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Instantons and fermion condensate in adjoint two-dimensional QCD

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We show that two-dimensional QCD with adjoint fermions involves instantons due to nontrivial $\pi_1[\mathrm{SU}(N)/Z_N] = Z_N$. At high temperatures, the quasiclassical approximation works and the action and the form of the effective (with account of quantum corrections) instanton solution can be evaluated. The instanton presents a localized configuration with a size $\propto g^{-1}$. At $N = 2$, it involves exactly 2 zero fermion modes and gives rise to the fermion condensate $\langle\bar\lambda^a\lambda^a\rangle_T$ which falls off $\propto \exp\{-\pi^{3/2}T/g\}$ at high T but remains finite. At low temperatures, both instanton and bosonization arguments also exhibit the appearance of the fermion condensate $\langle\bar\lambda^a\lambda^a\rangle_{T=0} \sim g$. For $N > 2$, the situation is paradoxical. There are $2(N-1)$ fermion zero modes in the instanton background which implies the absence of the condensate in the massless limit. On the other hand, bosonization arguments suggest the appearance of the condensate for any N. Possible ways to resolve this paradox (which occurs also in some four-dimemsional gauge theories) are discussed.

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I. INTRODUCTION

Two-dimensional QCD (QCD_2) with fermions belonging to the adjoint representation of the $\mathrm{SU}(N)$ group has attracted considerable attention lately. In very interesting recent works [1], the spectrum of the theory in the large N limit has been determined. It displayed the features which are strikingly analogous to the spectrum of four-dimensional QCD. In contrast with QCD_2 with fundamental fermions where the meson states lie on one Regge-like trajectory [2] so that

$$M_n^2 \sim g^2 N_c n \tag{1.1}$$

and the density of states rises linearly with mass $dn/dM \sim M$, here the number of such trajectories is infinite, and the density of states grows exponentially with mass.[1]

Of course, it is exactly the same behavior as in large N QCD_4 where the number of infinitely narrow resonances also rises exponentially with energy so that the Hagedorn phenomenon, the appearance of limiting temperature above which the system cannot be heated, takes place [3].

In this paper, we show that the adjoint QCD_2 bears much resemblance to four-dimensional QCD describing the real world also for finite N. The situation is clear and the analogy is straightforward for $N = 2$. In particular, we show that, in contrast with what happens in QCD_2 with fundamental quarks, a fermion condensate is generated here which falls down rapidly at high T. The physical picture is the same as in QCD_4 with only one light quark flavor [4] and in the Schwinger model [5–7].

The main effect leading to the appearance of the fermion condensate is the presense of instantons. They are specific for the theory with adjoint matter and were absent in $\mathrm{QCD}_2^{\mathrm{fund}}$. The topological reason for their existence is the nontrivial $\pi_1[\mathcal{G}]$ where the gauge group $\mathcal{G}$ is $\mathrm{SU}(N)/Z_N$ rather than just $\mathrm{SU}(N)$ (adjoint fields are not transformed under the action of the elements of the center), so that there are N topologically nonequivalent sectors.

Instantons appear by the same token as in the Schwinger model [8, 6, 9]. In the latter, the topological reason for the existence of instantons is the nontrivial $\pi_1[\mathrm{U}(1)] = Z$. The difference from the non-Abelian case is that, in the Schwinger model, the topological charge can be written as an integral invariant:

$$\nu = \frac{g}{4\pi} \int d^2x\ F_{\mu\nu}\epsilon_{\mu\nu} \tag{1.2}$$

(it is the two-dimensional analog of the four-dimensional Pontryagin class $\propto \int d^4x \mathrm{Tr}\{F_{\mu\nu}\tilde F_{\mu\nu}\}$). ν is an arbitrary integer which labels different topological sectors. In non-Abelian theory, no such integral invariant can be written ($\mathrm{Tr}\{F^a_{\mu\nu}t^a\} = 0$). That is understandable, of course. If such an integral invariant did exist, the number of topologically nontrivial sectors would be infinite, but it is *finite* in the non-Abelian case.

These new instantons which are specific for theories involving only adjoint fields occur also in four dimensions. Actually, they have been known for a long time as 't Hooft fluxes [10]. The difference with two dimensions is that, for $d = 4$, the corresponding configurations are not localized (they do not depend on two transverse directions), and their action is infinite. For high T, these

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[1]The notion of trajectories makes sense only for few first states with small enough mass. At larger masses, the trajectories begin to overlap, and the spectrum becomes stochastic [1].

"planar instantons" have been studied in [11] and also earlier in [12] (where they were, however, *misinterpreted* as real walls in Minkowski space separating different thermal vacua—we refer an interested reader to [11] for a detailed discussion of this question).

Topologically nontrivial fields appear in $\mathrm{QCD}_2^{\mathrm{adj}}$ both at low and at high temperatures. However, the high-T case is more "clean" because quantum fluctuations are small here, the quasiclassical approximation works, and a quantitative calculation for the instanton contribution in the partition function is possible.

One immediate effect related to instantons is the generation of the fermion condensate due to the presense of fermion zero modes in the instanton background. Recall the situation in QCD_4. Instantons involve there one complex zero mode for each light fermion flavor (one for ψ and one for $\bar{\psi}$). If $N_f = 1$, these zero modes are "absorbed" by external ψ operators in the Euclidean functional integral,

$$\langle \bar{\psi}\psi \rangle \sim \int \prod dA d\bar{\psi} d\psi \; \bar{\psi}\psi \exp \times \left\{ \int d^4x \left[-\frac{1}{2} \mathrm{Tr}(F_{\mu\nu}^2) + i\bar{\psi}\mathcal{D}_\mu \gamma_\mu \psi \right] \right\}, \tag{1.3}$$

and we get a finite result even for very large T. If $N_f \geq 2$, there are extra zero modes for extra flavors, and $\langle \bar{\psi}\psi \rangle_{T \gg \Lambda_{\mathrm{QCD}}}$ is zero for massless quarks. For small T, $\langle \bar{\psi}\psi \rangle$ is nonzero (this is an experimental fact; theoretically, its appearance can also be related to instanton zero modes but not in a direct way [13]), which means that the phase transition occurs.

The main observation of this paper is that the physics of $\mathrm{QCD}_2^{\mathrm{adj}}$ with $N = 2$ and one Majorana adjoint fermion flavor is essentially the same as that of QCD_4 with $N_f = 1$. A high-T instanton (the topologically nontrivial configuration which minimizes the effective action) involves exactly two zero modes which are absorbed by external fermion operators in the functional integral for $\langle \bar{\lambda}^a \lambda^a \rangle$ and leads to an exponentially suppressed $\propto \exp\{-\pi^{3/2} T/g\}$ but nonzero result.

What happens at low temperatures? Quantum fluctuations are large there and only dimensional estimates can be done. Still, these estimates display the presence of the condensate. Its value is of order g. The appearance of the condensate is also very clearly seen in the framework of the bosonization approach. It is very essential that, in contrast with $\mathrm{QCD}_2^{\mathrm{fund}}$, the bosonized version of $\mathrm{QCD}_2^{\mathrm{adj}}$ does not involve a massless field which smears away the condensate $\langle \bar{\psi}\psi \rangle$ in the former for any finite N.

Whereas for $N = 2$ the picture is rather clear and self-consistent, it is not so for $N \geq 3$. High-T instantons involve here $2(N-1)$ fermion zero modes which are "larger than necessary." Similarly to what happens in QCD_4 with $N_f \geq 2$, the extra zero modes lead to the suppression of the condensate in the massless limit. In QCD_4, the statement of the absence of the condensate at high T could not be extrapolated to low temperature region due to the presence of Goldstone bosons which display themselves in the low temperature partition function [14]. But in $\mathrm{QCD}_2^{\mathrm{adj}}$, Goldstone bosons are absent. They cannot appear in two dimensions due to the Coleman theorem [15] and they do not as the generation of the fermion condensate is *not* associated with spontaneous breaking of a continuous symmetry.

Assuming that *any* topologically nontrivial background involves exactly $2(N-1)$ fermion zero modes and the absence of massless modes in the spectrum, we have to conclude that the condensate is absent also at low T. On the other hand, bosonization arguments display the presence of the condensate universally for any N. This is a clear paradox. A possible way to resolve it which we suggest will be discussed later in this paper.

The structure of the paper is the following. In the next section, we fix our notations and discuss the symmetries of the theory considered. In Sec. III, the explicit form of the high-T instanton for $N = 2$ is obtained and the estimate for the fermion condensate is done. In Sec. IV, we discuss the low temperature region and show that both the instanton arguments and the bosonization arguments imply the appearance of fermion condensate. In Sec. V, we discuss characteristic field confugurations contributing to the partition function of the theory and show that the instantons are in some sense "confined" for strictly massless case and are "liberated" for any small but nonzero fermion mass. In Sec. VI, we analyze the case $N \geq 3$ and display the paradox. The paradox and possible ways for its resolution are discussed further in Sec. VII. Conclusive remarks are given in the last section.

II. QCD_2 WITH REAL ADJOINT FERMIONS

The Lagrangian of the model reads

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \frac{i}{2} \bar{\lambda}^a [\delta^{ab}\partial_\mu - g\epsilon^{abc} A_\mu^c] \gamma_\mu \lambda^b, \tag{2.1}$$

where λ_α^a is the two-dimensional Majorana (real) spinor, $\alpha = 1, 2$, and $\bar{\lambda}^a \equiv \lambda^a \gamma^0$. It is convenient to choose the representation $\gamma_0 = \sigma^2$, $\gamma_1 = i\sigma^1$. In that case, $\gamma_5 = \gamma_0\gamma_1 = \sigma^3$ and the left $\lambda_L = \frac{1}{2}(1+\gamma_5)\lambda$ and the right $\lambda_R = \frac{1}{2}(1-\gamma_5)\lambda$ components of the fermion field are described by the upper and lower components of the spinor λ_α, respectively. The fermion part of the Lagrangian can be written in terms of λ_L and λ_R as

$$\mathcal{L}_{\mathrm{ferm}} = \frac{i}{2} \{ \lambda_L^a [\delta^{ab}\partial_- - g\epsilon^{abc} A_-^c] \lambda_L^b + \lambda_R^a [\delta^{ab}\partial_+ - g\epsilon^{abc} A_+^c] \lambda_R^b \}, \tag{2.2}$$

with $\partial_\pm = \partial_0 \pm \partial_1$, $A_\pm^c = A_0^c \pm A_1^c$ (left fermions are the left movers and right fermions are the right movers).

Note (and this is very important) that, in contrast with the theory with fundamental Dirac fermions, the Lagrangian (2.1) does not enjoy any continuous global symmetry. The phase transformations $\lambda \to \exp\{i\alpha\}\lambda$ or $\lambda \to \exp\{i\beta\gamma_5\}\lambda$ are not allowed as they destroy the reality condition. The would-be currents corresponding to these transformations $\bar{\lambda}\gamma_\mu\lambda$ and $\bar{\lambda}\gamma_\mu\gamma_5\lambda$ are just zero for

Majorana fermions. One cannot also mix left and right components $\lambda_L \equiv \lambda_1$ and $\lambda_R \equiv \lambda_2$—the Lagrangian (2.2) is not invariant under such a transformation.

In this respect, the situation in two dimensions differs essentially from the four-dimensional case. The four-dimensional Majorana spinor can be expressed in terms of a complex two-component Weyl spinor w_α, and the chiral phase transformation $w_\alpha \to e^{i\phi} w_\alpha$ is the symmetry of the tree Lagrangian.

There is, however, a discrete two-dimensional remnant of this four-dimensional chiral symmetry.[2] Either of the transformations

$$\lambda_L \to -\lambda_L, \qquad \lambda_R \to -\lambda_R \tag{2.3}$$

leaves the Lagrangian (2.2) invariant. The mass term

$$m\bar{\lambda}\lambda = -2im\lambda_L\lambda_R \tag{2.4}$$

would break this $Z_2 \otimes Z_2$ symmetry down to Z_2 (only a *simultaneous* change of sign of λ_L and λ_R is now allowed). We shall see later that, even in the massless case, the symmetry ($\lambda_L \to -\lambda_L$, $\lambda_R \to \lambda_R$) is actually not there in the full quantum theory due to an *anomaly* (this is true, at least, for $N = 2$ theory which we understand well).

III. $N = 2$: INSTANTONS AND CONDENSATE AT HIGH T

A. Preliminaries

Let us consider the theory (2.1) with two colors. As was already mentioned, the gauge-symmetry group of this theory, $\mathcal{G}$, is $SU(2)/Z_2 = SO(3)$ with nontrivial $\pi_1[\mathcal{G}] = Z_2$. It admits therefore noncontractible topologically nontrivial field configurations $\equiv$ instantons. All nontrivial configurations belong to one and the same topological class. In this section, we are interested only with the high temperature case where a quasiclassical description works and quantitative estimates are possible. Euclidean path integrals are defined on the asymmetrical two-dimensional torus which is very long in the spatial direction, $L \gg g^{-1}$, and narrow in the Euclidean time direction, $\beta = 1/T \ll g^{-1}$.

To understand better how instantons appear, let us write down the high-T effective potential on the *constant* A_0 background in this theory. The evaluation of the one-loop fermion determinant (in two dimensions, there are no physical degrees of freedom associated with gauge fields, and the latter do not contribute; technically, the contribution of longitudinal degrees of freedom A_1^a cancels out the contribution of the ghosts) gives [16, 17]

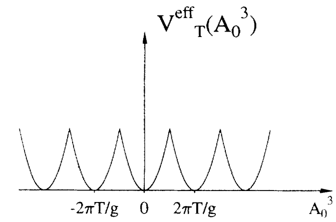

FIG. 1. Effective potential in adjoint SU(2) theory at high T.

$$V_T^{\text{eff}}(A_0^3) = \frac{g^2}{2\pi}\left[\left(A_0^3 + \frac{\pi T}{g}\right)_{\text{mod}\,\frac{2\pi T}{g}} - \frac{\pi T}{g}\right]^2, \tag{3.1}$$

where we directed A_0^a along the third isotopic axis for definiteness. The potential (3.1) is plotted in Fig. 1. It has exactly the same form as in Schwinger model [9] and is quite analogous to the similar potential $V_T^{\text{eff}}(A_0^3)$ for pure Yang-Mills theory in four dimensions [18].

The potential (3.1) is periodic with the period $2\pi T/g$. The periodicity is not causal. Really, the variable A_0^a is canonically conjugate to the Gauss law constraint, and the matrix

$$O^{ab} = \exp\{\beta g f^{abc} A_0^c\} \tag{3.2}$$

($f^{abc} \equiv \epsilon^{abc}$ for $N = 2$) may and should be thought of as the gauge transformation matrix acting on the dynamic variables A_1^a, λ_α^a. Now, the points $A_0^a = 0$ and $A_0^a = \delta^{a3}\, 2\pi T/g$ correspond to one and the same matrix $O^{ab} = \delta^{ab}$ and are therefore physically equivalent (see Ref. [11] for more detailed discussion).

One can consider, however, a field configuration which is x dependent and *interpolates* between the values $A_0^a = 0$ at $x = -\infty$ and $A_0^a = \delta^{a3}\, 2\pi T/g$ at $x = \infty$. It presents a noncontractible loop in SO(3) and cannot be trivialized. The instanton is the configuration belonging to this class with the minimal action. Usually, e.g., in four-dimensional Yang-Mills theory, the term "instanton" applies to a solution of classical equations of motion, i.e., to the configuration which minimizes the *tree* action. In two dimensions, this definition is not convenient for two reasons. First, such a classical solution does not have nice properties—it is just a constant field strength configuration which is smeared out over the whole volume $V = \beta L$ with the very small field strength $F_{01}^3 = 2\pi T/gL$ (A_0^3 interpolates between 0 at $x = -L/2$ and $2\pi T/g$ at $x = L/2$ with constant slope). Second, quantum corrections can be taken into account explicitly here—at high T, higher loop corrections to the potential (3.1) are small. (In the exactly soluble Schwinger model, they are just absent at any temperature.) And if so, why not do it? Thus, our definition of an instanton is the configuration which minimizes the *effective* action, quantum corrections being taken into account.

How does one do that? One may be tempted to al-

[2] I am indebted to I. Klebanov who brought this point to my attention.

low the argument A_0^3 in Eq. (3.1) to be x dependent, add the tree-level kinetic term $\frac{1}{2}(\partial_x A_0^a)^2$, and solve the equations of motion for the effective Lagrangian thus obtained. Though this naive procedure gives even the correct answer for the profile of the instanton, it cannot be justified—the expansion over derivatives of $A_0(x)$ breaks down at the point $A_0^3 = \pi T/g$ due to severe infrared singularities [11], and the true effective Lagrangian is highly nonlocal. One should proceed more accurately.

B. Fermion determinant and zero modes

As we have seen, the instanton presents a noncontractable loop $O^{ab}(x)$ in the SO(3) group. In the covering $SU(2) \equiv S^3$, it corresponds to a path which goes from the north pole $U \in SU(2) = 1$ at $x = -\infty$ to the south pole $U \in SU(2) = -1$ at $x = \infty$. By symmetry considerations, the path which minimizes the action should go along one of the meridians. Each such meridian corresponds to the *Ansatz*

$$A_0^a(x) = n^a a(x),$$
$$a(-\infty) = 0, \quad a(\infty) = 2\pi T/g, \tag{3.3}$$

where n^a is the unit color vector. Let us choose for definiteness $n^a = \delta^{a3}$ and calculate the fermion determinant on this background. Minimizing the effective action thus obtained, we will find the profile of the instanton $a(x)$ and evaluate its contribution to the partition function.

Right from the beginning, however, we run into a problem. The matter is, that the Lagrangian (2.1) is well defined in Minkowski space but not in Euclidean space. In Euclidean space, we cannot keep the fermion fields real—if we try to do so, the Euclidean counterpart of (2.2) with $\partial_0 \to i\partial_0$ becomes complex. This problem is well know in four dimensions [19] and its resolution is also known [20, 14]. One should *define* the integral over Majorana fields as the square root of the determinant of the full Dirac operator. The latter is well defined also in Euclidean space. The extraction of the square root also does not present problems here. The matter is that the spectrum of the eigenvalue equation for the Euclidean Dirac operator for complex adjoint fields,

$$\gamma_\mu^E(\partial_\mu \delta^{ab} \; - \; g\epsilon^{abc}A_\mu^c)\psi_n^b(x,\tau) = \mu_n \; \psi_n^a(x,\tau), \tag{3.4}$$

has a double-degenerate spectrum (γ_μ^E are anti-Hermitian Euclidean γ matrices). If $\psi_n(x,\tau)$ is a complex solution to (3.4), the function

$$\tilde{\psi}_n(x,\tau) = C\psi_n^*(x,\tau) \tag{3.5}$$

is also a solution with the same eigenvalue μ_n. [C is the charge conjugation matrix defined by $(\gamma_\mu^E)^* = -(\gamma_\mu^E)^T = C\gamma_\mu^E C^{-1}$. In two dimensions, $C = \sigma^2$, under the particular choice $\gamma_0^E = i\sigma^2, \gamma_1^E = i\sigma^1$.] In view of $C^*C = -1$, the two solutions are linearly independent. Hence, the square root is taken without pain:

$$[\text{Det } \|\hat{\mathcal{D}}\|]^{1/2} \; = \; \left[\prod_n \mu_n^2\right]^{1/2} \; = \; \prod_n \mu_n, \tag{3.6}$$

where only one of the double-degenerate eigenvalues μ_n is accounted for in the product. Let us write Eq. (3.4) on the Abelian background (3.3). It splits apart in two:

$$\gamma_\mu^E(\partial_\mu \; - \; ig\delta_{\mu 0}a)\psi_n^- = \mu_n \; \psi_n^-,$$
$$\gamma_\mu^E(\partial_\mu + ig\delta_{\mu 0}a)\psi_n^+ = \mu_n \; \psi_n^+, \tag{3.7}$$

where $\psi^\pm = \psi^1 \pm i\psi^2$. [There is also the third equation for ψ^3, but it is just a free one—the component ψ^3 decouples from the background (3.3).] It is explicitly seen that the solutions to these two equations are related by the transformation (3.5). The equations are exactly the same as for two-dimensional QED on the instanton background $A_\mu(x) = \delta_{\mu 0}a(x)$ for the fermions with the charges g and $-g$, respectively.[3] Thus, we need not calculate the determinant anew, but rather use the results of [6, 9] where the instanton Dirac determinant has been calculated for the Abelian theory:

$$\left[\text{Det}_{\text{Ab Ans}}\|i\hat{\mathcal{D}}\|\right]^{1/2} = \left\{\left[\text{Det}_{\text{QED}}\|i\hat{\mathcal{D}}\|\right]^2\right\}^{1/2}$$
$$= \text{Det}_{\text{QED}}\|i\hat{\mathcal{D}}\|. \tag{3.8}$$

Now, it is a proper time to note that all these determinants calculated on the instanton background just turn to zero for strictly massless fermions due to the presence of fermion zero modes in the spectrum. Each of the equations in (3.7) has exactly one normalizable solution with $\mu = 0$, the left one for ψ^- and the right one for ψ^+:

$$\psi_\alpha^{-(0)}(x,\tau) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-g\phi(x)}e^{i\pi T\tau},$$
$$\psi_\alpha^{+(0)}(x,\tau) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-g\phi(x)}e^{-i\pi T\tau}, \tag{3.9}$$

where

$$\partial\phi/\partial x \; = \; a(x) \; - \; \frac{\pi T}{g} \tag{3.10}$$

[the τ dependence provides the correct antiperiodic boundary conditions $\psi(\beta) = -\psi(0)$ for the fermion fields in Euclidean time direction]. We show in the Appendix that zero mode solutions are still there also for configurations involving small fluctuations around the Abelian *Ansatz* (3.3).

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[3]In the standard Abelian convention, this configuration should be called an anti-instanton rather than an instanton—its topological charge (1.2) is equal to -1. But in $\text{QCD}_2^{\text{adj}}$ with $N = 2$, all noncontractible configurations (3.3) belong to one and the same (the instanton) topological class. To make the analogy between the non-Abelian and Abelian theory more clear, we have changed the sign convention in the latter. Also the prepotential $\phi(x)$ defined in Eq. (3.10) has the opposite sign compared to that in Refs. [6, 11, 9].

The presence of fermion zero modes suppresses the contribution of topologically nontrivial sectors to the partition function exactly in the same way as it does in QCD_4. To get a nontrivial result, one should introduce a small but finite fermion mass $m \ll g$. In that case, the partition function involves $\mathrm{Det}\|i\hat{\mathcal{D}} - m\|$ rather than just $\mathrm{Det}\|i\hat{\mathcal{D}}\|$, and the whole result (3.8) will be proportional to m.

The accurate calculation of the determinant gives the result [6]

$$\left[\mathrm{Det}_{\mathrm{Ab\ ans}}\|i\hat{\mathcal{D}} - m\|\right]^{1/2} = \mathrm{Det}_{\mathrm{QED}}\|i\hat{\mathcal{D}} - m\| \propto \left(m\int dx\, e^{-2g\phi(x)}\right)\exp\left\{-\frac{\beta g^2}{2\pi}\int dy a^2(y)\right\}. \tag{3.11}$$

The second factor in the determinant comes from nonzero modes. In the Schwinger model, it was responsible for generating the photon mass. The first factor is due to the zero modes. The proportionality coefficient in (3.11) can be explicitly determined (in a finite box which provides infrared regularization) if choosing a particular convention for $\phi(x)$ [Eq. (3.10) defines $\phi(x)$ only up to an arbitrary constant]. We refer the reader to Ref. [6] for a detailed and accurate analysis.

If we substitute now the result (3.11) in the bosonic functional integral, calculate it, differentiate over mass, and divide over the similar functional integral for the partition function Z_0 in the topologically trivial sector, we obtain the expectation value for the operator $\bar{\lambda}\lambda$, i.e., the fermion condensate.

Let us recall how it has been done in the Schwinger model. The functional integral in the one-instanton topological sector had the form

$$Z_1 \propto Z_0 m \int \prod_y d\phi(y) \int dx\, e^{-2g\phi(x)} \times \exp\left\{-\frac{\beta}{2}\int dy\phi(y)\left[\frac{\partial^4}{\partial y^4} - \frac{g^2}{\pi}\frac{\partial^2}{\partial y^2}\right]\phi(y)\right\}. \tag{3.12}$$

The saddle points of this integral were determined from the equation

$$\left[\frac{\partial^4}{\partial y^4} - \frac{g^2}{\pi}\frac{\partial^2}{\partial y^2}\right]\phi(y) = -2gT\delta(y-x). \tag{3.13}$$

(The parameter x has the meaning of the collective coordinate describing the position of the instanton.) Substituting the solution of this equation in Eq. (3.10), we get the result [11, 9]

$$a(y) = \begin{bmatrix} \frac{\pi T}{g}\exp\{\mu(y-x)\}, & y \le x, \\ \frac{\pi T}{g}[2 - \exp\{\mu(x-y)\}], & y \ge x, \end{bmatrix} \tag{3.14}$$

where $\mu = g/\sqrt{\pi}$. The function $a(y)$ is plotted in Fig. 2. The field density $E(y) = -\partial a(y)/\partial y$ is localized at $|y - x| \sim \mu^{-1}$ so that the topological charge (1.2) is equal to -1 as expected. The characteristic quantum fluctuations

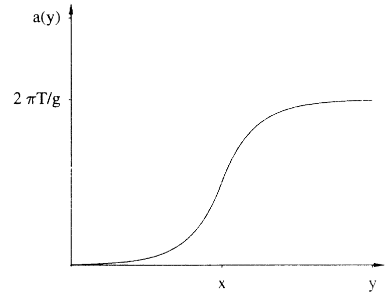

FIG. 2. High-T instanton.

determined by the integral (3.12) are $a^{qu} \sim \sqrt{T/g}$ [11, 9] which is much less than the characteristic amplitude of the solution (3.14) $a^{\mathrm{cl}} \sim T/g$ so that the quasiclasical picture works.

Calculating the whole integral (3.12) and adding the equal contribution from the one-anti-instanton sector (in the Abelian case, the relevant topology is $\pi_1[\mathrm{U}(1)] = Z$ and instanton and anti-instanton configurations are topologically nonequivalent), one obtains the following result for the fermion condensate [6]:

$$\langle\bar{\psi}\psi\rangle_{T\gg g} = -\frac{1}{\beta L Z_0}\frac{\partial}{\partial m}(Z_1 + Z_{-1}) = -2T\exp\left\{-\frac{\pi^{3/2}T}{g}\right\} \tag{3.15}$$

[the large factor L in the denominator cancels out the large factor L coming from the integration over translational zero mode of the instanton solution (3.14)].

Let us turn now to the non-Abelian case. In the framework of the *Ansatz* (3.3), the functional integral for Z_1 is basically the same as in the Schwinger model, and its saddle point is given by the same expression (3.14). However, two novel features appear. First of all, in addition to integrating over $\prod da(y)$ and dx, we should integrate also over dn^a in the *Ansatz* (3.3). n^a is the new collective coordinate describing the orientation of the instanton in color space. Naturally, the rotation of n^a does not change the action and corresponds to zero modes in the bosonic determinant. Let us make an estimate for the contribution of these zero modes. The general method for such a calculation is presenting the integral over quantum fluctuations over the classical solutions (3.3), (3.14) which do not change the action as the integral over collective coordinates n^a [21–23]. To this end, one should express $A_0^{qu(0)}(y)$ as a sum of two independent normalized zero modes:

$$A_0^{qu(0)}(y) = c_a^{(0)}\frac{\partial A_0^{\mathrm{cl}}(y)}{\partial n^b}\left[\int dy\left(\frac{\partial A_0^{\mathrm{cl}}(y)}{\partial n^c}\right)^2/2\right]^{-1/2} \times(1 - n^a n^b) \tag{3.16}$$

and then write

$$d^{(0)}A_0^{qu}(y) \sim dc^a dc^b(1 - n^a n^b) \sim d\mathbf{n}\int dy\left(\frac{\partial A_0^{\mathrm{cl}}(y)}{\partial n^a}\right)^2. \tag{3.17}$$

The representation (3.3) is, however, not convenient for this purpose because the zero modes $\partial A_0^{\rm cl}/\partial n^a$ appear to be not normalizable (this difficulty is also well known in four-dimensional theories [21]). The paradox can be resolved by noting that the proper measure in the functional integral is $\prod_y d^{\rm inv} O^{ab}(y)$, $O^{ab}(y)$ being given by Eq. (3.2), rather than just $\prod_{ya} dA_0^a(y)$. Thus, we have

$$\int d^{(0)} O^{qu}_{ab}(y) \sim \int d\mathbf{n} \int dy \left(\frac{\partial O^{\rm cl}_{ab}(y)}{\partial n^c}\right)^2 \sim \int dy \sin^2\left[\frac{g a^{\rm cl}(y)}{2T}\right] \sim \frac{1}{g}. \quad (3.18)$$

To find the condensate, we should divide Z_1 by Z_0. The latter may be estimated in the one-loop approximation (see, however, the discussion of the validity of this approximation later in the paper) in which case the fermion condensate depends on the ratio of two bosonic determinants—one calculated on the background (3.3), (3.14), and the other on the trivial background $O^{ab}(y) = \delta^{ab}$. Thus, one should divide the result (3.18) by the corresponding integral in the topologically trivial sector where only the constant harmonics of $A_0^1(y)$ and $A_0^2(y)$ should be taken into account [the integrals over y-dependent parts of $A_0^1(y)$ and $A_0^2(y)$ cancel out the contribution of *nonzero* modes in the bosonic determinant in the topologically trivial sector [18, 16, 17, 9]]. The range of y where this constant harmonic mode should be normalized is the characteristic size of the instanton $\propto g^{-1}$ ($O^{ab} \sim \delta^{ab}$ far away from the instanton center and the contributions of these distances in Z_1 and Z_0 cancel out). Hence, the denominator over which the integral (3.18) should be divided is

$$\sim \int_{|y-x|\sim g^{-1}} dy \left(\frac{\partial O^{ab}}{\partial A_0^c}\right)^2 \int dA_0^1\, dA_0^2 \quad (3.19)$$

$$\times \exp\left\{-\frac{\beta g^2}{2\pi}\int_{|y-x|\sim g^{-1}} dy\, [(A_0^1)^2 + (A_0^2)^2]\right\}$$

$$\sim \frac{1}{g}(\beta g)^2\, \frac{1}{\beta g} \sim \frac{1}{T}. \quad (3.20)$$

Thus, rotational zero modes provide the factor $\propto T/g$ in the condensate. In fact, this estimate could be obtained immediately using the rule of thumb coined in [22] (see also [23]): Each bosonic zero mode provides the factor $\sqrt{S_0}$ in the measure where S_0 is the instanton action. In our case, $S_0 = \pi^{3/2} T/g$ [see Eq. (3.15)], and there are two rotational zero modes. Our final result for the fermion condensate in $\mathrm{QCD}_2^{\rm adj}$ with $N = 2$ at high T is

$$\langle\bar\lambda\lambda\rangle_{T\gg g} = -\frac{1}{\beta L Z_0}\, \frac{\partial}{\partial m}(Z_1) = C\frac{T^2}{g}\exp\left\{-\frac{\pi^{3/2}T}{g}\right\}. \quad (3.21)$$

Unfortunately, the numerical coefficient C cannot be fixed here and a more accurate calculation for the ratio of determinants which could, in principle, be done would not help. The matter is (and this is the second and more serious nuisance which distinguishes the non-Abelian case compared to the exactly soluble Schwinger model) that the partition function Z_0 in the topologically trivial sector by which the integral for $\bar\lambda\lambda$ should be divided *cannot* not be determined analytically here—the one-loop approximation is not justified and higher-loop effects provide a comparable contribution in the free energy. We return to the discussion of this point in Sec. V.

It is convenient for us to adjourn now for a while the discussion of high-T instanton physics and look first what happens in the low temperature region.

IV. LOW TEMPERATURES

Consider now $\mathrm{QCD}_2^{\rm adj}$ with $N = 2$ at $T = 0$. Let us assume that the fields contributing to the Euclidean path integral tend to pure gauge at spatial infinity:

$$i\epsilon^{abc} A_\mu^a(x) \overset{r\to\infty}{\to} i\Omega^{-1}(x)\partial_\mu \Omega(x), \quad (4.1)$$

with $\Omega(x) \in \mathrm{SO}(3)$. All fields belong to one of two topological classes: the trivial class consisting of the fields which can be continuously deformed to zero and the instanton class for which $\Omega(x)$ presents a noncontractible loop in the SO(3) group when x goes around the large spatial circle. Another way to look at the problem is to define the theory on a large two-dimensional sphere. A topologically nontrivial field cannot be written as a uniform regular expression on the whole sphere. Such a field can be described by use of two different regular expressions defined on two patches, the northern and the southern hemispheres, which are glued together (related by a gauge transformation) on the equator. The transition matrix $\Omega(\phi)$ presents then a nontrivial loop in the SO(3) group (cf. the analogous description of the Schwinger model in Ref. [8]).

High-T analysis has taught us that the fields belonging to the instanton class involve two fermion zero modes related to each other by the transformation (3.5).[4] That means that the partition function in the topologically nontrivial sector Z_1 involves a factor m and the fermion condensate is generated:

$$\langle\bar\lambda\lambda\rangle_{T=0} = \mp \lim_{m\to 0}\frac{1}{V Z_0}\,\frac{\partial}{\partial m}(Z_1) \sim g, \quad (4.2)$$

where V is the volume of our two-dimensional sphere. In contrast with the high-T case, a one-loop calculation for Z_1 makes no sense here as quantum fluctuations are out of control. The estimate (4.2) has been done purely on dimensional grounds. The two signs in Eq. (4.2) correspond to two possible choices for the partition function:

$$Z_\pm = Z_0 \pm Z_1. \quad (4.3)$$

The freedom in choosing the sign is exactly analogous to the freedom of choice of the vacuum angle θ in QCD_4 or in the Schwinger model. The difference is that here

[4] We return to the discussion of this point in Sec. VII.

we have only two topologically distinct sectors and the "vacuum angle" can acquire only two values: 0 or π. In the Hamiltonian language, the choices (4.3) correspond to two possible superselection rules imposed on the wave functionals. The plus and minus sectors of the theory do not talk to each other: The matrix elements of all physical operators between the states from different sectors are zero.

The spectrum of the theory does not include massless states, the lowest excited state having the mass $M^{\rm gap}$ of order g [1]. That means that, for large volumes $Vg^2 \gg 1$, the partition functions $Z_\pm$ enjoy the extensive property [24]

$$Z_\pm \propto \exp\{-\epsilon_\pm^{\rm vac}(m,g)V\} \tag{4.4}$$

and the finite size corrections to the vacuum energy are exponentially small $\propto \exp\{-M^{\rm gap}R\}$. The presence of the condensate (4.2) implies that the function $\epsilon^{\rm vac}(m,g)$ involves a nonzero first-order term of the Taylor expansion in m, and we can write, for $m \ll g$,

$$Z_\pm \propto \exp\{-m\langle\bar\lambda\lambda\rangle_\pm V\}, \tag{4.5}$$

with $\langle\bar\lambda\lambda\rangle_- = -\langle\bar\lambda\lambda\rangle_+$, and hence

$$Z_0 \propto \cosh\{m\langle\bar\lambda\lambda\rangle_+ V\},$$
$$Z_1 \propto -\sinh\{m\langle\bar\lambda\lambda\rangle_+ V\}. \tag{4.6}$$

The result (4.6) is the analogue of the result $Z_\nu \propto I_\nu(m|\bar\psi\psi|V)$ for the partition function in the sector with a given topological charge ν for QCD_4 with *one* light fermion flavor derived in [14].

Note that the representations (4.5) and (4.6) are valid as long as $m \ll g$, $Vg^2 \gg 1$; the dimensionless combination $x = m|\langle\bar\lambda\lambda\rangle_\pm|V$ may be either large or small. The instanton zero modes are responsible for the formation of the condensate only in the limit when x is small and $Z_1 \propto x$. In the physical large volume limit (large x), the value of the condensate *is* the same but the mechanism for its formation is quite different being related to small $\propto 1/|\langle\bar\lambda\lambda\rangle_\pm|V$ but nonzero modes of the Dirac operator (see [14] for detailed explanations and discussions).

The presence of two fermion zero modes in the instanton background gives rise to the 't Hooft term $\sim \bar\lambda^a\lambda^a \sim \bar\lambda_L^a\lambda_R^a$ in the effective Lagrangian. That means that the $Z_2 \otimes Z_2$ symmetry (2.3) is in fact anomalous—quantum corrections break it down to Z_2 *explicitly.* And that means that the condensate $\langle\bar\lambda^a\lambda^a\rangle$ does not break spontaneously any symmetry of the full quantum theory. The appearance of two sectors of the theory (4.3) with opposite signs of the condensate should *not* be interpreted as a spontaneous breaking because, as we have already mentioned, these two sectors correspond to different superselection rules which should be imposed uniformly in the whole physical space and the formation of the "domains" separated by the "walls" so that $\langle\bar\lambda\lambda\rangle$ is negative to the left and positive to the right is not possible.[5]

Again, the situation is exactly the same as in QCD_4 with $N_f = 1$—the presence of the sectors with different θ in the theory should not be interpreted as a spontaneous breaking of U(1) symmetry. θ is one and the same in the whole physical space and the spatial fluctuations of θ (which would give rise to Goldstone bosons) are not possible.

The existence of the condensate is also clearly seen in the framework of the bosonization approach. Since [25], it is known that a theory involving Majorana fermion fields λ^a is dual to some other theory involving the bosonic field presenting an orthogonal matrix Φ^{ab}. The correlators of all fermion bilinears in the original theory coincide identically with the correlators of their bosonized counterparts in the bosonized theory. For the scalar bilinear $\bar\lambda^a\lambda^b$, the correspondence rule is just

$$\bar\lambda^a\lambda^b \equiv \mu\Phi^{ab}, \tag{4.7}$$

where μ depends on the normalization procedure for the operator Φ^{ab}. μ is of order g if the normalization convention $\langle\Phi^{ab}\rangle_{\rm vac} = \delta^{ab}$ is chosen. It is obvious then that

$$\langle\bar\lambda^a\lambda^a\rangle_{\rm vac} \sim \mu \sim g. \tag{4.8}$$

Note the difference with the theory involving fundamental Dirac fermions. For $QCD_2^{\rm fund}$, the bosonization rule is not (4.7) but rather

$$\bar\psi^i\psi^j \equiv \mu U^{ij}\exp\left(i\sqrt{\frac{4\pi}{N}}\phi\right), \tag{4.9}$$

where U is the unitary SU(N) matrix, and ϕ is a light color singlet. In that case, the normalization mass μ is not g but depends on the mass of the scalar singlet which in turn depends on the fermion mass m. Both μ and the light singlet mass tend to zero in the limit $m \to 0$ for any finite N (and the singlet becomes sterile) [26]. The condensate $\langle\bar\psi^i\psi^i\rangle_{\rm vac}$ also tends to zero in the massless limit. One can say that the light singlet ϕ smears the condensate away.[6]

But in the adjoint theory, all fields in the spectrum are massive and the condensate (4.8) survives.

The rapid falloff of the condensate at high temperature as given by Eq. (3.21) is also naturally explained in the bosonization language. Taking into account finite T effects, namely, the presence of excited states in the heat bath, the thermal average $\langle\Phi^{ab}\rangle_T$ is no longer δ^{ab}, but can acquire any value on the SO(3) group with almost equal (at high $T \gg g$) probability, and

$$\langle\Phi^{aa}\rangle_{T\to\infty} \to \int d^{\rm inv}\Phi\ \chi^{\rm adj}(\Phi) = 0. \tag{4.10}$$

As follows from Eq. (3.21), for high but finite T the direction δ^{ab} in the group is still a little bit preferred,

[5]Note that, if the plus and minus sectors could talk to each other, the walls between them would have a finite energy (due to the absence of transverse directions), and the condensate would vanish. (It is not easy to break not only a continuous but also a discrete symmetry in two dimensions.) But it does not.

[6]The absence of the condensate in $QCD_2^{\rm fund}$ is, of course, natural. The condensate would break spontaneously the global chiral symmetry, and such a breaking is not allowed in two dimensions [15].

and the condensate is still nonzero though exponentially small. The physical picture is exactly the same as in the Schwinger model where the quantitative calculation is possible at any temperature [7].

V. HIGH-T PARTITION FUNCTION

All the arguments of the previous section which have led to the results (4.5) and (4.6) can be repeated without change also for high temperatures. We only have to substitute βL for V and $\langle\bar{\lambda}\lambda\rangle_T$ for $\langle\bar{\lambda}\lambda\rangle_{\rm vac}$. Let us look how the partition functions (4.6) behave when the spatial volume L is very large:

$$x_T = m\beta L|\langle\bar{\lambda}\lambda\rangle_T| \gg 1. \tag{5.1}$$

The cosh and sinh functions in Eq. (4.6) can be expanded in the series and, if x_T is large, the number of the terms in the series to be taken into account is also large. Each such term is

$$Z^{(k)} = \frac{(-m\beta L\langle\bar{\lambda}\lambda\rangle_T)^k}{k!}, \tag{5.2}$$

where k is even for Z_0 and odd for Z_1. The series converge at $k \sim x_T$. The contribution (5.2) in the partition function can be interpreted as being due to k instantons (3.3). Each instanton brings about the factor m from the fermion zero mode and the factor L from the translational bosonic zero mode in the partition function. The instantons are very well spatially separated, the characteristic interinstanton distance being of order $L/k^{\rm char} \sim 1/(m\beta|\langle\bar{\lambda}\lambda\rangle_T|) \gg g^{-1}$. Thus, we see that the characteristic field configurations in the high-T partition function present a rarefied noninteracting instanton gas. Naturally, the total number of instantons is even for Z_0 (the configuration is topologically trivial) and odd for Z_1.

The same picture is valid in high-T Schwinger model [9] and in high-T QCD$_4$ [13]. Note that we cannot extrapolate it to low temperatures. When $T < g$, the characteristic separation between instantons is of the same order as their size $\sim g^{-1}$, and their interaction (as well as distortion of their form due to quantum fluctuations) cannot be neglected. Instead of a rarefied instanton gas, we have a dense strongly interacting instanton liquid [13, 27].

It is interesting to look also at the limit when L is kept large but finite and m tends to zero. In strictly massless theory, Z_1 vanishes and Z_0 has no trace of instantons at all. The explanation is simple. Consider the contribution of two well-separated instantons to Z_0. The zero modes of individual instantons are now shifted from zero, but the shift is tiny:

$$\mu^{\rm quasizero}(R) \sim \exp\{-\pi T R\}, \tag{5.3}$$

where R is the interinstanton separation. Thus, the large R configurations provide exponentially small contribution to the path integral, instantons are "confined," and cannot be separated from each other. (The same phenomenon occurs in the Schwinger model [11, 9]. For QCD$_2^{\rm adj}$; it has been actually observed in Ref. [17].)

If $m \neq 0$, the contribution of the two-instanton contribution to the path integral ceases to depend on R as soon as $\mu^{\rm quasizero}(R) \ll m$. If m is large enough [the condition (5.1) is satisfied], two-instanton contribution dominates over zero-instanton one: Instantons are "liberated."

Now, the time has come to pay our old debt and to discuss nonperturbative effects in Z_0 for the massless theory (one can forget about instantons until the end of this section). Let us estimate the free energy density $F = -TL^{-1}\ln Z_0$ of the theory at high temperature. In the leading order, it is given just by the free fermion loop and is of order T^2. But what are preasymptotic effects? It is instructive to consider first the Schwinger model. In the bosonized language, it is just a theory of free scalars with mass $\mu = g/\sqrt{\pi}$. At finite T, they are excited and the exact expression for F is

$$F_{\rm SM}(T) = \int_{-\infty}^{\infty}\frac{dp}{2\pi}\ln\left[1 - e^{-\beta\sqrt{p^2+\mu^2}}\right]. \tag{5.4}$$

Its high-T asymptotics is

$$F_{\rm SM}(T \gg \mu) = -\frac{\pi T^2}{6}\left[1 - \frac{3\mu}{\pi T} + o\left(\frac{\mu}{T}\right)\right]. \tag{5.5}$$

When $T \sim \mu$, subleading effects are essential.

For QCD$_2^{\rm adj}$, the qualitative estimate is the same, but we cannot determine now the coefficient of preasymptotic term exactly: The mass of bosons in the spectrum cannot be determined analytically, and their interaction cannot be neglected. Thus, we can only write

$$\Delta F^{\rm adj}_{\rm nonpert}(T) \sim gT. \tag{5.6}$$

That is the same uncertainty which prevented us from determining the exact coefficient in Eq. (3.21): The uncertainty in Z_0 is

$$\sim \exp\{-\beta\Delta F_{\rm nonpert}(T)\, g^{-1}\} \sim 1 \tag{5.7}$$

[g^{-1} is the instanton size where the background field (3.3) differs essentially from zero and the determinants of fluctuations in Z_1 and Z_0 are different].[7]

VI. $N \geq 3$

A. High T

Let us repeat the analysis of Sec. III for higher color groups. Consider first the case $N = 3$. The effective potential on the constant A_0 background has been calculated in Ref. [16]. For $N = 3$, it depends on two group invariants: $A_0^aA_0^a$ and $d^{abc}d^{ade}A_0^bA_0^cA_0^dA_0^e$. It is convenient to choose the matrix $A_0^at^a$ in the diagonal form

$$A_0^at^a = \text{diag}\,(a_1,\ a_2,\ a_3), \quad \sum_i a_i = 0, \tag{6.1}$$

---

[7]Note that uncertainties of essentially the same kind in the determination of the instanton measure appear also in QCD$_4$ *when the size of the instanton ρ becomes comparable with the characteristic scale of the theory.* The corrections to the measure are of order $\rho^4\epsilon^{\rm vac}_{\rm QCD} \sim \rho^4\Lambda^4_{\rm QCD}$ [28]. When $\rho\Lambda_{\rm QCD} \sim 1$, the situation is out of control.

and write the effective potential as a function of a_i (or, if you will, as a function of A_0^3 and A_0^8). The result is

$$V(a_i) = \frac{g^2}{2\pi} \sum_{i>j}^{3} \left[\left(a_i - a_j + \frac{\pi T}{g} \right)_{\mathrm{mod}\, \frac{2\pi T}{g}} - \frac{\pi T}{g} \right]^2 . \tag{6.2}$$

This potential has a hexagonal symmetry. The structure of its minima is shown in Fig. 3.

What is the proper range of integration over A_0^a in the functional integral? As was discussed earlier, the proper integration variable is not A_0^a but rather the adjoint gauge transformation matrix (3.2) (which was the orthogonal matrix in the case $N = 2$). To count each such matrix only once, we should restrict the range of integration by the "small" Weyl cell (marked out by the dashed lines inside the solid traingle in Fig. 3) which is spread out over the whole $SU(3)/Z_3$ group by the transformations from the torus of the group with nonzero $A_0^{1,2,4,5,6,7}$.

Note that in the general case where the theory involves also fundamental matter fields, the integration goes over unitary matrices $U = \exp\{i\beta g A_0^a t^a\}$ which are different in the three different classes of minima:

$$U_0 = 1, \quad U_\Box = e^{2\pi i/3}, \quad U_\triangle = e^{-2\pi i/3}. \tag{6.3}$$

In a theory with fundamental matter, these three sets of points mark out physically different gauge transformations; the counterpart of Eq. (6.2) would also be different at these points: $V^{\mathrm{fund}}(U_0) \neq V^{\mathrm{fund}}(U_\Box) \neq V^{\mathrm{fund}}(U_\triangle)$, and the proper integration region would be the standard Weyl cell (solid triangle in Fig. 3) + transformations from the torus.

But in $\mathrm{QCD}_2^{\mathrm{adj}}$, the proper gauge group is $SU(3)/Z_3$ rather than SU(3), and all minima of the potential (6.2) [which occur at the points (6.3)] should be identified. There are, however, noncontractible Euclidean configurations which interpolate between different center elements

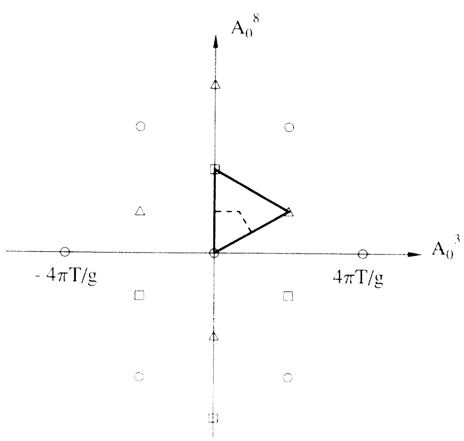

FIG. 3. Geometry of effective potential for high-T $\mathrm{QCD}_2^{\mathrm{adj}}$ with $N = 3$. The minima occur at the points ○, □, and △ which are related to each other by Z_3 transformations and are physically undistinguishable in the adjoint theory. The solid triangle marks out the standard "fundamental" Weyl cell and the dashed lines inside, the "adjoint" Weyl cell.

(6.3) of the unitary group so that, say,

$$U(x = -\infty) = 1, \qquad U(x = \infty) = e^{2\pi i/3}. \tag{6.4}$$

The configuration (6.4) presents a nontrivial loop in the $SU(3)/Z_3$ space. For $N = 3$, there are two different topologically nontrivial classes: the configurations (6.4) which may be called instantons and the configurations interpolating between 1 and $e^{-2\pi i/3}$ which may be called anti-instantons (double-instanton configurations are topologically equivalent to anti-instantons).

Consider a representative of the instanton class which has the form

$$t^a A_0^a(x) = \frac{1}{3} a(x)\, \mathrm{diag}(1, 1, -2),$$
$$a(-\infty) = 0, \quad a(\infty) = \frac{2\pi T}{g}. \tag{6.5}$$

It corresponds to going upwards along the vertical side of the solid triangle in Fig. 3 with the transformation from the torus being fixed to be trivial [so that different points on the side correspond to all different elements of the group $SU(3)/Z_3$; only the vertices are identified].

Let us estimate the fermion determinant in this background field. The eigenvalue equation for the Euclidean Dirac operator [the analogue of (3.7)] on the background (6.5) has the form

$$\gamma_\mu^E (\partial_\mu \pm iga(x)\delta_{\mu 0}) \psi_n^{4\pm i5} = \mu_n\, \psi_n^{4\pm i5},$$
$$\gamma_\mu^E (\partial_\mu \pm iga(x)\delta_{\mu 0}) \psi_n^{6\pm i7} = \mu_n\, \psi_n^{6\pm i7}, \tag{6.6}$$

and the components $\psi^{1,2,3,8}$ decouple from the background.

We see that the Dirac equation admits now not one but *two* pairs of zero modes (3.9). That means that the partition function in the instanton sector involves now the factor m^2 rather than just m. And that means that the contribution of topologically nontrivial sectors in the condensate is

$$\langle \bar\lambda^a \lambda^a \rangle_{T \gg g}^{N=3} = -\frac{1}{\beta L Z_0} \frac{\partial}{\partial m} (Z_I + Z_A) \propto m \tag{6.7}$$

and turns to zero in the massless limit. The situation looks the same as in QED_2 with two Dirac charged fermions where the fermion condensate is zero by the same reason.

A similar analysis can be done also for larger N. The generalization of the *Ansätze* (3.3), (6.5) for any N is

$$t^a A_0^a(x) = \frac{1}{N} a(x)\, \mathrm{diag}(1, 1, \ldots, 1 - N),$$
$$a(-\infty) = 0, \quad a(\infty) = \frac{2\pi T}{g}, \tag{6.8}$$

which supports $N - 1$ pairs of fermion zero modes. The determinant has the same structure as in the Schwinger model with $N - 1$ flavors, and the contribution to the condensate is

$$\langle \bar\lambda^a \lambda^a \rangle_{T \gg g}^{N} \propto m^{N-2}, \tag{6.9}$$

which vanishes in the massless limit.[8]

Thus, at high temperatures, fermion condensate seems not to be formed in QCD_2^{adj} with $N \geq 3$.

B. Low T: The paradox

The bosonization arguments of Sec. IV which have led to the conclusion of existence of the fermion condensate for $N = 2$ can be transferred without essential change to larger N. The theory involves now the set of $N^2 - 1$ Majorana fermion fields. Staying in the framework of the original Witten's paper [25] where only free fermions were discussed, we would have to put such a set of field into correspondence to the boson fields presenting orthogonal $SO(N^2 - 1)$ matrices. In the case when the fermions interact with gauge fields, it is more convenient, however, to write the bosonized theory in terms of the fields

$$\Phi^{ab} = \mathrm{Tr}\{t^a U t^b U^\dagger\}, \tag{6.10}$$

where $U \in SU(N)$ and $\Phi \in SU(N)/Z_N$. This modified bosonization procedure has been worked out in [29]. Φ^{ab} is dual to the scalar bilinear $\bar\lambda^a \lambda^b$ as written in Eq. (4.7). As earlier, $\mu \sim g$ and the estimate (4.8) for the fermion condensate is valid.

Again, the spectrum of the theory involves a gap and, in the limit $m \ll g$, $Vg^2 \gg 1$, the partition function can be written in the form

$$Z \propto \exp\{-m\langle\bar\lambda\lambda\rangle V\} \tag{6.11}$$

both for large and for small values of $m|\langle\bar\lambda\lambda\rangle|V$.

But that contradicts instanton arguments.

Let us consider for simplicity the case $N = 3$. There are three topological classes: the trivial, the instanton and the anti-instanton. In the topologically trivial sector, the partition function

$$Z_0 \propto \left\langle \prod_n (\lambda_n^2 + m^2) \right\rangle, \quad \lambda_n \neq 0, \tag{6.12}$$

is expanded in the even powers of m. The expansion of Z_I and Z_A in m also starts from the term $\propto m^2$ due to the presence of two pairs of fermion zero modes. It is absolutely not clear how the linear term in the expansion of the exponential (6.11) can appear.

Thus, bosonization arguments tell that the condensate *is* formed whereas the instanton arguments tell that it is *not* formed.

VII. CONFRONTING THE CONTROVERSY

The paradox appeared when putting together the following premises: (1) validity of topological classification; (2) the presence of $2(N-1)$ zero modes in the instanton sector; (3) bosonization arguments displaying the presence of condensate; (4) absence of massless states which allowed us to write the partition function in the extensive form (6.11) also for small values of the exponent.

The only way to resolve the paradox is to invalidate one of these premises.

For example, in the conventional QCD_4 with several flavors where instantons involve N_f zero modes, their contribution to the partition function is $\propto m^{N_f}$ but the condensate is still generated without any paradox because premise (4) is false. There are Goldstone states in the spectrum which lead to finite volume effects which are essential in the region of small $m|\langle\bar\psi\psi\rangle|V$ and the partition function cannot be written in the extensive form (6.11) but has a more complicated structure [14]. But in our case, no continuous symmetry is broken spontaneously and there are no Goldstone bosons.

At first sight, the weakest point is the second premise. We have obtained $2(N-1)$ zero modes by solving explicitly the Dirac equation in a particular background. We have also checked that the zero modes are stable with respect to small deformations of background (see the Appendix). But we cannot write down an index theorem which would enforce the presence of $2(N-1)$ zero modes for *any* background belonging to the instanton class. The "normal" index $n_L^0 - n_R^0 \propto \int \mathrm{Tr}\{F_{\mu\nu}^a t^a\}\epsilon_{\mu\nu} d^2x$ is just zero in QCD_2 [indeed, we have established the presence of $N-1$ left-handed and $N-1$ right-handed zero modes related to each other by the transformation (3.5)], and we do not know of any other relevant integral invariant.

Thus, we cannot rule out that, for some fields belonging to the instanton class and located at some finite distance from the Abelian *Ansatz* in Hilbert space, the number of zero modes is less which would allow the generation of the condensate.

We think, however, that it is not the case, and there *is* some index theorem prescribing the existence of exactly $2(N-1)$ zero modes; only we are not clever enough to unravel it. The reason why we believe it is the following.

The paradox discovered is actually not specific for QCD_2^{adj}. The same paradox appears also in some four-dimensional gauge theories where the conventional Atiah-Singer theorem works and the analogue of our premise (2) is certainly valid.

As we have already mentioned, there is no paradox in the conventional QCD. Consider, however, supersymmetric $d = 4$, $N = 1$ non-Abelian Yang-Mills theories involving a Majorana fermion in the adjoint representation of the gauge group. The paradox *does* not arise when the group is unitary. Let us understand why.

At first sight, it does. The fields belonging to the instanton class involve $2N_c$ fermion zero modes [the trace $\mathrm{Tr}\{T^a T^a\}$ which enters the index theorem differs, for the generators T^a in the adjoint representation, by the factor $2N_c$ from the analogous trace for the fundamental representation]. That means that the instanton contribution to the partition function involves a factor m^{N_c}. On the other hand, exact supersymmetric Ward identities tell us that the correlator $\langle\bar\lambda\lambda(x_1)\cdots\bar\lambda\lambda(x_{N_c})\rangle$ does not depend on x_i. The computation in the instanton background gives a nonzero result which implies that the correlator

[8]For $N > 3$, the leading contribution to the condensate comes not from instantons but just from the topologically trivial sector. The latter gives $\langle\bar\lambda\lambda\rangle \propto m$ for any N [cf. Eq. (8.22) in Ref. [14]].

does not vanish also when all $|x_i - x_j|$ tend to ∞ [30]. And that implies the presence of the condensate $\langle\bar{\lambda}\lambda\rangle$. As it does not break spontaneously any exact symmetry of the quantum theory, no massless states appear, the extensive representation (6.11) for the partition function is valid, and we cannot reproduce the linear in mass term in the expansion of Z when taking into consideration only the fields with integer winding number.

The paradox is resolved in this case by noting that, for a theory involving only adjoint fields, the fields carrying *fractional* winding numbers $\nu = \pm 1/N_c, \pm 2/N_c, \ldots$ are equally admissible [31–33, 14]. The reason is, again, that the gauge group here is actually $SU(N)/Z_N$ rather than $SU(N)$ and the gauge transformation matrices differing by an element of the center are undistinguishable. The configurations with $\nu = \pm 1/N_c$ involve only two fermion zero modes and *are* responsible for the formation of fermion condensate for small $m|\langle\bar{\lambda}\lambda\rangle|V$.

The situation is much worse, however, for higher orthogonal and exceptional groups. The simplest example where the problem appears is the super Yang-Mills (SYM) theory with SO(7) gauge group [34]. The instantons involve here $7 - 2 = 5$ pairs of zero modes and the corresponding contribution to the partition function is $\propto m^5$. The group SO(7) does not have a nontrivial center and, in contrast to what we had for $SU(N)$ groups, we cannot pinpoint a topological field configuration with winding number $\nu = 1/5$. Things are not better when $N > 7$. Thus, $SO(N \geq 7)$ four-dimensional SYM theories are as paradoxical as QCD_2^{adj} with $N \geq 3$.

We present here another very simple four-dimensional example where the paradox also appears. Consider the SU(2) Yang-Mills theory involving a Dirac fermion ψ belonging to the color representation with isospin $I = 3/2$. Suppose that the fermion condensate $\langle\bar{\psi}\psi\rangle$ is formed here. As in conventional QCD_4 with $N_f = 1$, it breaks only the $U_A(1)$ subgroup of the chiral symmetry group which is anyway anomalous, and no massless states appear. On the other hand, comparing $\mathrm{Tr}\{T^a T^a\} = I(I+1)(2I+1)$ in the representation with $I = 3/2$ with the same trace for $I = 1/2$, we see that the instantons involve here ten fermion zero modes and provide the contribution $\propto m^{10}$ to the partition function. There is no way to get the fermion condensate in the path integral framework.

Of course, the paradox here is not so prominent as in two other theories considered above. It appeared when *assuming* that the condensate is generated. The assumption looks natural—the dynamics of the theory is rather similar to that of conventional QCD with $N_f = 1$ where the condensate is formed, but there is also a distinction. The first coefficient in the Gell-Mann–Low function

$$b = \frac{22}{3} - \frac{2}{3} \times 10 = \frac{2}{3} \tag{7.1}$$

is comparatively small here (though the theory is still asymptotically free) which may after all prevent the formation of fermion condensate. And, in contrast with two previous cases, we cannot present solid independent theoretical arguments that the condensate is formed. Thus, this theory may serve only as an additional indication that something is grossly wrong in our understanding; we could not claim that solely on its basis.

But for $SO(N \geq 7)$ SYM theory and for QCD_2^{adj} with $N \geq 3$, the situation is really mysterious.

We cannot say that we understand how this mistery is resolved. But if there is a universal reason which resolves it in both theories, the only one we can think of is that premise (1) in the list in the beginning of this section is false. Perhaps, there are some singular field configurations which contribute to the path integral and which cannot be classified by topological considerations. If these unspecified configurations have only one pair of fermion zero modes, the condensate may be generated. One argument in favor of this guess comes from the observation that, in strong coupling theory, fields fluctuate wildly and the topological classification which is based on the assumption that the fields are smooth and regular may be not true.

Suggestions that this may happen can be found in the literature. In particular, Crewther [35] and Zhitnitsky [33] argued that, for the conventional QCD_4 with N_f light flavors with equal mass, field configurations carrying winding number $1/N_f$ (obviously, such fields cannot be described in topological terms) can be relevant. Actually, we do not see compelling reasons to assume this for standard QCD—the usual description including only the fields with integer winding numbers works perfectly well there. But for QCD_2^{adj} with $N \geq 3$, for SO(7) four-dimensional SYM theory, and maybe for SU(2) four-dimemsional gauge theory with Dirac fermions belonging to the representation $I = 3/2$ of the color group, we are kind of forced to think in this direction. What is absolutely unclear by now is in what respects path integral dynamics of these paradoxical theories differs from that in standard QCD and other well-studied theories where no need of invoking exotic nontopological fields arises.

VIII. CONCLUSIONS

The SO(3) QCD_2^{adj} which we analyzed first in this paper presents no problems. The picture is self-consistent: The instantons which are present there due to nontrivial $\pi_1[SO(3)] = Z_2$ involve two fermion zero modes and lead to the formation of the fermion condensate. This condensate falls down as the temperature increases [see Eq. (3.21)] but never turns to zero. Qualitatively, the same follows from bosonization arguments. This model can serve as a remarkably good playground which may allow us to understand better the physics of QCD (in particular, of QCD with only one quark flavor). For example, lattice simulations of this theory would be very interesting. One could try to calculate the fermion condensate on the lattice at zero and at high temperature and compare the numerical results with theoretical prediction (3.21). Such simulations are *much* simpler than in four dimensions and could provide an independent test for the whole lattice technology.

For $N \geq 3$, we encountered an explicit paradox: The existence of the condensate follows from bosonization arguments but we could not get it in the path integral approach. As was discussed in details in Sec. VII of this paper, a similar paradox displays itself also in some four-dimensional gauge theories. Its satisfactory resolution could bring about a progress in our understanding of quantum field theory in general.

In conclusion, we note that, if we would believe in the bosonization arguments at low temperatures and in the instanton arguments at high temperatures (at high T, quasiclassical approximation works and one could think that it still suffices to consider only smooth topological field configurations), the conclusion of the existence of the *phase transition* in the theories with $N \geq 3$ would follow—at some temperature T_c, the condensate would vanish and stay zero beyond it. But at the present level of understanding, we cannot really claim it is true.

If nontopological fields contribute to the path integral also at high temperatures, there is no phase transition but only a crossover where the condensate falls down but never turns to zero (as *is* the case for $N = 2$). As N grows, the crossover is expected to become more and more sharp. Its temperature is estimated as

$$T^* \sim g\sqrt{N} \tag{8.1}$$

(a natural mass scale of the theory). In the limit $N \to \infty$, $T^* \to T_H$, the Hagedorn limiting temperature.

Note added in proof. After this work had been completed, I got acquainted with the paper [36] where the idea of the nontrivial topological structure of QCD_2 with adjoint matter has been put forward for the first time.

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APPENDIX

We want to show here that the zero modes (3.9) and their conterparts for larger N are stable under small deformations of the Abelian high-T instanton background (3.3), (6.8). Consider first the case $N = 2$. Choose as earlier $n^a = \delta^{a3}$ in Eq. (3.3) and deform it in the transverse direction in the color space so that

$$A_0^a(x) = \delta^{a3}a(x) + (1 - \delta^{a3})b^a(x), \tag{A1}$$

with $b^a(-\infty) = b^a(\infty) = 0$ and $b \ll a$ for all x. Then the deformation $b^a(x)$ has no projection on the global gauge rotation modes discussed at length in Sec. III.

For $b^a(x) = 0$, the Dirac eigenvalue equation (3.7) had two zero mode solutions (3.9). With $b \neq 0$, the solutions are modified. Unfortunately, in contrast with the more simple Abelian case [8, 6], we cannot solve the zero mode equation explicitly for any gauge field background. What we can do is to develop a perturbation theory in the small parameter b/a and find the solution as the series in this parameter:

$$\psi^{a(\text{zero})} = \psi_0^{a(\text{zero})} + \psi_1^{a(\text{zero})} + \psi_2^{a(\text{zero})} + \cdots . \tag{A2}$$

Let us start, for definiteness, from the solution $\psi_0^{-(\text{zero})}(x,\tau)$ and find the corresponding $\psi_1^{a(\text{zero})}(x,\tau)$. It satisfies the equation

$$\left[(\partial_0\sigma_2 + \partial_x\sigma_1)\delta^{ab} - g\epsilon^{ab3}a(x)\sigma_2\right]\psi_1^{b(\text{zero})}(x,\tau) = -i\frac{g}{2}\delta^{a3}b^+(x)\sigma_2\psi_0^{-(\text{zero})}(x,\tau). \tag{A3}$$

We see that only the component $\psi_1^{3(\text{zero})}$ appears. It is left handed as $\psi_0^{-(\text{zero})}$ was and also has the same τ dependence $\propto \exp(i\pi T\tau)$. The solution of (A3) is

$$\psi_1^{3(\text{zero})}(x,\tau) = -\frac{g}{2}e^{\pi Tx}\int_x^\infty b^+(y)\psi_0^{-(\text{zero})}(y,\tau) \times e^{-\pi Ty}dy. \tag{A4}$$

It is easy to see that $\psi_1^{3(\text{zero})}$ has the same asymptotics $\propto \exp\{-\pi T|x|\}$ at $|x| \to \infty$ and is normalizable.

Generally, the nth term of the series (A2), $\psi_n^{a(\text{zero})}(x)$, is related to $\psi_{n-1}^{a(\text{zero})}(x)$ by a similar integral kernel which provides the asymptotics $\propto \exp\{-\pi T|x|\}$ for ψ_n if ψ_{n-1} had such, and the normalizability of the deformed zero mode is proved by induction.

For larger N, the analysis is quite similar. The integral kernels are a little bit different for different $\psi_0^{a(\text{zero})}$—the different color components of the deformation $b^a(x)$ enter, but the result is the same: If the perturbation is small, all $2(N-1)$ different zero modes remain normalizable and are there in the spectrum.

Certainly, this analysis cannot rule out bifurcations in the space of zero modes when the perturbation is large enough so that the number of zero modes would be less than $2(N-1)$ for some b, but we do not think that this possibility is realized (see the main text for more detailed discussion).

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