Some properties of the finite temperature chiral phase transition

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We study the phase transition of the (3+1)-dimensional Yukawa model at finite temperature. We calculate the critical exponents in the $1/N$ expansion and clarify certain subtleties involved in such a calculation. In the leading order we do not find the presence of any of the metastable states which were claimed in the literature. To this order, the exponents are the mean field, but corrections shift them to the usual nontrivial values. Dimensional reduction of this model is studied with special attention paid to the discrete symmetries of the Lagrangian before and after reduction. In the reduced $d=3$ theory there are two possible types of mass terms, one of which is allowed by chiral symmetry. It is the different discrete symmetries of these two mass terms which force the finite temperature 3 + 1 Yukawa Lagrangian to reduce to the usual scalar universality class (characterized by a conformally invariant σ model) rather than the chiral universality class (characterized by $d=3$ conformally invariant NJL-type model).

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I. INTRODUCTION AND SUMMARY

The finite temperature phase transition in QCD has been a matter of much interest and controversy. One physical picture of the phase transition emphasizes confinement as the root cause of the phase transition [1] and follows closely the confinement-deconfinement transition in pure Yang-Mills theory. In this approach chiral symmetry breaking is not emphasized. An entirely different description of the critical phenomenon, on the other hand, uses explicitly a chiral Lagrangian [2]. This approach relies solely on the chiral-symmetry-breaking pattern to describe the transition. Confinement does not play any role and in fact the σ model may describe *any* gauge or nongauge theory with the same symmetry-breaking pattern, with or without the presence of fermions. In between these two descriptions one finds a picture in which the phase transition is thought of as being fairly well described by the Yukawa or Nambu–Jona-Lasinio (NJL) model [3].

In the chiral Lagrangian point of view, which is supported by lattice simulations [4], the phase transition is second order for $N=2$ flavors, but becomes first order in the presence of additional massless quarks. The singularity structure of the thermodynamic quantities are universal and are defined by the critical exponents of the system. The $N=2$ flavor phase transition belongs to the $O(4)\to O(3)$ universality class, meaning that its critical properties are well approximated by the (Euclidean) $d=3$, $O(4)\to O(3)$ conformally invariant (critical) σ model. According to standard dimensional "reduction" arguments (see, for example, [5]), the fermions themselves, even if they are massless at zero temperature, do not influence the nature of the phase transition at finite temperature. It is rather the presence of their bosonic composites, Goldstone bosons, which are of importance. This follows directly from the universality of second order phase transitions [6]. The commonly held assumption is that all the possible universality classes (or equivalently, $d=3$ conformal field theories) are variations of the σ model and one need only match the correct symmetry-breaking patterns.

Recently, however, it was pointed out that there exist different $d=3$ conformal field theories with the same symmetry-breaking pattern [7]. These conformal field theories are the critical four-fermion interaction models of the NJL type. The critical indices are different and were recently calculated using the $1/N$ expansion [8], lattice [9], and $4-\epsilon$ and $2+\epsilon$ expansions [10]. They depend on N and are clearly different from the scalar ones. Physically, this corresponds to the fact that on the *chirally symmetric* side of the phase transition there are N massless fermions whose effect is felt even in the IR fixed point, just like the effect of Goldstone bosons. With the presence of more than one universality class in $d=3$, the standard "reduction" procedure becomes ambiguous and it is now uncertain to which critical conformal field theory the (3+1)-dimensional, finite temperature quantum field theory will reduce. In this situation, one must explicitly follow the reduction process.

The argument in favor of the bosonic universality class goes as follows. Loosely speaking, at finite temperature the $d=4$ fermion field reduces to a collection of $d=3$ fermions which may be considered as having a mass $\omega_n=2\pi(n+1/2)T$ [5,11]. Unlike for bosons, there is no zero mode for which this frequency vanishes. There are, however, an *infinity* of such temperature harmonics. One can then imagine that even if a single massive field does not influence the phase transition, the cumulative effects of an infinite number of such fields may have an appreciable impact. In order to see whether or not this happens, all the harmonics should be summed and their cumulative effects studied.

In this paper we shall study explicitly the finite temperature phase transition of the simplest chirally invariant theory, the Yukawa model, using the $1/N$ expansion. A careful discussion of the dimensional reduction process for the Yukawa model, so as to determine explicitly to which universality class the finite temperature chiral Lagrangian belongs, is given.

In Sec. II we shall calculate the critical exponents α, β, and δ and clarify a few subtle issues involved in this calculation. In particular, we show that certain spurious metastable states claimed in the literature [12] do not actually exist. To the leading order, we find that the critical exponents are mean field. This is similar to what happens in lower dimensions [13] and may appear to be a bit puzzling within the framework of the $1/N$ expansion. According to conventional wisdom, the critical exponents should be completely independent of the number of fermions as long as the symmetry-breaking pattern is preserved. Later in Sec. IV we shall show how the nontrivial critical exponents are recovered. Dependence of the critical exponents on N disappears completely. The higher order terms in $1/N$ encode corresponding higher orders in the so-called fixed ($d=3$) dimension renormalization group (RG) calculation of the critical exponents [6].

In Sec. III the dimensional reduction is performed, concentrating on the relationship between the symmetries in $d=4$ dimensions and those in $d=3$ dimensions. As is well known by now [14], the notions of parity and chiral symmetry in three dimensions are quite different from those in four. In particular, in $d=3$ dimensions there exists a charge conserving, chirally invariant mass, while such an object does not exist in $d=4$. Although this term does break $d=3$ parity, this parity was never a symmetry in the reduced $d=4$ theory to begin with. Rather, parity in $3+1$ dimensions becomes full spatial inversion in $d=3$ dimensions. Since these mass terms are not forbidden by chiral symmetry, they appear as a result of dimensional reduction. This is the reason why the parity invariant four-Fermi model does not appear as an IR fixed point of the finite temperature chiral phase transition: unlike at $T=0$, there are no massless fermions on the chirally symmetric side of the phase transition.[1]

II. FINITE TEMPERATURE PHASE TRANSITION IN THE YUKAWA MODEL

We shall consider the simplest Z_2 chirally invariant Yukawa model

$$\mathcal{L}=\tfrac{1}{2}(\partial_\mu\phi)^2+i\bar{\psi}^a\partial\!\!\!/\psi^a-g\bar{\psi}^a\psi^a\phi-V(\phi)\ , \tag{1}$$

where $a=1,\ldots,N$ labels the fermionic species and the potential $V(\phi)$ is the standard Mexican hat potential

$$V(\phi)=-\frac{\mu^2}{2}\phi^2+\frac{\lambda}{4}\phi^4\ . \tag{2}$$

As is well known, the weak coupling expansion for the finite temperature phase transition has the problem that the effective potential is imaginary [15]. We shall use instead the $1/N$ expansion, even though, as we shall see, it itself has its own subtleties. Although we consider only the $Z_2\to 1$ case, the more complicated chiral-symmetry-breaking patterns like $O(4)\to O(3)$ are quite analogous. This type of calculation has been partially done previously to leading order under the name of mean field approximation [12], which is in fact equivalent to the lowest order term in the $1/N$ expansion. Kawati and Miyata [12] correctly estimated the critical exponent β to be $\frac{1}{2}$, the mean field value. In their calculation, however, they completely neglected an important logarithmic term in the effective Lagrangian which led them to a lengthy discussion of a nonexistent critical point "just below T_c." A similar problem also occurs at finite fermion density (see Appendix A). We now demonstrate that there is, in fact, no such complication in the phase diagram and will derive β and other critical exponents using the method developed for scalar fields at finite temperature.

The $1/N$ expansion can be conveniently developed by integrating over the ψ field using the path integral formalism. The partition function is then

$$Z=\int\mathcal{D}\phi\exp\left\{iN\left[\mathrm{Tr}\ln(i\partial\!\!\!/-\tilde{g}\phi)-\frac{1}{2}(\partial_\mu\phi)^2+\frac{\tilde{\mu}^2}{2}\phi^2-\frac{\tilde{\lambda}}{4}\phi^4\right]\right\}, \tag{3}$$

where rescaled couplings are defined by $\tilde{g}=g\sqrt{N}$, $\tilde{\mu}^2=\mu^2N$, and $\tilde{\lambda}=\lambda N$. At finite temperature the integration over the fourth component of the momentum becomes a sum over Matsubara frequencies. The lowest order term in the $1/N$ expansion is simply the steepest descent approximation to the exponential. As usual, this involves an expansion of the action about its extremum, which is determined by the gap equation

$$0=-\Lambda\sqrt{\Lambda^2+m^2}+m^2\ln\left[\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right]-\frac{\tilde{\mu}^2}{\tilde{g}^2}+\frac{\tilde{\lambda}}{\tilde{g}^4}m^2+4T^2f(x)\ , \tag{4}$$

where

$$f(x)\equiv\int_0^\infty dy\frac{y^2}{\sqrt{x^2+y^2}}\left[\frac{1}{1+e^{\sqrt{x^2+y^2}}}\right] \tag{5}$$

and $m\equiv\tilde{g}\phi$ is the order parameter while $x\equiv m/T$.[2] Here Λ is an UV cutoff. Evaluating the argument of the exponential at this point gives us the effective potential

$$V_{\mathrm{eff}}=-\frac{1}{4\pi^2}\left\{\left[\Lambda^3+\frac{\Lambda m^2}{2}\right]\sqrt{\Lambda^2+m^2}-\frac{m^4}{2}\ln\left[\frac{\Lambda+\sqrt{\Lambda^2+m^2}}{m}\right]\right\}+V(m/\tilde{g})-\frac{2T^4}{\pi^2}\int_0^\infty y^2dy\,\ln(1+e^{-\sqrt{x^2+y^2}})\ . \tag{6}$$

[1] In the presence of a chemical potential μ this fact does not change. Only a *real* constant will be added to the fermion mass and the frequencies will be shifted by a complex value $\omega_n=2\pi(n+\frac{1}{2})T+i\mu$.

[2] To avoid confusion with the critical index β we shall avoid using inverse temperature.

Renormalization at finite temperature will be the same as at $T=0$, as usual. The triviality features such as the appearance of the cutoff and the exponentially negligible metastability of the vacuum are basically the same as in ϕ^4 [16]. To find the temperature dependence on the order parameter ϕ we solve the gap equation near criticality. Notice in particular the presence of the logarithmic term in Eq. (4). It is this term that was dropped in [12]. With its presence the system seemingly will not have a power law dependence in its critical properties. Actually, we shall see that this logarithmic term will always be canceled by a corresponding term in $f(x)$ near the transition temperature.

The critical temperature T_c is given by

$$T_c^2 f(0)=\frac{1}{4}\left[\Lambda^2+\frac{\tilde{\mu}^2}{\tilde{g}^2}\right], \tag{7}$$

where $f(0)=\pi^2/12$. We next need an approximation for $f(x)$ near $x=0$. Since $f(x)$ is not an analytic function of x at this point, doing so is nontrivial. Nevertheless, following the methods of [15], it is enough for our purposes to use the asymptotic formula

$$f(x)\approx f(0)+\frac{x^2}{4}\ln\left[\frac{x}{\pi}\right]-\frac{x^2}{8}+\frac{\gamma}{4}x^2 \tag{8}$$

where γ is the Euler number. The somewhat lengthy derivation of this expression is given in Appendix B. It then follows from Eq. (4) that, for small order parameters near T_c,

$$\frac{T_c-T}{T_c}=\frac{m^2}{8T_c^2}\left[\frac{\tilde{\lambda}}{\tilde{g}^4}+\gamma-1+\ln\left[\frac{2\Lambda}{\pi T_c}\right]\right]. \tag{9}$$

In arriving at this simple expression we benefited from the fact that the logarithmic terms from the second and the last terms cancel in Eq. (4). We immediately see that the critical exponent β is one-half. Notice also that the metastable state that was found in [12] disappears due to a similar cancellation of the logarithmic term in the effective potential Eq. (6).

Other common critical exponents can be calculated in a similar manner. To calculate α, we find the specific heat near T_c to be

$$C=\frac{24T^3}{V}\int_0^\infty y^2\ln(1+e^{-y})dy \tag{10}$$

and continuous at all T. This gives $\alpha=0$. As for δ, we first introduce a source term. In the current context this is just an explicit chiral-symmetry-breaking mass term $M\bar{\psi}\psi$. The gap equation then becomes

$$\begin{aligned}0=(m-M)\Bigg[&-\Lambda\sqrt{\Lambda^2+(m-M)^2}\\&+(m-M)^2\ln\left[\frac{\Lambda+\sqrt{\Lambda^2+(m-M)^2}}{m-M}\right]\Bigg]\\&+4T^2(m-M)f((m-M)/T)\\&-\frac{\tilde{\mu}}{\tilde{g}^2}m+\frac{\tilde{\lambda}}{\tilde{g}^4}m^3 .\end{aligned} \tag{11}$$

Near the transition temperature, both m and M are small, and once again we use the expansion of f for small arguments. The logarithm again cancels and we find that

$$M=\frac{\tilde{g}^2}{\tilde{\mu}}\left[\frac{\tilde{\lambda}}{\tilde{g}^4}+\gamma-1+\ln\left[\frac{2\Lambda}{\pi T_c}\right]\right]m^3, \tag{12}$$

from which we can see explicitly that $\delta=3$.

All three critical exponents that we have calculated are mean field critical exponents and satisfy hyperscaling relations. Actually, one will get the same critical exponents in this approximation in $d=3$ [13], although they will obviously not satisfy the hyperscaling relations in this dimension. Notice also that they are far from the Ising critical exponents $\beta\approx0.3$, $\alpha\approx0.15$ at $d=3$ [6]. This would seem to contradict conventional wisdom according to which only the symmetry-breaking pattern plays a role in determining the values of the critical exponents. The number of fermion multiplets does not influence this pattern. One then wonders how nontrivial critical exponents could be recovered from this expansion. Moreover, since these critical exponents are obviously not the chiral critical exponents $\beta=1$, $\alpha=0$ [8] either, there is a question as to which critical exponents will ultimately be recovered: the scalar Ising ones, or the chiral ones. Using symmetry arguments in Sec. III we shall show that the scalar critical exponents are the ones which will be recovered. Then in Sec. IV we shall address the issue of how the scalar critical exponents can be obtained within the framework of the $1/N$ expansion.

III. WHAT HAPPENS TO VARIOUS SYMMETRIES UPON DIMENSIONAL REDUCTION

We begin with the standard reduction procedure paying special attention to the $3+1$ and $d=3$ symmetries. The $3+1$ theory is invariant under the special Lorentz group, *CPT*, *P*, and *T* and chiral symmetry. For convenience, we use the Weyl representation of the γ matrices in Minkowski $(3+1)$-dimensional space:

$$\gamma^0=\begin{bmatrix}0 & -I\\ -I & 0\end{bmatrix},\quad \boldsymbol{\gamma}=\begin{bmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{bmatrix}, \tag{13}$$

where $\boldsymbol{\sigma}$ are the Pauli matrices. In this representation the discrete symmetries are

$$\begin{aligned}&P\psi P^\dagger=\gamma^0\psi,\quad C\psi C^\dagger=\begin{bmatrix}-i\sigma^2 & 0\\ 0 & i\sigma^2\end{bmatrix}\psi,\\&T\psi T^\dagger=-i\gamma_5\begin{bmatrix}-i\sigma^2 & 0\\ 0 & i\sigma^2\end{bmatrix}\psi .\end{aligned} \tag{14}$$

At finite temperature, Lorentz boosts and T invariance are obviously lost due to the presence of the thermostat. Three-dimensional rotations, *PC*, and chirality are left undisturbed, however.

To perform dimensional reduction, we first Wick rotate $t\to-i\tau$. Then the fermionic piece of the Lagrangian Eq. (1) becomes

$$\mathcal{L}_f=\psi^\dagger(\gamma^0)^2\frac{\partial\psi}{\partial\tau}-i\psi^\dagger\gamma^0\gamma^j\frac{\partial\psi}{\partial x^j}-g\phi\psi^\dagger\psi, \tag{15}$$

where for clarity we have considered only one of the N fields. Writting ψ in terms of its "right" and "left" components $\psi=(\psi^R,\psi^L)^t$, we then expand ψ^R and ψ^L in a Fourier series in τ:

$$\psi^{R,L}(\mathbf{x},\tau)=T^{1/2}\sum_{n=-\infty}^{\infty} e^{i\omega_n\tau}\psi_n^{R,L}(\mathbf{x}) \tag{16}$$

where $\omega_n=(2n+1)\pi T$ are the fermionic Matsubara frequencies. Then

$$\mathcal{L}_f=\sum_{n=-\infty}^{\infty}[-i\psi_n^{R\dagger}\sigma\cdot\partial\psi_n^R+i\psi_n^{L\dagger}\sigma\cdot\partial\psi_n^L +i\omega_n(\psi_n^{R\dagger}\psi_n^R+\psi_n^{L\dagger}\psi_n^L) +g\phi_0(\psi_n^{R\dagger}\psi_n^L+\psi_n^{L\dagger}\psi_n^R)] \tag{17}$$

and the Lagrangian becomes a sum over an infinite number of fields each having a different "mass" ω_n. We have retained only the $n=0$ temperature harmonic of the scalar field ϕ. Notice, however, the relative sign difference between the kinetic terms of ψ_n^R and ψ_n^L, which would seem to be peculiar. It is known, however, that the Dirac algebra in three dimensions (Euclidean or Minkowski) does not have a "γ_5" matrix. Rather, the role of "γ_5" is played by $-I$, the identity matrix, and the algebra decomposes into two inequivalent representations. The two different kinetic pieces of Eq. (17) are due to the two different representations of the Dirac algebra. Under reduction, the "right" fields ψ^R are mapped into one of the representations, while the "left" fields ψ^L are mapped into the other representation.

In order to compare the Lagrangian Eq. (17) with the Z_2 chirally invariant Gross-Neveu model,

$$\mathcal{L}=i\bar\psi\partial\!\!\!/\psi+\frac{g}{2}(\bar\psi\psi)^2\ , \tag{18}$$

in 2+1 dimensions (where ψ is a four component field) undergoing the chiral phase transition [7], we shall have to rotate the $d=3$ Euclidean Lagrangian Eq. (17) back into the (2+1)-dimensional Minkowski space. To do so, we shall choose x_3 as our "time" coordinate and define our (2+1)-dimensional Minkowski γ matrices as

$$\gamma^0_{(2+1)}=i\sigma_3\ ,\quad \gamma^1_{(2+1)}=i\sigma_1,\quad \gamma^2_{(2+1)}=i\sigma_2 \tag{19}$$

while $\bar\psi^{R,L}\equiv\psi^{R,L\dagger}\sigma_3$. Then the Minkowski space form of Eq. (17) becomes (see [17])

$$\mathcal{L}_f=\sum_{n=-\infty}^{\infty}[-i\bar\psi_n^R\gamma_{(2+1)}\cdot\partial\psi_n^R+i\bar\psi_n^L\gamma_{(2+1)}\cdot\partial\psi_n^L +\omega_n(\bar\psi_n^R\psi_n^R+\bar\psi_n^L\psi_n^L)+g\phi_0(\bar\psi_n^R\psi_n^L+\bar\psi_n^L\psi_n^R]\ . \tag{20}$$

The parity operation in 2+1 dimensions corresponds to $x\to -x, y\to y, t\to t$. Therefore, for the spinors the (2+1)-dimensional parity $P_{(2+1)}$ are

$$P^\dagger_{(2+1)}\psi^L P_{(2+1)}=\sigma_1\psi^L \tag{21}$$

while because the right fields are in the other representation of the Dirac algebra,

$$P^\dagger_{(2+1)}\psi^R P_{(2+1)}=-\sigma_1\psi^R\ . \tag{22}$$

From this we can see explicitly that the thermal "mass" terms in Eq. (20), which are precisely the Dirac-type mass terms one expects from fermions in 2+1 dimensions, are *not* invariant under (2+1)-dimensionsional parity [14]. On the other hand, the dimensionally reduced 3+1 chirally noninvariant mass term $\bar\psi\psi$, when written in terms of the two component Weyl spinors, is $\bar\psi_n^R\psi_n^L+\bar\psi_n^L\psi_n^R$. This term *is* invariant under (2+1)-dimensional parity.

The original *(3+1)-dimensional* parity reduces to the complete inversion of all the three-dimensional coordinates which are *not* equivalent to (2+1)-dimensional parity. The mass terms obtained from dimensional reduction would therefore not be allowed in the $d=3$ parity invariant Gross-Neveu model exhibiting chiral critical exponents. It is for this reason that under dimensional reduction one will not get the $d=3$ chiral critical exponents, but will, instead, obtain the Ising critical exponents. Simply put, the discrete symmetries of the two Lagrangians are different. In one, $d=3$ parity is preserved, while in the other it is explicitly broken by the temperature mass terms.

One can see this explicitly within the calculation performed in Sec. II. To develop the $1/N$ expansion of the reduced Lagrangian Eq. (17), we consider the partition function in terms of the modes ω_n:

$$Z=\int \mathcal{D}\phi_0\mathcal{D}\bar\psi_n\mathcal{D}\psi_n\exp\left\{N\left[\sum_{n=-\infty}^{\infty}(-i\psi_n^{R\dagger}\sigma\cdot\partial\psi_n^R+i\omega_n\psi_n^{R\dagger}\psi_n^R+i\psi_n^{L\dagger}\sigma\cdot\partial_j\psi_n^L+i\omega_n\psi_n^{L\dagger}\psi_n^L) +g\sum_{n=-\infty}^{\infty}(\psi_n^{R\dagger}\psi_n^L+\psi_n^{L\dagger}\psi_n^R)\phi_0-\frac{1}{2}(\partial_j\phi_0)^2-\frac{\tilde\mu^2}{2}\phi_0^2+\frac{\tilde\lambda}{4}\phi_0^4\right]\right\}, \tag{23}$$

where we have neglected the $n\neq 0$ modes of ϕ. Retaining these terms does not alter the final result. The integration over each individual fermionic frequency mode can then be done straightforwardly, and we obtain, as the effective potential,

$$V_{\rm eff}=\sum_{n=-\infty}^{\infty}|\langle\phi_0\rangle+i\omega_n|^3-\frac{\tilde\mu^2}{2}\langle\phi_0\rangle_0^2+\frac{\tilde\lambda}{4}\langle\phi_0\rangle^4\ . \tag{24}$$

Notice that the sum is over all n, positive and negative, and consequently the expression is a real number.

We then observe that, since $\omega_n\neq 0$ for any n, the cubic terms in $V_{\rm eff}$ become irrelevant. Consequently, $V_{\rm eff}$ is analytic in $\langle\phi_0\rangle$ and can be expanded as a power series in $\langle\phi_0\rangle$. In fact, because of the presence of the imaginary mass terms, we find that $V\sim\langle\phi_0\rangle^2$ for small $\langle\phi_0\rangle$. Therefore, $\beta=\frac{1}{2}$ and we have recovered explicitly the mean field result. (How the correct Ising critical exponents can be restored from this mean field result shall be addressed shortly.) This should be contrasted with the Gross-Neveu [8] theory in which the effective potential is not analytic in $\langle \phi_0 \rangle$ due precisely to these cubic terms in the effective potential, as can be seen explicitly by setting all the $\omega_n = 0$ by hand in Eq. (24). In this case one immediately obtains $\beta = 1$ as the *chiral* critical exponent.

IV. CONCLUDING REMARKS

We now address the question of how the nontrivial critical exponents are restored from the mean field values that we have obtained in the leading order in $1/N$. These mean field critical exponents sharply disagree with the Ising ones. This would appear to be a problem since if the usual universality argument is applied only the symmetry-breaking pattern plays a role in determining the values of the critical exponents. The number of fermion multiplets does not influence this pattern.[3]

In many ways the $1/N$ expansion parameter functions in much the same way that $\hbar$ does in the loop expansion of the ϕ^4 model. In the partition function in the $1/N$ formalism, Eq. (3), the factor of N appears as an overall factor multiplying the Lagrangian. Although in the $d=3$ RG "loop expansion" one formally corrects the values of critical exponents by taking into account higher loop diagrams [6], the values of the exponents are actually independent of the "expansion parameter" $\hbar$. Obviously, $\hbar$, as is $1/N$ in our case, cannot appear in the values of the *universal* critical exponents. The calculation of the critical exponents in either scheme is not truly an expansion.

From the symmetry arguments we know that these mean field critical exponents cannot be restored to the chiral critical exponents. Consequently, they must be restored to the Ising critical exponents and therefore cannot be functions of $1/N$. Furthermore, one observes that the $1/N$ expansion for the finite temperature Yukawa model and the expansion in $\hbar$ of the $d=3$ ϕ^4 model coincide after dropping irrelevant terms. We can see this explicitly by first calculating the β function for the scalar field coupling constant λ in the Yukawa model by using the $1/N$ expansion. It has the form $\bar{\beta}_I(\lambda) \sim \lambda - a(\lambda^2/N) + \cdots$, where we have restored the original factor of N in this expression and a is a constant. To find the values of the critical exponents, we then determine the fix point of the β function $\lambda^* \sim N$. Clearly, λ^* is not an analytic function of $1/N$. Because of this, to the lowest order in $1/N$, the Ising β function does not have a nontrivial fix point and one obtains mean field values as was done in Sec. II. Then the next order term in the $1/N$ expansion is very significant and one finds that (to this order) λ^* suddenly becomes proportional to N. When we then incorporate λ^* in the expression for the critical exponents, say $\beta \sim \frac{1}{2} - b\lambda^*/N$, the dependence on N cancels and a nontrivial value of β will be obtained.

Let us now contrast this situation with what happens for the calculation of the $1/N$ expansion for the β function of the coupling constant g in the chirally invariant four-fermion Lagrangian in $d=3$ [8]. One now finds that $\bar{\beta}_{\rm chi}(g) \sim g(1-g) + [h(g)/N] + \cdots$ where h is some known function of g. Its specific value is unimportant. What is important is that this β function has a nontrivial fixed point even to the lowest order in $1/N$. Consequently, a perturbative solution of the fixed point for the chiral β function is well defined and analytic about $g^*=1$, unlike the previous case.[4]

Another remark is that the infinite number of modes in the reduction does not spoil the argument about the irrelevance of the fermionic modes. *A priori* one might expect that this term will have an effect but, as the explicit four-dimensional calculation has shown, this is not the case. As we have seen in Sec. III, this is basically due to the discrete symmetries of the Lagrangian.

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APPENDIX A

In this appendix we shall consider effects of adding in the chemical potential to the system. The addition of a chemical potential μ is straightforward. One only has to modify the distribution function $f(x)$ in Eq. (5) so that

$$f(x,\tilde{\mu}) = \frac{1}{2}\int_0^\infty \frac{y^2}{\sqrt{x^2+y^2}} \left[\frac{1}{\exp(-\sqrt{x^2+y^2}+\tilde{\mu})+1} + \frac{1}{\exp(-\sqrt{x^2+y^2}-\tilde{\mu})+1} \right] dy \tag{A1}$$

where $\tilde{\mu} = \mu/T$. The phase diagram was studied to leading order in $1/N$ (or, equivalently, in the mean field) approximation in [12]. Actually the authors of [12] studied the four-Fermi (Nambu–Jona-Lasinio) model rather than the Yukawa model, but any differences are irrelevant for our purposes. They established a phase diagram in the $T-\mu$ plane with the critical line satisfying a simple equation

$$T_c^2(\mu) + \mu_c^2 = \text{const} . \tag{A2}$$

In their analysis, however, they once again found another pole in the thermal propagator of the Goldstone bosons,

[3]The situation is very different from the $1/M$ expansion for critical exponents in scalar theories, where M is the number of scalar fields. In this case the symmetry-breaking pattern does depend on the expansion parameter $1/M$. See Ref. [6].

[4]This also resolves a similar "paradox" in three dimensions [13].

leading them to complicate the situation with the addition of a metastable state just below the phase transition. Moreover, they seemingly found the presence of a tachyonic ghost between these two temperatures.

We have reconsidered the calculation and have found that this additional pole of the propagator arises from a rather unnecessary approximation they made. Once again, as in the case of zero chemical potential, they ignored the logarithmic term in the gap equation. As before, this contribution will be canceled by a corresponding term in $f(x,\bar{\mu})$ even in the presence of a finite chemical potential. Because of this the tachyon pole disappears and the metastable state they alluded to does not actually exist. There is only one phase transition.

It is also interesting to note that the simple spherical shape of the phase transition line in the $T-\mu$ plane in mean field is rather generic. It follows from Eq. (A2) that there is a symmetry

$$\begin{aligned} T_c' &= \cos(\alpha)T_c + \sin(\alpha)\mu_c \ , \\ \mu_c' &= \cos(\alpha)\mu_c - \sin(\alpha)T_c \ . \end{aligned} \tag{A3}$$

This symmetry, however, does not survive higher order corrections in $1/N$.

As we have seen, the critical exponents of Yukawa theory Eq. (1) will fall within the Ising universal class in the absence of any chemical potentials. This is true even with the addition of a real chemical potential. As was observed in Sec. III, under dimensional reduction fermions gain masses which cannot be canceled by any real chemical potential. Only nonuniversal quantities such as the critical temperature will be changed, and as was calculated in [12], the critical temperature is related to the chemical potential by Eq. (A2).

APPENDIX B

In this appendix we shall derive the small x approximation for $f(x)$ in Eq. (5) following the technique of [15]. Although $f(x)$ and its derivative are well defined functions of x for all $x \geq 0$, one finds that the second derivative of $f(x)$ is singular at $x=0$. Consequently, a straightforward Taylor series expansion of $f(x)$ for small x is ill defined. We are instead forced to use the method developed by [15] to find an approximate expression for $f(x)$ which is valid for small x.

We first differentiate $f(x)$ and separate the resultant integral into two pieces:

$$f'(x) = -\frac{x}{2}\lim_{\epsilon\to 0}(I_\epsilon^1 + I_\epsilon^2) \tag{B1}$$

where

$$I_\epsilon^1 = \int_0^\infty \frac{y^{-\epsilon}}{\sqrt{x^2+y^2}}dy, \quad I_\epsilon^2 = \int_0^\infty \tanh(y/2)\frac{y^{-\epsilon}}{\sqrt{x^2+y^2}}dy, \tag{B2}$$

and ϵ is introduced as a regulator for the integrands at large y. In its absence, each integral separately is infinite, although since $f'(x)$ is a well defined function, their sum is finite. The first integral is straightforward to do, giving

$$I_\epsilon^1 = \frac{1}{2\epsilon} - \frac{1}{2}\ln 2x \ . \tag{B3}$$

Now we can see explicitly the singularity when $\epsilon \to 0$. There is, however, a logarithmic singularity in x which shows explicitly that $f''(x)$ is singular at $x=0$. As for the second integral, we first make use of the expansion

$$\tanh\left[\frac{\pi z}{2}\right] = \frac{4z}{\pi}\sum_{n=1}^{\infty}\frac{1}{z^2+(2n+1)^2} \ , \tag{B4}$$

then integrate the series term by term giving

$$I_\epsilon^2 \approx -\frac{1}{2\epsilon} + \frac{1}{2}\ln 2\pi - \frac{\gamma}{2} + \cdots \ , \tag{B5}$$

where γ is the Euler number and we have only included terms to order x^2. We can now see that the singular terms in ϵ in the sum $I_\epsilon^1 + I_\epsilon^2$ cancel and after integration we obtain the expression for $f(x)$ valid for small x:

$$f(x) \approx f(0) + \frac{x^2}{4}\ln\left[\frac{x}{\pi}\right] - \frac{x^2}{8} + \frac{\gamma}{4}x^2 \ . \tag{B6}$$

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