac{B}{\mu}\right) \simeq 0.084 \left[\left(\frac{m_p}{M_\odot}\right) - \left(\frac{m_c}{M_\odot}\right)\right] = 0.005, \tag{4.42}$$

which is consistent with the results obtained from Arnett and Bowers.

Equation (4.42) highlights an important feature of binary systems such as PSR 1913+16, where the pulsar and its companion may be similar in mass and composition. In such cases the difference in charge-to-mass ratios can be relatively small, which reduces the radiated power from such sources and hence their sensitivity to new forces. By contrast, the difference in charge-to-mass ratios would be significantly larger if the companion were a white dwarf or a black hole. In the former case, its gravitational binding energy would be negligible ($B_c/\mu_c \sim 1$), which gives $\Delta(B/\mu) \simeq 0.1$, a factor of 20 larger than for a neutron star companion. If the companion were a black hole, then the implication of various "no hair" theorems [43,44] is that $B_c = 0$, and hence $\Delta(B/\mu) \simeq 1.1$ for a long- (but finite-) range field coupled to baryon number. Since the limits on the coupling strength ξ_V (or ξ_S) in Eq. (4.27a) are proportional to $[\Delta(B/\mu)]^2$, the sensitivity of a binary system increases dramatically when the pulsar and its companion are more dissimilar, as we see from Table III.

To obtain numerical limits on the dimensionless coupling strength ξ_5 we apply Eqs. (4.27a) and (4.28a) using the above results for $\Delta(B/\mu)$. However, since $g_V(\mu,\epsilon) \leq g_V(\mu=0,\epsilon)$, the most stringent limits come from the massless, infinite-range interaction, and hence these are the constraints that are tabulated in Table III for three possible pulsar companions. The radiation limits for an infinite-range force coupled to baryon number are still much weaker than those obtained by other methods. For example, torsion balance experiments by Adelberger *et al.* [45] using the Earth as a source have found $\xi_5 \lesssim 10^{-8}$, while Braginskii and Panov [46] performing a similar experiment using the Sun as the source found $\xi_5 \lesssim 10^{-9}$ [45]. If the range of the interaction is finite, then the radiation limits obtained by the pulsar will be weaker still than those obtained by more conventional experiments.

The results for the vector field case can be taken over to set limits on the dimensionless coupling ξ_5 of a scalar field coupling (approximately) to baryon number. In the limit $\mu \to \infty$, we have seen in Eq. (4.24) that $g_V \simeq g_S$, so it follows from Eqs. (4.27a) and (4.27b) that the limits on a scalar coupling to baryon number differ from the vector coupling by a factor of 2 (Table III).

V. CONCLUSIONS AND OUTLOOK

In this paper a formalism for computing multipole radiation for massive vector and scalar fields has been developed. In addition to confirming some results of earlier workers, we have developed for the first time relatively simple formulas for monopole, dipole, and quadrupole radiation. As we have shown, one application of this formalism is to obtain constraints on new weak forces from

TABLE III. The constraints on the massless baryon coupling constant ξ_5 for vector and scalar couplings of the three possible companion objects have been calculated using Eq. (4.28) and the present 2σ value $\delta \simeq 10^{-2}$.

Companion	$\Delta(B/\mu)$	$\xi_{5,\max}$ (vector)	$\xi_{5,\max}$ (scalar)
Neutron star	0.005	2.5×10^{-2}	1.3×10^{-2}
White dwarf	0.1	6.2×10^{-5}	3.1×10^{-5}
Black hole	1.1	5.1×10^{-7}	2.6×10^{-7}

radiation by binary pulsar systems. Unfortunately the best known binary system, PSR 1913+16, has limited sensitivity to new forces if the companion is also a neutron star. At the same time our results also demonstrate that for less symmetric binary systems (e.g., neutron star-black hole) very stringent limits can be obtained. We have also emphasized that the binary systems will provide ever-refined limits on new forces as data accumulate over time. It is also worth noting that we can invert the line of reasoning we have been pursuing to infer from Table III that the radiation rate expected from new forces cannot upset the existing agreement between observed and expected rates for gravitational radiation. This follows by observing that the limits for various new fields that are obtained from laboratory experiments [1–5] are more stringent then those appearing in Table III. Hence if we were to combine these results with our formalism, the rate of energy loss via new fields would be smaller then the present uncertainties in the comparision of theory and experiment for gravitational radiation.

As new binary systems are discovered, and data on existing binaries accumulate, the radiation losses into possible new fields will provide an ever more refined tool for setting limits on the couplings of new forces. Eventually these could complement those obtained from laboratory experiments in a significant way or, perhaps, even exceed them.

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