Exact results on the space of vacua of four-dimensional SUSY gauge theories

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We consider four-dimensional quantum field theories which have a continuous manifold of inequivalent exact ground states—a moduli space of vacua. Classically, the singular points on the moduli space are associated with extra massless particles. Quantum mechanically these singularities can be smoothed out. Alternatively, new massless states appear there. These may be the elementary massless particles or new massless bound states.

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I. INTRODUCTION

Supersymmetric quantum field theories are easier to analyze and are more tractable than nonsupersymmetric theories. The main tool which makes them simple are the constraints which follow from supersymmetry. In particular, the holomorphicity of the superpotential when combined with global symmetries enables one to find many exact results. For example, recently it has been shown that both the perturbative nonrenormalization theorems and a new nonperturbative generalization of them are simple consequences of these principles [1]. Other exact results about the superpotential which do not follow trivially from the symmetries will be presented in [2].

One application of these theories is for dynamical supersymmetry (SUSY) breaking. We will have nothing new to say about it here. Instead, we will focus on another application. Since some observables in these theories are exactly calculable, these theories are interesting arenas for the study of dynamical effects in strongly coupled four-dimensional quantum field theories. For example, here we will present theories which exhibit surprising patterns of chiral symmetry breaking and massless bound states and will argue that some theories have a nontrivial critical behavior associated with massless interacting gluons.

Many of these theories have classical flat directions and hence the classical theory has a space of inequivalent ground states. We will refer to this space as the "classical moduli space." It is singular at the points where the number of massless fields is increased. The degeneracy between these states cannot be lifted perturbatively. In some cases nonperturbative effects generate a superpotential on the space which destabilizes these vacua. In other cases, no superpotential is generated and the vacuum degeneracy cannot be lifted. Then it is of interest to study the quantum moduli space. Of particular interest is the fate of the singularities on the classical space.

This situation is analogous to two different problems in string theory. First, classical string theory has moduli spaces which can be studied in the α' (large radius) expansion. It is known that the classical (in the sigma model sense) moduli space is different than the quantum one. World sheet instantons modify the space, can change its topology and can even connect it to a different space. The second stringy analogue of our moduli spaces are in the space time interpretation. Some of the string moduli spaces might be exactly stable in the full quantum string theory (for example, if there are several unbroken supersymmetries in space time). Then, it is interesting to know how the classical moduli space is modified quantum mechanically. For example, it is known that classically the dilaton Kahler potential is $K=\ln(S+S^{\dagger})$. If the number of space-time supersymmetries is larger than one the vacuum degeneracy associated with the expectation value of S might not be removed. What is then the quantum Kahler potential? Can the strong coupling region of small ReS be absent?

We do not have a complete theory of these moduli spaces and their nature in many cases is not clear to us. However, we will discuss here three examples. In the first, the quantum moduli space is different than the classical one and the classical singularities are smoothed out. In the second example, the quantum space is the same as the classical one but the physical nature of the singularities is different. In the classical theory the singularities correspond to massless gluons and in the quantum theory it is associated with massless bound states. In our third example the classical and quantum moduli spaces are identical. Furthermore, both the classical and the quantum theory have massless gluons and quarks at the singular points. In quantum theory these massless interacting fields correspond to a nontrivial four-dimensional critical point.

In Sec. II we review the classical field theory of supersymmetric QCD and present the classical moduli space of the massless quark theory. In Sec. III we review some known results about the quantum theory. For massless quarks the quantum theory with fewer flavors than colors has no ground state. When the number of flavors is larger or equal the number of colors the theory has a quantum moduli space of inequivalent ground states. Section IV discusses the situation for equal numbers of flavors and colors where the classical singularities on the moduli space are blown up. On this space we find points with unusual patterns of chiral symmetry breaking. Section V discusses the case when the number of flavors is one plus the number of colors. The quantum moduli space is the same as the classical one but the interpretation of the singularities is different. At the singular points there are new massless bound states and some or even all of the chiral symmetry of the model is unbroken. Our understanding of the situation with larger number of flavors is limited. We present it in Sec. VI. We speculate that the singular points might be associated with massless interacting gluons. Unfortunately, we can prove that this is the case only for some range of N_c and N_f.

After completing this work we received a paper [3] which partially overlaps with ours.

II. CLASSICAL MODULI SPACES

We will be studying supersymmetric QCD. The theory is based on an $SU(N_c)$ gauge theory with N_f flavors of quarks, Q^i in the N_c representation and $\tilde{Q}_{\bar{i}}$ in the $\bar{N}_c$ representation ($i,\bar{i}=1,\ldots,N_f$). The anomaly free global symmetry is

$$SU(N_f)\times SU(N_f)\times U(1)_B\times U(1)_R \ , \tag{2.1}$$

where the $U(1)_R$ charge of Q and $\tilde{Q}$ is $(N_f-N_c)/N_f$. For $N_c=2$ there are $2N_f$ quarks, Q^i. The global symmetry is

$$SU(2N_f)\times U(1)_R \tag{2.2}$$

and the $U(1)_R$ charge of Q is $(N_f-2)/N_f$.

An important property of these theories is the existence of classical flat directions. For $N_c=2$ the classical flat directions (up to gauge and global symmetries) are

$$Q=\begin{pmatrix} a & \\ & a \end{pmatrix} . \tag{2.3}$$

They can be described by the gauge-invariant combinations

$$V^{ij}=Q^iQ^j \ . \tag{2.4}$$

The classical moduli space can be described as the space of V's subject to

$$\epsilon_{i_1,\ldots,i_{2N_f}} V^{i_1i_2}V^{i_3i_4}=0 \tag{2.5}$$

(which is meaningful only for $N_f\geq 2$). This constraint equation can also be understood as a trivial consequence of the Bose statistics of the underlying quark superfields.

For nonzero V the gauge symmetry is completely broken and the global symmetry (2.2) is broken to

$$SU(2)\times SU(2N_f-2)\times U(1)_R \ , \tag{2.6}$$

where this $U(1)_R$ is a linear combination of the original R charge and a generator in $SU(2N_f)$. The massless components in V are in the $(2,2N_f-2)_{(N_f-2)/(N_f-1)}+(1,1)_0$ representation of the unbroken global symmetry (2.6). One of the scalars in $(1,1)_0$ represents the inequivalent flat directions labeled by a and all the other scalars are Goldstone bosons.

Similarly, for $N_c>2$ the flat directions can be labeled by the gauge invariant combinations ("mesons" M and "baryons" B, $\tilde{B}$)

$$M_{\bar{i}}^{i}=Q^i\tilde{Q}_{\bar{i}} \ ,$$

$$B_{i_{N_c+1},i_{N_c+2},\ldots,i_{N_f}}=\frac{1}{N_c!}\epsilon_{i_1,\ldots,i_{N_f}}Q^{i_1}Q^{i_2}\cdots Q^{i_{N_c}} \ , \tag{2.7}$$

$$\tilde{B}^{\bar{i}_{N_c+1},\bar{i}_{N_c+2},\ldots,\bar{i}_{N_f}}=\frac{1}{N_c!}\epsilon^{\bar{i}_1,\ldots,\bar{i}_{N_c}}\tilde{Q}_{\bar{i}_1}\tilde{Q}_{\bar{i}_2}\cdots\tilde{Q}_{\bar{i}_{N_c}} \ ,$$

whose $U(1)_B\times U(1)_R$ charges are $[0,2(N_f-N_c/N_f)]$, $[N_c,N_c(N_f-N_c/N_f)]$ and $[-N_c,N_c(N_f-N_c/N_f)]$. For $N_f<N_c$ the baryons B and $\tilde{B}$ do not exist and the flat directions are the space of M's. For $N_f\geq N_c$ flat directions are the space of M, B, and $\tilde{B}$ subject to the constraints following from Bose statistics of the fundamental quarks. For $N_f=N_c$ there is only one constraint:

$$\det M-B\tilde{B}=0 \ . \tag{2.8}$$

For $N_f=N_c+1$ there are three constraints:

$$\begin{aligned} &\frac{1}{N_c!}\epsilon_{i_1,\ldots,i_{N_f}}\epsilon^{\bar{i}_1,\ldots,\bar{i}_{N_f}}M_{\bar{i}_1}^{i_1}\cdots M_{\bar{i}_{N_c}}^{i_{N_c}}-B_{i_{N_f}}\tilde{B}^{\bar{i}_{N_f}}=0 \ , \\ &B_iM_{\bar{j}}^{i}=0 \ , \\ &M_{\bar{i}}^{j}\tilde{B}^{\bar{i}}=0 \ . \end{aligned} \tag{2.9}$$

Various points on the classical moduli spaces exhibit a different unbroken gauge and global symmetry. The unbroken global symmetry can easily be identified by examining the expectation values of the gauge invariant order parameters (2.4) and (2.7). The generic point on the moduli space has an unbroken $SU(N_c-N_f)$ gauge symmetry which is absent for $N_f\geq N_c-1$. Special points where $B=\tilde{B}=0$ and M has fewer than N_c-1 nonzero eigenvalues ($V=0$ for $N_c=2$) have enhanced gauge symmetries. At these points the classical moduli spaces are singular.

III. THE QUANTUM THEORY

The quantum theory was studied by several groups using different techniques. The authors of [4] advocated the use of an effective Lagrangian involving the light meson fields M and the glueball field $S=(1/32\pi^2)W_\alpha^2$ and imposed the anomalous Ward identities on the superpotential. On the other hand, Ref. [5] followed the Wilsonian approach and focused only on the light fields. Then, the anomalous Ward identities should certainly not be imposed. Since we are interested here in the moduli space, we should follow the point of view of [5] and keep only the light fields. In general, a Wilsonian effective action with at most two space-time derivatives must include all the massless fields. It may include some of the massive fields after others have been integrated out. In this case it reproduces the correct dynamics of the light fields but might lead to incorrect answers for the massive ones.

Dynamical calculations in these theories were performed using instanton methods. Reference [6] studied these theories along the flat directions and performed the instanton calculations in the Higgs picture. The CERN group [7] applied the instanton method of [8] and considered the theory near the origin in field space $Q=0$. For $N_f<N_c-1$ the two approaches lead to qualitatively consistent answers (Ref. [9] claims that the details are not consistent). For $N_f \geq N_c$ the authors of [7] did not agree with the conclusions of [6].

To control the theory along the flat directions we can add mass terms to the superpotential $m_i^{\tilde{\imath}} Q^i \tilde{Q}_{\tilde{\imath}}$. Later we will also add terms of the form $bB+\tilde{b}\tilde{B}$ for $N_f \geq N_c$.

For $\det m \neq 0$ all the flat directions are lifted and classically $M=B=\tilde{B}=0$. Quantum mechanically the expectation values are different. Expectation values of lower components of chiral superfields are holomorphic in m and must respect selection rules under the global symmetry (2.1) of the massless theory [7]. This leads to

$$M_{\tilde{\imath}}^i = \langle Q^i \tilde{Q}_{\tilde{\imath}} \rangle \sim \Lambda^{(3N_c-N_f)/N_c} (\det m)^{1/N_c} \left[\frac{1}{m} \right]_{\tilde{\imath}}^i , \qquad B=\tilde{B}=0 . \tag{3.1}$$

If we also add $bB+\tilde{b}\tilde{B}$ the expectation values of B and $\tilde{B}$ do not vanish. The phases from the fractional power $1/N_c$ correspond to N_c different ground states—exactly as predicted by the Witten index. For $N_c=2$ with the mass term $m_{ij}Q^iQ^j$ these become

$$V^{ij} = \langle Q^i Q^j \rangle \sim \Lambda^{(6-N_f)/2} (\text{Pf} m)^{1/2} \left[\frac{1}{m} \right]^{ij} . \tag{3.2}$$

Explicit calculations [6,7] show that the coefficients of order one in these relations do not vanish. Therefore, we can redefine Λ such that

$$M_{\tilde{\imath}}^i \langle Q^i \tilde{Q}_{\tilde{\imath}} \rangle = \Lambda^{(3N_c-N_f)/N_c} (\det m)^{1/N_c} \left[\frac{1}{m} \right]_{\tilde{\imath}}^i , \qquad V^{ij} = \langle Q^i Q^j \rangle = \Lambda^{(6-N_f)/2} (\text{Pf} m)^{1/2} \left[\frac{1}{m} \right]^{ij} . \tag{3.3}$$

It is crucial that these expectation values are exact and are not just approximate.

For $N_f<N_c$ the light fields can be represented by M. Its expectation value (3.3) can be obtained from the effective Lagrangian [5]

$$W_{\text{dyn}} = (N_c-N_f) \frac{\Lambda^{(3N_c-N_f)/(N_c-N_f)}}{(\det M)^{1/(N_c-N_f)}} + mM . \tag{3.4}$$

Again, we should stress that this expression for the superpotential is exact [1].

For $N_f \geq N_c$ an effective Lagrangian description is more subtle. It is easy to see that using the constrained light fields or the elementary quark superfields Q and $\tilde{Q}$ no invariant superpotential can be written in the massless theory. Therefore, the classical vacuum degeneracy cannot be lifted [5,6] and the quantum theory has a moduli space of ground states. This is the point which was questioned in [7].

It is of interest to find the quantum moduli space of the massless theory. By taking m to zero we should find a point on that space. It is enough to diagonalize m (for $N_c=2$ bring m to a block diagonal form with every block proportional to σ^2) with eigenvalues m_i. One way to see that a nontrivial quantum moduli space exists is to note that the limit $m_i \rightarrow 0$ is not smooth and the limit of the expectation values depends on the way it is taken. Below we analyze this problem for different values of N_f.

IV. THE QUANTUM MODULI SPACE FOR $N_f=N_c$

It is easy to see using (3.3) that the classical constraint (2.8) [or (2.5) for $N_c=2$] is modified quantum mechanically to

$$\det M - B\tilde{B} = \Lambda^{2N_c} , \qquad \text{Pf} V = \Lambda^4 \quad \text{for } N_c=2 . \tag{4.1}$$

This modification is due to a one instanton effect. As mentioned above, the classical constraint follows from Bose statistics of the quark superfields. However, such a condition does not necessarily apply quantum mechanically. Note that these relations are true for every m and not only in the limit $m \rightarrow 0$.

We therefore conclude that the classical moduli space which was defined by (2.5) and (2.8) is modified quantum mechanically to the space defined by (4.1). Far from the singular points of the classical moduli space where semiclassical analysis is reliable the quantum space is very similar to the classical one. However, the quantum modification is crucial as it "blows up" the singularities; the singular points $B=\tilde{B}=0$ with at least two vanishing eigenvalues of M ($V=0$ for $N_c=2$) are not on the quantum moduli space.

It is amusing to note that since the quantum moduli space is different than the classical one, the low energy effective Lagrangian may have solitons. Such states cannot be understood as solitons on the classical moduli space.

Some of the points on the manifold (4.1) have an enhanced global symmetry.

(1) For $N_c=2$ the points related to

$$V = \Lambda^2 \begin{bmatrix} \sigma^2 & \\ & \sigma^2 \end{bmatrix} \tag{4.2}$$

by the SU(4) symmetry break it to Sp(4). This is the natural guess for the pattern of chiral symmetry breaking in theories with matter in a pseudoreal representation. What is somewhat unusual is that the R symmetry is unbroken. Therefore we should check the 't Hooft anomaly conditions associated with it. The high energy fermions are in $2\times\mathbf{4}_{-1}+3\times\mathbf{1}_1$ of Sp(4)$\times$U$(1)_R$. The low energy fields are the fluctuations of V around the expectation value (4.2) subject to the constraint (4.1). Their fermion components transform like $\mathbf{5}_{-1}$ under the unbroken group. The nontrivial anomalies which should be checked are

$$\begin{aligned} &\mathrm{Sp}(4)^2\mathrm{U}(1)_R && -2d^{(2)}(4)=-d^{(2)}(5)\ , \\ &\mathrm{U}(1)_R^2 && 2\times4\times(-1)^3+3=5\times(-1)^3\ , \\ &\mathrm{U}(1)_R && 2\times4\times(-1)+3=5\times(-1)\ , \end{aligned} \tag{4.3}$$

where $d^{(2)}(r)$ is the quadratic Sp(4) Casimir operators in the r representation. Note that the anomalies at the macroscopic and microscopic levels are the same.

(2) One generalization of these points to arbitrary N_c is

$$\begin{aligned} &B=\tilde{B}=0\ , \\ &M_{\tilde{i}}^{i}=\Lambda^2\delta_{\tilde{i}}^{i}\ , \end{aligned} \tag{4.4}$$

corresponding to the breaking of the flavor $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ symmetry to the diagonal $\mathrm{SU}(N_f)$. Again, since $\mathrm{U}(1)_R$ is unbroken, we should check the 't Hooft anomaly conditions. The high energy fermions are in $N_f\times(N_f)_{1,-1}+N_f\times(\bar{N}_f)_{-1,-1}+(N_f^2-1)\times 1_{0,1}$ under $\mathrm{SU}(N_f)\times\mathrm{U}(1)_B\times\mathrm{U}(1)_R$. The low energy fields are the fluctuations of M, B, and $\tilde{B}$ around the expectation values (4.4) subject to the constraint (4.1). Their fermion components transform like $(N_f^2-1)_{0,-1}+1_{-N_f,-1}+1_{N_f,-1}$ under the unbroken symmetry. The nontrivial anomalies are

$$\begin{aligned} &\mathrm{SU}(N_f)^2\mathrm{U}(1)_R && -N_f d^{(2)}(N_f)-N_f d^{(2)}(\bar{N}_f) \\ & && \quad =-d^{(2)}(N_f^2-1)\ , \\ &\mathrm{U}(1)_R^3 && 2N_f^2(-1)^3+(N_f^2-1) \\ & && \quad =(N_f^2-1)(-1)^3-2\ , \\ &\mathrm{U}(1)_B^2\mathrm{U}(1)_R && -2N_f^2\ , \\ &\mathrm{U}(1)_R && -2N_f^2+(N_f^2-1)=-(N_f^2-1)-2\ , \end{aligned} \tag{4.5}$$

where $d^{(2)}(r)$ is the quadratic $\mathrm{SU}(N_f)$ Casimir operators in the r representation. Again, the anomalies match.

(3) Another generalization to arbitrary N_c has

$$\begin{aligned} &B=-\tilde{B}=\Lambda^{N_f}\ , \\ &M_{\tilde{i}}^{i}=0\ , \end{aligned} \tag{4.6}$$

where the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ chiral symmetry is unbroken and only the baryon number symmetry $\mathrm{U}(1)_B$ is spontaneously broken. As before, at these points the 't Hooft anomaly conditions provide a powerful consistency check. The low energy fermions are in the $(N_f,\bar{N}_f)_{-1}+(1,1)_{-1}$ representation of $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)\times\mathrm{U}(1)_R$. The relevant anomalies which should be checked are

$$\begin{aligned} &\mathrm{SU}(N_f)^3 && N_f d^{(3)}(N_f)\ , \\ &\mathrm{SU}(N_f)^2\mathrm{U}(1)_R && -N_f d^{(2)}(N_f)\ , \\ &\mathrm{U}(1)_R^3 && -N_f^2-1\ , \\ &\mathrm{U}(1)_R && -N_f^2-1\ , \end{aligned} \tag{4.7}$$

where $d^{(3)}(r)$ is the cubic $\mathrm{SU}(N_f)$ Casimir operators in the r representation. Again, the anomalies match between the macroscopic and microscopic levels.

The authors of [7] have already noticed some of these anomaly matching conditions which led them to conjecture that states with these quantum numbers could be massless. The novelty of our analysis is the identification of the quantum moduli space which includes both these special points and the semiclassical flat directions of [6].

In order to obtain these results from an effective Lagrangian we need to impose the constraint (4.1). One way of doing that is using a Lagrange multiplier field X with the superpotential

$$\begin{aligned} &W=X(\det M-B\tilde{B}-\Lambda^{2N_f})\ , \\ &W=X(\mathrm{Pf}V-\Lambda^4)\ \text{ for } N_c=2\ . \end{aligned} \tag{4.8}$$

We now study some perturbations on the quantum moduli space.

(1) Example 1: $W_t=m_{\tilde{i}i}M^{i\tilde{i}}$. If the superpotential (4.8) is perturbed by mass terms, i.e., the tree level superpotential $W_t=m_{\tilde{i}i}M^{i\tilde{i}}$ (or $m_{ij}V^{ij}$ for $N_c=2$) is added to it, the expectation values (3.1), (3.2) are found. In the special case where $n<N_f$ of the masses vanish we can integrate out the massive quarks and X in the effective Lagrangian and find $W_{\mathrm{eff}}=(N_c-n)\Lambda_L^{(3N_c-n)/(N_c-n)}/\det' M$ where the determinant is only over the massless modes and the low energy scale $\Lambda_L=\Lambda^{2N_f/(3N_f-n)}(\prod_i m_i)^{1/(3N_f-n)}$, exactly as in (3.4).

(2) Example 2: $W_t=bB+\tilde{b}\tilde{B}$. A less trivial application with a nontrivial moduli space arises when the massless theory is perturbed by $bB+\tilde{b}\tilde{B}$. Adding this to (4.8) we find

$$\begin{aligned} &B=\pm i\Lambda^{N_c}\left[\frac{\tilde{b}}{b}\right]^{1/2}\ , \\ &\tilde{B}=\pm i\Lambda^{N_c}\left[\frac{b}{\tilde{b}}\right]^{1/2}\ , \\ &\det M(M^{-1})=0\ , \end{aligned} \tag{4.9}$$

which means that there is a moduli space of solutions where M is an arbitrary matrix with at most N_f-2 nonzero eigenvalues.

Semiclassically, at large field strength, this moduli space can be understood as follows. The expectation value $Q=\tilde{Q}$ with only N_f-2 nonzero eigenvalues is a flat direction of the theory with the superpotential. It breaks the gauge group to SU(2). N_f^2-4 chiral superfields acquire masses in the Higgs mechanism and eight more from the superpotential. The remaining N_f^2-4 light fields include the Goldstone bosons of the broken generators in the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ global symmetry and the parameters which label the flat directions. Since the unbroken SU(2) gauge theory has no light quarks, it can easily be integrated out. Its scale Λ_L is determined by $\Lambda^{2N_f}\sim Q^{2N_f-4}/b\tilde{b}(Q^{N_f-2})^2\Lambda_L^6$ where the numerator arises from the Higgs mechanism and the denominator from the mass term in the superpotential. Since Λ_L is independent of Q, gluino condensation in the unbroken gauge group cannot lead to a superpotential for the light fields. Furthermore, it is easy to use the symmetries of the problem and to show that no invariant superpotential can be generated. Therefore, the flat directions are not lifted.

The gauge invariant description of this moduli space is in terms of the matrix M constrained to have at most N_f-2 nonzero eigenvalues. This manifold is singular when M has fewer than N_f-2 nonzero eigenvalues. The most singular point is $M=0$. Classically, these singularities represent enhanced gauge symmetry at these points. Quantum mechanically, we expect the gluons of these unbroken gauge symmetry to confine and not to be massless. Instead, there might be other massless fields. Since our order parameter is M, we should find an effective Lagrangian for M. The symmetries including the explicitly broken ones as in [1] lead to

$$W=\sqrt{\tilde{b}b}\,\Lambda^{N_f}h\left[\frac{\det M}{\Lambda^{2N_f}}\right] \tag{4.10}$$

for some function h. In the next example we will integrate out B and $\tilde{B}$ more carefully and will show that $h(t)=\pm 2\sqrt{t-1}$. The two signs correspond to the two values of B and $\tilde{B}$ (for a more detailed discussion of such phenomena see [2]). The supersymmetric ground states which follow from this Lagrangian satisfy $\det N(1/M)=0$ exactly as in the full theory and as expected in the semiclassical region. Furthermore, at the singular points of this manifold, the unbroken global symmetry is enhanced and more components of M become massless. They join the other massless fields to representations of the unbroken symmetry. In particular, for $M=0$ the global $SU(N_f)\times SU(N_f)$ is unbroken and all the components of M are massless.

(3) Example 3: $W_t=m_i^{\bar{\imath}}M_{\bar{\imath}}^i+bB+\tilde{b}\tilde{B}$. We now combine the previous two examples and consider the tree level superpotential $W_t=m_i^{\bar{\imath}}M_{\bar{\imath}}^i+bB+\tilde{b}\tilde{B}$. For simplicity, we present the answers only for $N_c=N_f=3$. Classically, there is one vacuum at the origin $M=B=\tilde{B}=0$ where the gauge group is unbroken and another ground state with $M=(\det m/b\tilde{b})(1/m)$, $B=-(\det m/b^2\tilde{b})$, $\tilde{B}=-(\det m/\tilde{b}^2 b)$ where the gauge group is completely broken. The expectation values in the quantum theory are determined by

$$W=X(\det M-B\tilde{B}-\Lambda^6)+m_i^{\bar{\imath}}M_{\bar{\imath}}^i+bB+\tilde{b}\tilde{B}\ . \tag{4.11}$$

It is easy to see that the expectation values are

$$B=-\frac{\Lambda^6\tilde{b}}{\det m}y^2\ ,\qquad \tilde{B}=-\frac{\Lambda^6 b}{\det m}y^2\ ,\qquad M=\Lambda^3 y\frac{1}{m}\ , \tag{4.12}$$

where y satisfies

$$b\tilde{b}y^4-\frac{\det m}{\Lambda^3}y^3+\left[\frac{\det m}{\Lambda^3}\right]^2=0\ . \tag{4.13}$$

This equation has four solutions. For small $\det m/\Lambda^3$ they are

$$y_{1234}=\xi\left[\frac{\det m}{\Lambda^3}\right]^{1/2}(b\tilde{b})^{-1/4}+\frac{\det m}{4\Lambda^3 b\tilde{b}}+\cdots\ , \tag{4.14}$$

with $\xi^4=-1$ and, for small $b\tilde{b}$,

$$y_{123}=\omega\left[\frac{\det m}{\Lambda^3}\right]^{1/3}+\frac{1}{3}\omega^2\left[\frac{\det m}{\Lambda^3}\right]^{-1/3}b\tilde{b}+\cdots\ ,\qquad y_4=\frac{\det m}{b\tilde{b}\Lambda^3}+\cdots\ , \tag{4.15}$$

with $\omega^3=1$.

This example demonstrates that not only can there be a continuous manifold of inequivalent ground states that can also be inequivalent discrete vacua. Semiclassically we found two different ground states. They are most easily related to the situation for small $b\tilde{b}$. For $b\tilde{b}=0$ the model has only the first three states. The anomaly free Z_6 R symmetry is spontaneously broken by gluino condensation to Z_2 and the three ground states are related by the symmetry. These three states correspond to the unique state we found semiclassically near the origin. For small $b\tilde{b}$ there are four inequivalent ground states. The first three solutions near the origin are no longer related by symmetry. The fourth one which was also observed semiclassically is at large field strength.

We can now use the equations of motion of all the fields to integrate out B, $\tilde{B}$, and X and to find an effective superpotential for M only and thus determine the function $h(t)$ in (4.10) Generalizing to arbitrary $N_f=N_c$, it is straightforward to find the equations of motion and to show that they are reproduced by $W_{\rm eff}=\pm 2\sqrt{b\tilde{b}}\,\Lambda^{N_f}\sqrt{(\det M/\Lambda^{2N_f})-1}+mM$ and therefore, $h(t)=\pm 2\sqrt{t-1}$.

V. THE QUANTUM MODULI SPACE FOR $N_f=N_c+1$

As for $N_f=N_c$, the expectation values (3.3) do not satisfy the classical constraints (2.5), (2.9):

$$\det M\left[\frac{1}{M}\right]_i^{\bar{\imath}}-B_i\tilde{B}^{\bar{\imath}}=\Lambda^{2N_c-1}m_i^{\bar{\imath}}\ ,\qquad \mathrm{Pf}V\left[\frac{1}{V}\right]_{ij}=\Lambda^3 m_{ij}\ . \tag{5.1}$$

However, unlike $N_f=N_c$ case the classical constraints satisfied in the $m_i\rightarrow 0$ limit. Therefore, the quantum moduli space of the massless theory is the same as for the classical theory. The only exception is at the singular points where different light fields might be present. Note that for $m_i\neq 0$ all values of M and V (and with $bB+\tilde{b}\tilde{B}$ also of B and $\tilde{B}$) can be found. Again, this is unlike the $N_f=N_c$ case. This suggests that a complete description for nonzero m needs all the fields M, B, and $\tilde{B}$ (and all the components of V for $N_c=2$) and not only the constrained ones.

We now discuss the behavior of the massless theory at the singular points $V=0$ and its higher N_c analogues such as $M=B=\tilde{B}=0$. Since the expectation values of all our order parameters vanish there it seems like the full chiral symmetry should be unbroken there. Since we are planning to use all the components of M, B, and $\tilde{B}$ (all the components of V for $N_c=2$), it is natural to expect that all of them are massless there.

For $N_c>2$ the massless quarks in the microscopic theory transform under the global symmetry (2.1) as $N_c\times(N_f,1)_{1,1/N_f}+N_c\times(1,\bar{N}_f)_{-1,1/N_f}$ and there are also N_c^2-1 gauge fields. The fields M, B, and $\tilde{B}$ transform like $(N_f,\bar{N}_f)_{0,2/N_f}$, $(\bar{N}_f,1)_{N_f-1,(N_f-1)/N_f}$ and $(1,N_f)_{-N_f+1,(N_f-1)/N_f}$, respectively. As a first test we should check the 't Hooft anomaly equations. The nontrivial ones are

$$\begin{aligned}
&SU(N_f)^3 && (N_f-1)d^{(3)}(N_f)\ ,\\
&SU(N_f)^2U(1)_R && -\frac{(N_f-1)^2}{N_f}d^{(2)}(N_f)\ ,\\
&U(1)_R^3 && -N_f^2+6N_f-12+\frac{8}{N_f}-\frac{2}{N_f^2}\ ,\\
&U(1)_B^2U(1)_R && -2(N_f-1)^2\ ,\\
&SU(N_f)^2U(1)_B && (N_f-1)d^{(2)}(N_f)\ ,\\
&U(1)_R^2U(1)_B && 0\ ,\\
&U(1)_R && -N_f^2+2N_f-2\ ,
\end{aligned}\tag{5.2}$$

and they match between the low energy and the high energy spectra. Reference [7] noticed this anomaly matching and conjectured the existence of a ground state with these massless particles.

For $N_c=2$, the fundamental quarks are in $\mathbf{6}_{1/3}$ and the field V is in $\mathbf{15}_{2/3}$ of $SU(6)\times U(1)_R$. The 't Hooft anomaly conditions are satisfied:

$$\begin{aligned}
&SU(6)^3 && 2d^{(3)}(6)=d^{(3)}(15)\ ,\\
&SU(6)^2U(1)_R && 2(-\tfrac{2}{3})d^{(2)}(6)=-\tfrac{1}{3}d^{(2)}(15)\ ,\\
&U(1)_R^3 && 12(-\tfrac{2}{3})^3+3=15(-\tfrac{1}{3})^3\ ,\\
&U(1)_R && 12(-\tfrac{2}{3})+3=15(-\tfrac{1}{3})\ .
\end{aligned}\tag{5.3}$$

Given that there are more massless fields at the origin, we should be able to find a low energy effective Lagrangian which describes all these fields. There is a unique superpotential

$$\begin{aligned}
W_{\text{eff}}&=\frac{1}{\Lambda^{2N_f-3}}(B_iM_{\bar{\imath}}^i\tilde{B}^{\bar{\imath}}-\det M)\ ,\\
W_{\text{eff}}&=-\frac{1}{\Lambda^3}\text{Pf}V\ ,
\end{aligned}\tag{5.4}$$

which satisfies the following properties.

(1) It is invariant under all the symmetries in of the problem including the $U(1)_R$ symmetry.

(2) Its flat directions are exactly as in the microscopic theory. The classical constraint equations (2.5) (2.9) which are not modified quantum mechanically arise here as the equations of motion from (5.4).

(3) At the origin all the fields are massless. However, away from the origin only some of the fields remain massless in a way consistent with the semiclassical treatment.

(4) By adding $m_i^{\bar{\imath}}M_{\bar{\imath}}^i+b^iB_i+\tilde{b}_{\bar{\imath}}\tilde{B}^{\bar{\imath}}$ and solving for the fields we recover all the results for $N_f<N_c+1$. In particular, we can add masses to some of the fields, integrate them out and find the low energy Lagrangian for fewer flavors.

This theory is similar to the second example in Sec. IV. There we also found a singular moduli space where the singularity was associated with new massless fields. As in that example, the superpotential leads to masses to some of the fields away from the origin and its equation of motion are the defining equations of the moduli space.

VI. THE QUANTUM MODULI SPACE FOR $N_f\geq N_c+2$

It is not easy to extend the previous descriptions for $N_f|=N_c+2$. As for $N_f=N_c+1$, all values of V, M, B, and $\tilde{B}$ can be obtained and they should all be considered independent fields. By examining the massless limit of the expectation values we see that the classical constraints are satisfied quantum mechanically. Therefore, as for $N_f=N_c+1$, the quantum moduli space is the same as the classical one. However, unlike $N_f=N_c+1$, the singularities cannot be associated with unbroken global symmetry and massless V, M, B, and $\tilde{B}$ fields. This can be seen in several ways: (1) The 't Hooft anomaly conditions are not satisfied there; (2) an effective Lagrangian description depending only on our mesons and baryons is singular. Consider for simplicity the case of $N_c=2$. There is a unique invariant superpotential for V:

$$W=\frac{2-N_f}{\Lambda^{(6-N_f)/(N_f-2)}}\text{Pf}V^{1/N_f-2}\ .\tag{6.1}$$

Although it leads to the correct expectation values of V in the massive theory it is singular at $V=0$. We could try, following [7] to add the field S. Even without imposing the anomalous Ward identities the symmetries lead to $W=Sf[\text{Pf}V/(S^{N_f-2}\Lambda^{6-N_f})]$ which is always singular at $V=S=0$.

We conclude that a complete gauge invariant description near the origin needs more fields. We could not find a simple set of fields which could resolve the singularity. Perhaps the Kahler potential is singular there and the origin is infinitely far from every point on the moduli space.

An alternative is that at least some of the elementary colored fields are massless at the origin. One possibility is that the gauge group breaks to an Abelian subgroup there. A more interesting possibility is that the spectrum at the origin is identical to that in the classical theory. Clearly, this is the simplest solution of the 't Hooft equations.

Massless quarks and gluons are possible only if the theory is scale invariant. We cannot show that such a scale invariant theory exists for every $N_f \geq N_c+2$. However, it is easy to establish it at least for some range of N_f and N_c.

The two-loop β function of this theory is [10]

$$\beta(g)=-\frac{g^3}{16\pi^2}(3N_c-N_f)+\frac{g^5}{128\pi^4}\left[2N_cN_f-3N_c^2-\frac{N_f}{N_c}\right]+O(g^7)\ . \quad (6.2)$$

The authors of [11] argued that the exact β function satisfies

$$\beta(g)=-\frac{g^3}{16\pi^2}\frac{3N_c-N_f+N_f\gamma(g)}{1-N_c(g^2/8\pi^2)}\ ,$$
$$\gamma(g)=-\frac{g^2}{8\pi^2}\frac{N_c^2-1}{N_c}+O(g^4)\ , \quad (6.3)$$

where $\gamma(g)$ is the anomalous dimension of the mass [Eq. (6.3) is consistent with (6.2)].

Since there are values of N_f and N_c where the one-loop β function is negative but the two-loop contribution is positive, there might be a nontrivial critical point [12]. Several people noticed that by taking N_c and N_f to infinity holding N_cg^2 and $N_f/N_c=3-\epsilon$ fixed, one can establish the existence of a critical point at $N_cg_*^2=8\pi^2/3\epsilon+O(\epsilon^2)$. Therefore, at least for large N_c and $\epsilon=3-(N_f/N_c)\ll 1$ and perhaps even for every $N_f\geq N_c+2$ there are massless interacting gluons and quarks at the origin.

We would like to make a few comments about these scale invariant theories.

(1) At the origin the operators have anomalous dimensions. For example, Eq. (6.3) shows that $\gamma(g_*)=1-(3N_c/N_f)$ [in deriving this result we only need the numerator of (6.3) which is better motivated than the full expression] and therefore the dimension of the operator $\tilde{Q}Q$ is $3(N_f-N_c)/N_f$.

(2) Λ in the formulas is the scale at which the theory crosses over from the UV to the IR critical behavior. Using Λ we can relate the UV expression for the mass to the IR expression which has an anomalous dimension.

(3) By adding mass terms for the quarks one can flow between these critical points by reducing N_f.

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[1] N. Seiberg, Phys. Lett. B **318**, 469 (1993).
[2] K. Intriligator, R. Leigh, and N. Seiberg, Rutgers report (unpublished).
[3] V. Kaplunovski and J. Louis, Report No. UTTG-94-1 and No. LUM-TPW-94-1 (unpublished).
[4] G. Veneziano and S. Yankielowicz, Phys. Lett. **113B**, 321 (1982); T. R. Taylor, G. Veneziano, and S. Yankielowicz, Nucl. Phys. **B218**, 493 (1983).
[5] A. C. Davis, M. Dine, and N. Seiberg, Phys. Lett. **125B**, 487 (1983).
[6] I. Affleck, M. Dine, and N. Seiberg, Nucl. Phys. **B241**, 493 (1984); **B256**, 557 (1985).
[7] D. Amati, K. Konishi, Y. Meurice, G. C. Rossi, and G. Veneziano, Phys. Rep. **162**, 169 (1988), and references therein.
[8] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B223**, 445 (1983).
[9] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B260**, 157 (1985).
[10] D. R. T. Jones, Nucl. Phys. **B75**, 531 (1974); R. Barbieri, S. Ferrara, L. Maiani, F. Palumbo, and C. Savoy, Phys. Lett. **115B**, 212 (1982); A. Parkes and P. West, *ibid.* **138B**, 199 (1984).
[11] V. Novikov, M. Shifman, A. Vainshtein, M. Voloshin, and V. Zakharov, Nucl. Phys. **B223**, 394 (1983); M. A. Shifman and A. I. Vainshtein, *ibid.* **B227**, 456 (1986).
[12] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982).