# Hot gauge theories and $Z_N$ phases

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In this paper several aspects of $Z_N$ symmetry in gauge theories at high temperatures are discussed. The metastable $Z_N$ bubbles in SU($N$) gauge theories with fermions may have, generically, unacceptable thermodynamic behavior. Their free energy $F \propto T^4$, with a positive proportionality constant. This leads not only to negative pressure but also to negative specific heat and, more seriously, to negative entropy. We argue that although such domains are important in the Euclidean theory, they cannot be interpreted as physical domains in Minkowski space. A related problem is connected with the analysis of the high-temperature limit of the confining phase. Using two-dimensional QCD with adjoint fermions as a toy model we shall demonstrate that in the light fermion limit there is no breaking of the $Z_N$ symmetry in the high-temperature limit and thus there are no $Z_N$ bubbles.

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## I. INTRODUCTION

The $Z_N$ symmetry of pure Yang-Mills theories plays an important role in the study of their thermal properties [1]. It is known both from perturbative studies [2,3] and from lattice simulations [4] that at high temperature the $Z_N$ symmetry is spontaneously broken and the Euclidean theory has $N$ degenerate vacua distinguished by different vacuum expectation values of the Polyakov line [1]:

$$\langle L \rangle = \left\langle \frac{1}{N} \mathrm{tr} P \exp\left[ i \int_0^\beta A_0 d\tau \right] \right\rangle . \tag{1.1}$$

In the presence of any matter which transforms as the fundamental representation of SU($N$) (for example, quarks in QCD or quarks, leptons, and Higgs particles in the electroweak theory), this $Z_N$ symmetry is no longer present and all but one of these $N$ degenerate vacua become either metastable or unstable (depending on $N$ and the number of flavors). Following work on the computation of the interface tension between phases of different $Z_N$ "vacua" in pure gluonic theories [5] it has recently been argued [6] that these metastable vacua may lead to interesting cosmological consequences.

However it has been found in [7] that $Z_N$ domains, generically, have unacceptable thermodynamic behavior; for example, their free energy $F \propto T^4$ with a positive proportionality constant. This leads not only to negative pressure but also to negative specific heat and, more seriously, to negative entropy which means that something is definitely wrong in our understanding of the hot gauge theories structure. Using the above-mentioned thermodynamical arguments one must conclude that the $Z_N$ bubbles, i.e., different $Z_N$ phases coexisting in space, do not exist in models with metastable vacua. However the thermodynamical arguments do not forbid the existence of the different degenerate $Z_N$ phases in the theories with unbroken $Z_N$ symmetry. It is completely unclear what will be wrong with the theory if one adds some matter in a fundamental representation. How does this matter destroy the existence of (meta)stable states? In some sense the situation would be much more clear if one could assume that these phases cannot coexist in the space even in the case of unbroken $Z_N$ symmetry. The problem of $Z_N$ bubbles has been recently discussed in [8].

Recently Smilga argued in a paper [9] that this point of view may be indeed correct and that "different $Z_N$ thermal vacua of hot pure Yang-Mills theory distinguished in the standard approach by different values of Polyakov loop average correspond actually to one and the same physical state." He presented different arguments supporting this statement including the possible role of the infrared divergences in the calculation [5] of the surface tension of the walls separating different $Z_N$ phases as well as the difference between strong-coupling lattice SU($N$) gauge theories, where $Z_N$ bubbles indeed exist [1], and the weak-coupling continuum limit.

In addition to these general arguments he considered an interesting example of the $(1+1)$-dimensional hot QED, the Schwinger model, where a similar problem appears. Instead of $Z_N = \pi_1[\mathrm{SU}(N)/Z_N]$ different vacuum states in a pure SU($N$) gauge theory one has $Z = \pi_1[\mathrm{U}(1)]$ different states in the Schwinger model. However it was shown in [9] that in this case there are no domain wall solutions with finite surface tension.

In this paper we shall consider another $(1+1)$-dimensional model which, contrary to the Schwinger model, shares some common features with realistic $(3+1)$-dimensional gauge theories. This is the $(1+1)$-dimensional SU($N$) gauge theory with Majorana fermions in the adjoint representation with the action

$$S_{\mathrm{adj}} = \int d^2x \, \mathrm{Tr} \left[ -\frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i \bar{\Psi} \gamma^\mu D_\mu \Psi + m \bar{\Psi} \Psi \right] \tag{1.2}$$

which obviously has $Z_N$ symmetry. The light-cone

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quantization of this theory was considered in a large $N$ limit in [10]. The spectrum consists of closed-string excitations. Contrary to the 't Hooft model [11] with fermions in the fundamental representation of SU($N$) describing the open-string excitations with the only meson Regge trajectory, in this theory there is an infinite number of the closed-string Regge trajectories and the density of particle states increases exponentially with energy [12,13],

$$n(m) \sim m^{-\alpha} \exp(\beta_H m) , \tag{1.3}$$

which means that there is the Hagedorn temperature $T_H = \beta_H^{-1}$ and the model undergoes a confinement-deconfinement transition at this temperature which must be the simplest analogue of the real confinement-deconfinement transition in QCD. The numerical value of $\beta_H \approx (0.7-0.75)\sqrt{\pi/(g^2N)}$ in the large $N$ limit was calculated in [13]. The same picture was obtained in a recent paper [14] where the Hagedorn spectrum was obtained for (1+1)-dimensional QCD with adjoint scalar matter. The numerical value of inverse Hagedorn temperature in this case is $\beta_H \approx (0.65-0.7)\sqrt{\pi/(g^2N)}$.

In string theory language the Hagedorn transition occurs due to the fact that some winding modes in the imaginary time direction become tachyonic at high temperature [15,16] (see also [17]). In a string description of the gauge theory this winding modes are generated by the Polyakov line operators

$$L_k(\mathbf{x}) = \frac{1}{N} \mathrm{tr} P \exp \left[ i \int_0^{k\beta} A_0(\tau, \mathbf{x}) d\tau \right] \tag{1.4}$$

wrapping $k$ times around the imaginary-time direction $\tau$. In a very interesting paper [18] Polchinski studied the high-temperature limit of the confining phase and calculated in large $N$ limit the mass of the tachyonic winding modes at high temperatures. His method was used by Kutasov in [12] to study the stability of the confining phase in the two-dimensional QCD coupled to adjoint matter. We would like to note that both the Polchinski and Kutasov analyses were based on the form of effective potential which leads to the existence of the $Z_N$ bubbles and these two problems—$Z_N$ bubbles and tachyonic winding modes seem to be ultimately related.

In Sec. II we shall discuss the effective potential for the Polyakov line and the thermodynamical properties of the $Z_N$ phases. In Sec. III we consider the Polchinski method and discuss its relation to the existence of the $Z_N$ phases. In Sec. IV the 1+1 gauge theory with adjoint fermions will be considered. Using the results obtained in [19] we shall demonstrate that there are no coexisting $Z_N$ phases in this model by the same reasons which have been found by Smilga for Schwinger model [9]. Moreover, there is no $Z_N$ symmetry breaking in this model in the light fermion limit $m \ll \sqrt{g^2N}$. In conclusion we shall discuss the obtained results and unsolved problems. In particular we shall consider the possibility that the metastable $Z_N$ phases (if they do exist in the four-dimensional theories) may have interpretation as states with inverse population, i.e, with negative temperatures.

## II. $Z_N$ DOMAINS IN GAUGE THEORIES

Let us briefly review the origin of the $Z_N$ structure in gauge theories. It is most illuminating to begin with the Hamiltonian theory in $A_0=0$ gauge and impose Gauss' law $D_iE_i - g\psi^\dagger\psi = 0$ as a constraint on the Hilbert space of states. The projection operator $P$ onto these states is just the projection operator onto gauge invariant states:

$$P = \frac{\int \mathcal{D}g U_g}{\int \mathcal{D}g} \tag{2.1}$$

where the integral is over all gauge transformations $g(\mathbf{r})$ with $g(\mathbf{r}) \in \mathrm{SU}(N)$ using the Haar measure $\mathcal{D}g$. $U_g$ is the representation of the gauge transformation $g$ on the Hilbert space of states. The partition function at nonzero temperature $1/\beta$ is given by

$$Z = \mathrm{Tr}\{e^{-\beta H}P\} = \int \mathcal{D}g \, \mathrm{Tr}\{e^{-\beta H}U_g\} \tag{2.2}$$

If we compute this trace in a basis $\{|A_i, \xi\rangle\}$ where $\xi$ represents an appropriate fermionic state then

$$Z = \int \mathcal{D}A_i \mathcal{D}\xi \int \mathcal{D}g / \langle A_i, \xi | e^{-\beta H} U_g | A_i, \xi \rangle . \tag{2.3}$$

One can now proceed with the usual derivation of the Euclidean functional integral at nonzero temperature except to note that the presence of the factor $U_g$ modifies the boundary conditions on both the gauge fields and the fermions. Thus apart from an overall normalization

$$\begin{aligned} Z &= \int \mathcal{D}g(\mathbf{r}) \int_{BC} \mathcal{D}A_i(\mathbf{r},\tau) \mathcal{D}\psi(\mathbf{r},\tau) \\ &\quad \times \mathcal{D}\psi^\dagger(\mathbf{r},\tau) \exp\left[ -\int_0^\beta d\tau \int d^3r \, \mathcal{L}_{\mathcal{E}} \right] , \\ BC&: \; A_i(\mathbf{r},\beta) = g[A_i(\mathbf{r},0)] , \\ &\quad \psi(\mathbf{r},\beta) = -g[\psi(\mathbf{r},0)] \end{aligned} \tag{2.4}$$

where $\mathcal{L}_{\mathcal{E}}$ is the usual Euclidean Lagrangian for a gauge theory coupled to fermions, $g[A] \equiv gAg^{-1} - \partial g g^{-1}$ and $g[\psi] \equiv g\psi$. In other words we derive the usual functional integral in the $A_0=0$ gauge but where the periodic (antiperiodic) boundary conditions are modified to be periodic (antiperiodic) up to an arbitrary gauge transformation which we then integrate over.

It is of course possible to remove these strange boundary conditions by performing a gauge transformation in the functional integral. For each value of the integrand $g$ we may introduce any gauge transformation $V(\mathbf{r},\tau)$ with the property that $V(\mathbf{r},0)=I$ and $V(\mathbf{r},\beta)=g^{-1}(\mathbf{r})$. This will force the introduction of a temporal gauge field $A_0 = \partial_0 V V^{-1}$. The integral over $g$ will then become an integral over $A_0$ with the appropriate measure. In fact if we integrate over all possible such $V$'s we recover the usual path integral over all gauge fields $A_\mu(\mathbf{r},\tau)$ with periodic boundary conditions and over all fermionic fields with antiperiodic boundary conditions.

If we consider the spatially constant gauge transformation $g \in Z_N$, i.e., $g = \exp(2\pi i k/N)I$ (where $k$ is an integer and $I$ is the identity matrix). Then $g[A_i] = A_i$ but $g[\psi] = \exp(2\pi i k/N)\psi$. Thus in the absence of fermions there are $N$ degenerate vacua. When fermions are present, however, a value of $g \in Z_N$ corresponds to a path integral in which the fermions have the "twisted" boundary conditions $\psi(\beta) = -\exp(2\pi i k/N)\psi(0)$. One can say that this describes fermions having an imaginary chemical potential (for more detailed discussion see, for example, [20]).

The main tool for analyzing the $Z_N$ structure of gauge theories is the calculation of the effective potential in a constant, background temporal gauge field $A_0$. For definiteness we begin by considering a four-dimensional SU($N$) gauge theory with $N_f$ flavors of massless Dirac fermions. Although at finite temperature it is not possible to choose a gauge in which $A_0 = 0$, it is possible to choose a gauge in which $A_0$ is independent of (Euclidean) time $\tau$ and in which it is diagonal. $A_0$ can then be written as

$$\Theta = \beta A_0 = \begin{pmatrix} \theta_1 & & \\ & \ddots & \\ & & \theta_N \end{pmatrix} \tag{2.5}$$

where for SU($N$) one has $\theta_1 + \cdots + \theta_N = 0 \pmod{2\pi/\beta}$ and there are $N-1$ independent $\theta_i$, the number of independent diagonal generators of SU($N$) [i.e., the elements of the Cartan subalgebra of the Lie algebra of SU($N$)]. In this gauge the Polyakov line is given by

$$L = \frac{1}{N} \sum_{i=1}^{N} e^{i\theta_i} . \tag{2.6}$$

The effective potential for $A_0$ has been calculated up to two loops[1] [2,3,21]. Here we shall consider only the one loop result [2,3]. For gluons the effective potential is

$$V_G(\theta_1, \ldots, \theta_N) = -\frac{\pi^2 T^4}{45}(N^2-1) + \frac{\pi^2 T^4}{24} \sum_{i,k=1}^{N} \left[ \frac{\theta_i}{\pi} - \frac{\theta_k}{\pi} \right]^2_{\mathrm{mod}\,2} \left[ 2 - \left[ \frac{\theta_i}{\pi} - \frac{\theta_k}{\pi} \right]_{\mathrm{mod}2} \right]^2 \tag{2.7}$$

and for each fermion flavor in fundamental representation it is

$$V_F(\theta_1, \ldots, \theta_N) = \frac{2\pi^2 T^4}{45} N - \frac{\pi^2 T^4}{12} \sum_{i=1}^{N} \left\{ 1 - \left[ \left[ \frac{\theta_i}{\pi} + 1 \right]_{\mathrm{mod}2} - 1 \right]^2 \right\}^2 . \tag{2.8}$$

We would like to mention here that what we considered was the effective action in a very specific external field. This action was not obtained by the standard method of Legendre transform and one cannot use in this case the general results [25] that the effective potential obtained via a Legendre transform is convex and has a unique minimum—on the contrary, the most interesting situation for us is when there are $Z_N$ degenerate minima (or quasidegenerate in the case of matter in the fundamental representation).

The values of the effective potentials at zero field $V_G(0, \ldots, 0) = -(\pi^2 T^4/45)(N^2-1)$ and $V_F(0, \ldots, 0) = -\frac{7}{4}(\pi^2 T^4/45)N$. It is easy to see that it is the free energy density of an ideal gas of gluons and the fermions at a temperature $T$ which is equal to $-\pi^2 T^4 \kappa/90$, where each bosonic degrees of freedom contributes 1 to $\kappa$ and each fermionic degree of freedom contributes $\frac{7}{8}$ to $\kappa$. Thus the gauge fields contribute $2(N^2-1)$ and the fermions contribute $4(\frac{7}{8})N_f N$ to $\kappa$ which reproduces $V_G(0, \ldots, 0)$ and $V_F(0, \ldots, 0)$.

The gluon effective potential (2.7) has the $Z_N$ symmetry

$$\theta_i \to \theta_i + \frac{2\pi k}{N} \tag{2.9}$$

where the integer $k = 0, \ldots, N-1$ must be the same for all $\theta_i$. Then $\sum \theta_i = 0 \pmod{2\pi/\beta}$ and without fermions there are $N$ minima at $\theta_i = (2\pi/\beta N)k$ where $k = 0, \ldots, N-1$. The fermion potential (2.8) obviously violates the $Z_N$ symmetry (2.9) and only $\theta_i = 0$ is the global minimum of the total effective potential. All other $Z_N$ vacuua become either local minima (metastable states) or unstable states depending on the number of fermion flavors $N_f$.

In general one can consider an $N-1$ parameter dependent configuration of the external field $\Theta$. However if we are looking for the domain-wall-like configuration interpolating between two minima with neighboring values of the Polyakov line $\langle L \rangle$, say $\langle L \rangle = 1$ and $\langle L \rangle = \exp(2\pi i/N)$ one can consider a more simple, one-parameter representation for $\Theta$ in the form

$$\Theta = \frac{2\pi}{N} q \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -(N-1) \end{pmatrix} \tag{2.10}$$

where the transition from $\langle L \rangle = 1$ to $\langle L \rangle = \exp(2\pi i/N)$ is described by interpolating between $q=0$ and $q=1$. In the general case of an interpolation between $\langle L \rangle = \exp(2\pi i l/N)$ and $\langle L \rangle = \exp[2\pi i(l+1)/N]$ one must take $q$ in an interval $g \in [l, l+1)$. Let us note that for SU(2) and SU(3) groups any two minima are the neighbors [$\langle L \rangle = \pm 1$ for SU(2) and $\langle L \rangle = 1, \exp(\pm 2\pi i/3)$ for SU(3)]. Only starting from SU(4) we have pairs which are not neighbors and for which the parametrization (2.10) is wrong. In the case of SU(4) there are two such pairs $\langle L \rangle = \pm 1$ and $\langle L \rangle = \pm i$.

[1]Let us note that a similar problem arises when one considers the effective potential in some Kaluza-Klein models and dynamical breaking of gauge symmetry, the so-called "Hosotani mechanism" [22]. The significance of the adjoint representation for fermions was discussed in detail in recent papers [23,24].

In this paper we shall not consider such a configurations for the sake of simplicity, but it will not be a conceptual problem to repeat the same analysis for a general external field $\Theta$.

The total free energy as a function of $q$ is the sum of a gluon (2.7) and $N_f$ fermion (2.8) effective potentials and can be written as

$$\frac{F(q)}{\pi^2 T^4}=\tfrac{4}{3}(N-1)[V_G(q)+N_f V_F(q)] - \tfrac{1}{45}[(N^2-1)+\tfrac{7}{4}NN_f] \tag{2.11}$$

where the bosonic and fermionic contributions are given in terms of the function

$$f(x)=(x_{\text{mod}1})^2(1-x_{\text{mod}1})^2 \tag{2.12}$$

by

$$\begin{aligned} V_G(q)&=f(q)\ , \\ V_F(q)&=\frac{1}{16}\frac{N}{N-1}-f\left[\frac{q}{N}+\frac{1}{2}\right] - \frac{1}{N-1}f\left[\frac{q}{N}+\frac{1}{2}-q\right] . \end{aligned} \tag{2.13}$$

Notice that the function $f$ is periodic with period 1. Thus $V_G(q)$ is periodic with period 1 but $V_F(q)$ is periodic with period $N$. Note also that a constant has been added to $V_F$ so that it vanishes at $q=0$ which is the perturbative vacuum of the theory. The last term in (2.11) is the free energy density of an ideal gas of gluons and $N_f$ fermions at a temperature $T$.

Now let us repeat the arguments which were used in [7] to demonstrate very serious problems arising in the thermodynamical description of the metastable $Z_N$ vacua. For example for $N=3$ it is easy to show that the metastable minimum at $q=1$ remains metastable for $N_f<18$ at which point it becomes unstable. Notice, however that for $N_f>3$ the free energy density becomes *positive* in these metastable states. Since $F\propto T^4$ this poses a very serious problem which we shall now discuss.

First note that the positivity of the free energy at nonzero values of $q$ is entirely due to the fermions. For integer $q\le N/2$ one gets

$$\frac{F(q\ \text{integer}\le N/2)}{\pi^2T^4}=NN_f\left\{\frac{2}{3}\frac{q^2}{N^2}\left[1-\frac{2q^2}{N^2}\right]-\frac{7}{180}\right\} - \frac{N^2-1}{45} . \tag{2.14}$$

This will be positive provided

$$N_f>\frac{N^2-1}{45N}\left[\frac{2}{3}\frac{q^2}{N^2}\left[1-\frac{2q^2}{N^2}\right]-\frac{7}{180}\right]^{-1} . \tag{2.15}$$

For $N=3$ and $q=1$, for example, we find that $F$ is positive (and proportional to $T^4$) for $N_f>3$. For even $N$ and $q=N/2$ one has positive $F(N/2)$ for $N_f>(N^2-1)/2N$. This point will be a local minimum for $N_f<N$. Thus for $N=4$ and $N_2=2,3$ the point $q=2$ is both a minimum of $F$ and has a positive value $F$ proportional to $T^4$. In more detail this was discussed in [7].

It follows from the above discussion that for a large class of models with both gauge fields and fermions we can write the free energy density as

$$F=+|\gamma|T^4 . \tag{2.16}$$

Such a situation is *impossible* for the metastable state of a real physical system. To see this let us remember that the free energy of any physical system at temperature $T$ is defined as

$$F(T)=-T\ln\sum_n e^{-E_n/T} . \tag{2.17}$$

Shifting $E_n$ by a constant $C$ one can add $C$ to $F(T)$, but the $T$-dependent part must be negative as one can see defining energy levels $E_n$ in a such way that the ground-state energy $E_0=0$ and $F(T)=-T\ln(1+\cdots)<0$. In our case we got positive $F(T)$ which leads to physically senseless thermodynamic quantities, namely, the negative entropy density

$$S=\frac{E-F}{T}=-\frac{dF(T)}{dT}=-4|\gamma|T^3 , \tag{2.18}$$

the negative internal energy density

$$E=F+TS=-3|\gamma|T^4 , \tag{2.19}$$

the negative pressure

$$p=-|\gamma|^{T^4} , \tag{2.20}$$

and the negative specific heat

$$c=-12|\gamma|T^3 . \tag{2.21}$$

Such a metastable vacuum thus has not only a negative pressure but also a negative specific heat and worst of all a negative entropy. It is clear that no physical systems with positive temperature of this type can exist. However, one cannot exclude the metastable states with inverse population (the well-known examples are lasers). We shall briefly discuss such an intriguing possibility in a conclusion.

There *are* interesting cases in which the free energy density at the metastable minimum has the correct, negative sign for $F$. One such example is given in Ref. [6] in which the standard electroweak model is considered well above the QCD phase transition point. In this case the base free energy density of the leptons, the Higgs and the weak gauge bosons contribute to the total free energy density and make it negative. It is clear, however, that if a subsystem of the full system (namely, the quarks and gluons) has the disease discussed above, namely a positive free energy density which grows like $T^4$, then including the leptons, Higgs, and weak gauge bosons which couple weakly to it cannot save the situation. This will be clarified below but if we imagine a system whose total entropy is positive but that some identifiable subsystem has a negative entropy then any statistical description of the full system fails since the subsystem has no states available to it. More discussion about thermodynamical properties of these states as well as difficulties arising in interpretation of $Z_N$ domains in Minkowski space can be found in [7].

Let us note that the real problems arise only when we are assuming that these metastable states appear in our space as $Z_N$ bubbles. To get such bubbles it was usually assumed that one could use the effective potentials (2.7) and (2.8) not only for a constant $A_0$ for which the potentials had been really found in [2] and [3], but also for a spatially dependent $A_0(\mathbf{x})$. Let us note that the $Z_N$ symmetry (neglecting the matter in a fundamental representation) exists only for coordinate-independent part $A_0$ and generally speaking it is not necessary that potential for nonconstant modes $A_0(\mathbf{x})$ is the same. Moreover, as has been demonstrated by Smilga in the Schwinger model [9] and as we shall demonstrate later in $(1+1)$-dimensional QCD with adjoint matter, there are situations when the total effective potential is the sum of two independent potentials for constant and nonconstant modes. In this case the zero mode $A_0$ becomes a quantum mechanical variable and no $Z_N$ bubbles exist in space; however the price for this is the unbroken $Z_N$ symmetry. Before we start this discussion let us consider how one can use the effective potential (2.7) to analyze the high-temperature limit of the confining phase following the ideas suggested by Polchinski in [18].

## III. $Z_N$ PHASES AND HIGH-TEMPERATURE LIMIT OF THE CONFINING PHASE

Let us consider the two-point correlation function of Polyakov lines (1.4) at low temperature in pure gluodynamics:

$$\langle L_k(\mathbf{x})L_{-k}(0)\rangle \sim \exp(-M_k(\beta)x), \quad x\to\infty \ . \tag{3.1}$$

The correlation function vanishes at infinity because of the confinement, so the expectation value of Polyakov line is zero $\langle L_k\rangle=0$ and $Z_N$ symmetry is unbroken. The usual interpretation of the Polyakov line is the world line of an external source in fundamental (for $k=1$) or in higher ($k>1$) representations. However there is a dual description when one can consider compact Euclidean time $\tau$ as a spatial coordinate and $L_k$ is the creation operator of a winding state with an electric flux in the periodic direction and $M_k(\beta)$ is a temperature-dependent mass of the winding state. In the deconfinement phase the winding modes become tachyonic and the theory makes a transition to a phase with broken $Z_N$ symmetry.

Polchinski in [18] used the high-temperature effective potential (2.7) to calculate the mass of these tachyonic states. To find them he considered the effective action

$$\begin{aligned} S_{\text{eff}}=\int d^3x\frac{1}{2g^2\beta}\sum_{i=1}^{N}(\nabla\theta_i)^2 &+\frac{1}{24\pi^2\beta^3}\sum_{i,k=1}^{N}(\theta_i-\theta_k)^2_{\text{mod}2\pi}\\ &\times[2\pi-(\theta_i-\theta_k)_{\text{mod}2\pi}]^2 \ . \end{aligned}\tag{3.2}$$

The gradient term corresponds to the square of the electric field $\mathbf{E}^2$ in the bare action and the second term can be obtained from the effective potential (2.7) after omitting the $\theta$-independent first term. In the large $N$ limit one can introduce the normalized density

$$\rho(\theta,\mathbf{x})=\frac{1}{N}\sum_i\delta(\theta-\theta_i(\mathbf{x})) \ . \tag{3.3}$$

In the $N\to\infty$ limit $Z_N$ symmetry is transformed into U(1) symmetry $\theta\to\theta+\text{const}$. It is easy to see that $L_k(\mathbf{x})$ are the Fourier coefficients of the density $\rho(\theta,\mathbf{x})$ [see (2.6)]:

$$\begin{aligned} \rho(\theta,\mathbf{x})&=\frac{1}{2\pi}\left[1+\sum_{k=1}^{\infty}L_k(\mathbf{x})\exp(-ik\theta)+L_{-k}(\mathbf{x})\exp(ik\theta)\right],\\ L_k(\mathbf{x})&=\int_0^{2\pi}\rho(\theta,\mathbf{x})e^{ik\theta}d\theta=\frac{1}{N}\sum_{i=1}^{N}\exp(ik\theta_i(\mathbf{x}))=\frac{1}{N}\text{tr}P\exp\left[i\int_0^{k\beta}A_0(\tau,\mathbf{x})d\tau\right] \ . \end{aligned}\tag{3.4}$$

The action (3.2) takes the form

$$S_{\text{eff}}=\frac{N}{2g^2\beta}\int d^3x\int_0^{2\pi}d\theta\frac{1}{\rho(\theta,\mathbf{x})}(\partial_\theta^{-1}\nabla\rho(\theta,\mathbf{x}))^2+\frac{N^2}{24\pi^2\beta^3}\int d^3x\int_0^{2\pi}d\theta_1\int_0^{2\pi}d\theta_2\rho(\theta_1,\mathbf{x})\rho(\theta_2,\mathbf{x})(\theta_1-\theta_2)[2\pi-(\theta_1-\theta_2)]^2 \tag{3.5}$$

where $\partial_\theta^{-1}$ is defined as $\partial_\theta^{-1}\exp(ik\theta)=\exp(ik\theta)/ik$ and in large $N$ limit we must keep $g^2N$ fixed. The minimum of the potential is when all the eigenvalues are equal which gives us the spectral density in the high-temperature phase with broken $Z_N$ [here is U(1) in large $N$ limit] symmetry:

$$\rho_{\text{broken}}(\theta,\mathbf{x})=\delta(\theta-\theta_0) \tag{3.6}$$

for some $\theta_0$. The symmetric confining phase is defined by the U(1) invariant distribution

$$\rho_c(\theta,\mathbf{x})=\frac{1}{2\pi} \tag{3.7}$$

which is unstable. One can easily find that in quadratic approximation in $L_k$ the action (3.5) takes the form

$$S=N^2\sum_{k=-\infty}^{\infty}\int d^3x\left[\frac{1}{2g^2N\beta k^2}\nabla L_k\nabla L_{-k}+\frac{1}{24\pi^2\beta^3}V_kL_kL_{-k}\right] \tag{3.8}$$

where $V_k$ is the Fourier transform of the potential $V(\theta_1-\theta_2)=(\theta_1-\theta_2)^2_{\mathrm{mod}2\pi}[2\pi-(\theta_1-\theta_2)_{\mathrm{mod}2\pi}]^2$,

$$V_k=\frac{1}{2\pi}\int_0^{2\pi}d\theta e^{ik\theta}V(\theta)$$
$$=\frac{1}{2\pi}\int_0^{2\pi}d\theta e^{ik\theta}\theta^2(2\pi-\theta)^2=-\frac{24}{k^4} \tag{3.9}$$

and one gets the tachyonic winding modes with masses

$$M_k^2=-2g^2N/\pi^2\beta^2k^2\ . \tag{3.10}$$

One can make an assumption that the same spectrum of the tachyon masses must be reproduced in the string theory (if any) describing QCD. Let us note that this spectrum is different from the spectrum obtained in the usual string theory [15] $\alpha'M_k^2=-n/6+\beta^2/4\pi^2\alpha'$ where $n$ is some effective constant proportional to the number of the world-sheet degrees of freedom (24 for critical bosonic string). Using (3.10) Polchinski made a conclusion that for QCD string the effective number of degrees of freedom grows with temperature as $n_{\mathrm{eff}}(\beta)\sim g^2(\beta)N/\beta^2$ and the main conclusion of his analysis is that a string theory describing QCD in the large $N$ limit must have a number of world-sheet degrees of freedom which diverge at short distances [18].

Let us note that this analysis can be repeated even in the case when there are $N_f$ fermions in the fundamental representation. Then using the (2.8) one can see that the fermion contribution to the effective action (3.2) will be

$$S_f=-\frac{N_f}{12\pi^2\beta^3}\int d^3x\sum_{i=1}^{N}[\pi^2-[(\theta_i+\pi)_{\mathrm{mod}2\pi}-\pi]^2]^2\ . \tag{3.11}$$

This corresponds to the linear in $\rho(\theta,\mathbf{x})$ contribution to the effective action for the density $\rho$:

$$S_f=-\frac{NN_f}{12\pi^2\beta^3}\int d^3x\int_{-\pi}^{\pi}d\theta\rho(\theta,\mathbf{x})(\theta^2-\pi^2)^2$$
$$=NN_f\sum_k\int d^3x\frac{(-1)^k}{k^4\pi^2\beta^3}L_k(x)\ . \tag{3.12}$$

Including this term into (3.8) we get

$$S=N^2\sum_{k=-\infty}^{\infty}\int d^3x\left[\frac{1}{2g^2N\beta k^2}\nabla L_k\nabla L_{-k}-\frac{1}{24\pi^2\beta^3k^4}L_kL_{-k}+\frac{N_f}{N}\frac{(-1)^k}{k^4\pi^2\beta^3}L_k\right] \tag{3.13}$$

and it is easy to see that linear terms do not change the values of the tachyon masses (3.10), but shift the fields $L_k$:

$$L_k\to L_k+\frac{(-1)^k}{2}\frac{N_f}{N}\ . \tag{3.14}$$

For small $N_f/N$ this shift is small and the quadratic approximation for the total action (3.5) is still valid. One cannot use this approximation when $N_f\approx N$, i.e., precisely when the suppression factor for nonplanar diagrams $N_f/N$ is of order one and nonplanar diagrams have the same order of magnitude as the planar ones. In this case the one-loop approximation is invalid and one can no longer use the method of an effective potential in the external field to get any information about the (possible) string instabilities at high temperature. However one must take into account that in the case when nonplanar diagrams are not suppressed, the connection between large $N$ gauge theory and a string theory is much more problematic.

Thus we see that effective high-temperature theory gives us important information about the (possible) string description of the low-temperature confinement phase. This information was based on the form of the effective potential which may lead in some situation to physically senseless results. So we must be very accurate in dealing with this potential and for this reason let us study the simplest model where one can hope to have a confinement-deconfinement transition: $(1+1)$-dimensional QCD with adjoint matter [10,12–14].

## IV. TWO-DIMENSIONAL QCD COUPLED TO ADJOINT MATTER AT HIGH TEMPERATURES

Let us repeat the following Kutasov [12] the analysis of the previous section in the case of $(1+1)$-dimensional QCD interacting with Majorana fermions in the adjoint representation described by the action

$$S_{\mathrm{adj}}=\int dx\int_0^{\beta}d\tau\,\mathrm{Tr}\left[\frac{1}{2g^2}F_{\mu\nu}F^{\mu\nu}+i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi+m\bar{\Psi}\Psi\right] \tag{4.1}$$

defined on Euclidean space-time with periodic Euclidean time $\tau\sim\tau+\beta$ with periodic boundary conditions for gauge fields and antiperiodic for fermions. One can again choose the gauge when $A_0$ is diagonal and independent of time $\tau$: $A_0=(1/\beta)\mathrm{diag}(\theta_1,\ldots,\theta_N)$. Then the one-loop effective action for the $\theta_i$ takes the form

$$S_{\mathrm{eff}}=\frac{1}{2g^2\beta}\int dx\sum_{i=1}^{N}\left[\frac{d\theta_i}{dx}\right]^2+V(\theta_1,\ldots,\theta_N) \tag{4.2}$$

where the effective potential is nothing but the determinant of the Dirac operator in adjoint representation $\gamma_\mu D_\mu^{\mathrm{adj}}$ in an external $A_0$ field:

$$V(\theta_1,\ldots,\theta_N)=-\ln\det[\gamma_\mu D_\mu^{\mathrm{adj}}[A]+m]$$
$$=-\frac{1}{2}\sum_{i,j=1}^{N}\ln\det[\gamma_\mu D_\mu[\theta_i-\theta_j]+m]\ . \tag{4.3}$$

Using the proper time representation of the fermion determinant one gets

$$V(\theta_i-\theta_j)=\frac{\beta}{2\pi}\int dx\sum_{k=1}^{\infty}(-)^k \times\int_0^{\infty}\frac{d\tau}{\tau^2}\exp\left[-\frac{k^2\beta^2}{4\tau}-m^2\tau\right]\times\cos k(\theta_i-\theta_j)\,,$$

$$V(\theta_i-\theta_j)=\frac{1}{2\pi\beta}\int dx[(\theta+\pi)_{\mathrm{mod}2\pi}-\pi]^2,\quad m=0\,, \tag{4.4}$$

and in the high-temperature limit $\beta\to 0$ one can neglect the mass term in the leading $1/\beta$ approximation. The sum over $k$ is the sum over winding of the particle trajectory around the compact imaginary time and in the high-temperature limit $\beta\to 0$ one can neglect the mass term[2] in the leading $1/\beta$ approximation. This representation of the determinant can be easily generalized to higher dimensions where one must substitute $d\tau/\tau^2$ by $d\tau/\tau^{d+1}$ in the case of $(d+1)$-dimensional theory. Also in the case of bosonic degrees of freedom the factor $(-1)^k$ is missing. One can easily reproduce (2.7) and (2.8) using the proper time representation of the boson and fermion determinants.

Thus for small $\beta$ the effective potential for *constant* $\theta_i$ is known and assuming, as usual, that for slowly varying $\theta$ one can simply substitute $\theta_i$ by $\theta_i(x)$ one gets the effective action (see [12])

$$S_{\mathrm{eff}}=\frac{1}{2g^2\beta}\int dx\sum_{i=1}^{N}\left[\frac{d\theta_i}{dx}\right]^2+\frac{2}{\pi\beta}\int dx\sum_{i,j=1}^{N}\sum_{k=1}^{\infty}(-1)^k\frac{1}{k^2}\cos k[\theta_i(x)-\theta_j(x)]\,. \tag{4.5}$$

Obviously this action has $Z_N$ symmetry $\theta_i\to\theta_i+2\pi m/N$, $m=1,2,\ldots,N$ and we get the same $Z_N$ bubbles as in four-dimensional case. One can also consider the density (3.3) $\rho(\theta,x)$ and using the same method as in [18] Kutasov obtained the effective action for $L_k(x)$ in quadratic approximation[3]

$$S=N^2\sum_{k=-\infty}^{\infty}\int dx\left[\frac{1}{2g^2N\beta k^2}\frac{dL_k}{dx}\frac{dL_{-k}}{dx}+(-1)^k\frac{2}{\pi\beta k^2}L_kL_{-k}\right] \tag{4.6}$$

and got the masses of the winding modes,

$$M_k^2(\beta\to 0)=\frac{4g^2N}{\pi}(-1)^k\,, \tag{4.7}$$

where only the odd $k$ winding modes are tachyonic.

However in the massless limit the form of the effective potential (4.5) which leads both to the existence of the $Z_N$ domain walls (and in the case of the fermions in the fundamental representation they will be metastable bubbles) and gives the imaginary mass of the unstable winding states is wrong. As we shall demonstrate now in the massless theory there are no domain walls and one cannot make simple predictions about the condensation of the winding states—and all this because the potential for the $x$-dependent modes of the fields $\theta_i$ is not periodic, contrary to the constant mode part which was really calculated in (4.4). The two-dimensional fermion determinant on cylinder in arbitrary gauge field was calculated in [19] where it was shown that it is factorized into product of two independent terms. The first one is the periodical potential but for constant modes $\theta_i$, not for the fields $\theta_i(x)$. The second term depends on $x$-dependent fields $\theta_i(x)$; however, there is no reason why this part must have $Z_N$ symmetry, and it does not. As a result any configuration with a domain wall has an energy proportional to the one-dimensional system length $\int dx$ and these configurations are absolutely irrelevant.

To make this statement more clear let us repeat the calculation of the determinant of the two-dimensional Dirac operator and demonstrate the factorization property following [19]. We shall not consider here the case of an arbitrary gauge field $A_\mu$, where $A_\mu$ is a Hermitian matrix, instead we shall consider interesting for us case when this matrix is diagonal [see (2.5)]:

$$A^{\mu}(\tau,x)=\begin{bmatrix} A_1^{\mu}(\tau,x) & & \\ & \ddots & \\ & & A_N^{\mu}(\tau,x)\end{bmatrix}\,. \tag{4.8}$$

Then the fermion action from (4.1) takes the form

$$\begin{aligned}S_f&=\int dx\int_0^{\beta}d\tau\,\mathrm{Tr}\{\bar\Psi(i\gamma_\mu\partial^\mu+m)\Psi+\bar\Psi\gamma_\mu[A^\mu\Psi]\}\\&=\int dx\int_0^{\beta}d\tau\sum_{i,j}\bar\Psi_{ij}(i\gamma_\mu\partial^\mu+\gamma_\mu[A_i^\mu-A_j^\mu]+m)\Psi_{ij}\end{aligned} \tag{4.9}$$

and we see that the total determinant for the adjoint fermions,

$$\det[\gamma_\mu D^{\mu}_{\mathrm{adj}}(A)+m]=\prod_{i,j}\det(\gamma_\mu[i\partial^\mu+A_{ij}^\mu]+m)\,, \tag{4.10}$$

is the product of Abelian determinants of fermions interacting with Abelian fields $A_{ij}^\mu=A_i^\mu-A_j^\mu$.

As discussed before in the high-temperature limit one

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[2]It is easy to see that the corrections will be by order of $m\beta$.

[3]To get the effective action in quadratic approximation one must simply repeat the same procedure as in a (3+1)-dimensional case. Let us note also that because we need Fourier transformed effective potential $V_k$ it is not necessary to take the sum over $k$ in (4.5) to get $V(\theta)$.

can omit the mass term and what we are trying to calculate now is the determinant of the Dirac operator for the massless fermion in an Abelian gauge field $A^{\mu}$ on the cylinder $S^1 \times R^1$ or on the torus $S^1 \times S^1$ if the space is a circle too. Let us consider the Hodge decomposition of the vector potential (in the sector with zero topological charge),

$$A^{\mu}=\frac{1}{R_{\mu}}\theta^{\mu}+\epsilon^{\mu\nu}\partial_{\nu}\chi+\partial^{\mu}\xi\ , \tag{4.11}$$

where the last term is the pure gauge and can be neglected. The $R_{\mu}$ factors are the radii of two $S_1$ and $R_{\tau}=\beta$. In the case of the noncompact space, i.e., when we are considering the cylinder $S^1 \times R^1$ one can put $R_x \to \infty$. Constant modes $\theta^{\mu}$ are the angle variables with periodicity $2\pi$ and $\chi$ is a coordinate-dependent mode. Substituting the Hodge decomposition into Dirac operator and using the well-known property of two-dimensional $\gamma$ matrices $\gamma_{\mu}\epsilon^{\mu\nu}=i\gamma^{\nu}\gamma_5$ one can get

$$i\gamma_{\mu}(\partial^{\mu}-iA^{\mu})=i\gamma_{\mu}\left[\partial^{\mu}-\frac{i}{R_{\mu}}\theta^{\mu}-i\epsilon^{\mu\nu}\partial_{\nu}\chi\right]$$

$$=ie^{\gamma_5\chi}\gamma_{\mu}\left[\partial^{\mu}-\frac{i}{R_{\mu}}\theta^{\mu}\right]e^{\gamma_5\chi}\ . \tag{4.12}$$

Now let us consider a family of operators:

$$\mathcal{D}_{\tau}=ie^{\tau\gamma_5\chi}\gamma_{\mu}\left[\partial^{\mu}-\frac{i}{R_{\mu}}\theta^{\mu}\right]e^{\tau\gamma_5\chi}\ . \tag{4.13}$$

We are looking for $D_1$ and $D_0=\gamma_{\mu}[\partial^{\mu}-(i/R_{\mu})\theta^{\mu}]$ is the operator in a constant field whose determinant is periodic in $\theta^{\mu}$.

A very elegant formula was obtained by Blau, Visser, and Wipf in [19] for a family of general first-order elliptic self-adjoint operators depending on a parameter $\tau$ of the type $\mathcal{O}_{\tau}=\sqrt{g_0/g_{\tau}}\exp(\tau f^{\dagger})\mathcal{O}_0\exp(\tau f)$ where $f(x)$ can be in general some matrix-valued function and $g_{\tau}$ is the parameter-dependent metric on the manifold. In our case (4.13) we have $f(x)=\gamma_5\chi(x)$ and flat metric independent on $\tau$. We shall present here the result for this case only, more general expressions including general non-Abelian field can be found in original paper [19]. For a family of operators (4.13) one gets

$$\frac{d}{d\tau}\ln\det\mathcal{D}_{\tau}=\frac{\tau}{\pi}\int d^2x\,\chi\partial^2\chi \tag{4.14}$$

which can be easily integrated and finally we get

$$\det\gamma_{\mu}(\partial^{\mu}-iA^{\mu})=\det\gamma_{\mu}\left[\partial^{\mu}-\frac{i}{R_{\mu}}\theta^{\mu}\right]\times\exp\left[\frac{1}{2\pi}\int d^2x\,\chi\partial^2\chi\right] \tag{4.15}$$

and we see that the contributions of constant modes $\theta^{\mu}$ and nonconstant modes $\chi(x)$ are factorizable.

Here we neglect the possible contributions of the zero modes. If they are present the determinant itself is zero and one has the expression for $\det'\mathcal{D}_{\tau}/\det K_{\tau}$, where $K_{\tau}$ is the matrix of scalar products of zero modes. However if we consider the sector with a zero topological charge there are no zero modes and one can simplify life a little. Because we restricted the topological charge to zero we are not allowed to consider a single domain wall which carries a nonzero topological charge. However we can consider wall-antiwall configurations (which one may call soliton-antisoliton or instanton–anti-instanton configurations) which is nothing but a one-dimensional $Z_N$ bubble [see discussion after Eq. (4.18)]. Thus we see that the restriction to the sector with a zero topological charge is not too restrictive indeed. Moreover, it is easy to generalize the analysis to the case of nonzero topological charge—the only new elements will be the presence of several zero fermion modes and we have to calculate not the vacuum-vacuum amplitudes in the presence of the external field $\langle 0|0\rangle_{A_0}=\exp(-S_{\rm eff}(A_0))$, but the matrix elements of the transitions with the production of $n$ fermion zero modes $\langle\prod_i^n\bar{\Psi}_i\Psi_i\rangle$. It is evident that there are no contributions from nontrivial topological sectors to the vacuum-vacuum amplitude we are looking for.

Thus the effective potential is the sum of two terms. One is the effective potential for the constant field which we had calculated and which is periodic in $\theta$ with a period $2\pi$. Another one is the potential the $x$-dependent part of the gauge field which *is not* periodic at all; this is the ordinary Schwinger mass term proportional to $\int d^2x\,\chi\partial^2\chi=(1/4)\int d^2x\,\epsilon_{\mu\nu}F_{\mu\nu}(1/\partial^2)\epsilon_{\mu\nu}F_{\mu\nu}$.

Now let us consider the effective action when only $A_0$ component in (4.8) is nonzero and $A_0$ depends only on $x$. This is precisely the case which is relevant both for studying $Z_N$ bubbles and instability of the confining phase at high temperatures as we have discussed before. Then one can immediately write the effective action in which the constant modes $\theta_i$ are completely independent from $x$ dependent fields $\tilde{A}_i$, where the tilde means that we extracted the zero mode from $A_i(x)$:

$$S_{\rm eff}=\frac{L}{2\pi\beta}\sum_{i,j}[(\theta_i-\theta_j+\pi)_{{\rm mod}2\pi}-\pi]^2+\frac{\beta}{2g^2}\int dx\sum_{i=1}^{N}\left[\frac{dA_i}{dx}\right]^2+\frac{\beta}{2\pi}\int dx\sum_{i,j=1}^{N}[\tilde{A}_i(x)-\tilde{A}_j(x)]^2 \tag{4.16}$$

where $L$ is the size of the one-dimensional system. The $Z_N$ symmetry of this action $\theta_i\to\theta_i+2\pi m/N$, $m=1,2,\ldots,N$ does not affect the $x$-dependent fields $\tilde{A}_i(x)$ at all.

It is clear now that for such potential any $Z_N$ bubbles will have infinite energy in the thermodynamical limit. To see it let us consider the field [see (2.5)]

$$A_0(x)=\frac{2\pi}{\beta N}q(x)\begin{pmatrix}1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & -(N-1)\end{pmatrix} \tag{4.17}$$

where $q=0$ and $q=1$ corresponds to two different $Z_N$ vacuua. The effective action for one interpolating field $q(x)$ is

$$S[q]=(N-1)\left\{\frac{2\pi^2}{\beta g^2 N}\int dx\left[\frac{dq}{dx}\right]^2+\frac{2\pi}{\beta}\int dx\tilde{q}^2(x)+\frac{2\pi L}{\beta}\left[\left[q_0+\tfrac{1}{2}\right]_{\mathrm{mod}\,1}-\tfrac{1}{2}\right]^2\right\} \tag{4.18}$$

where $q(x)=q_0+\tilde{q}(x)$ and zero mode $q_0=(1/L)\int_0^L dx\, q(x)$. Let us consider the $Z_N$ bubble, i.e., the region with $q(x)=1$ with size $R$. The rest part of the space, i.e., the region with size $L-R$ has $q(x)=0$ and we assumed that both $L$ and $R$ are large in comparison with domain wall width $1/\sqrt{Ng^2}$, so one can neglect the contributions from the regions where $q(x)$ interpolates between 0 and 1. It is easy to see that in this case $q_0=R/L$ and $\tilde{q}(x)=q(x)-R/L$ and one must have $R/L<\frac{1}{2}$; in the opposite case it is better to say that there is a bubble of $q=0$ phase in the space with $q=1$. Then the second and third terms in $S[q]$ contribute to the action

$$\begin{aligned}S(L,R)&=(N-1)\frac{2\pi}{\beta}\left[R\left[1-\frac{R}{L}\right]^2+(L-R)\left[\frac{R}{L}\right]^2\right.\\&\qquad\left.+L\left[\frac{R}{L}\right]^2\right]\\&=(N-1)\frac{2\pi}{\beta}R\end{aligned} \tag{4.19}$$

and we see that the action is proportional to the bubble size $R$. To consider the single domain wall one must put $R\sim L$ sending the "anti"wall far apart, then in the thermodynamical limit $L\to\infty$ and single domain wall is infinitely heavy.

Thus there are no $Z_N$ bubbles in this theory; moreover, we are not allowed to have any $Z_N$ vacua. The reason is very simple—in this theory the $Z_N$ symmetry cannot be broken spontaneously. The reason is obvious—due to the decoupling of constant modes from all other modes we have to average over all $Z_N$ vacua and cannot restrict ourselves for only one, as would be possible in the case without factorization. Let us consider, for example, SU(2) theory with the symmetry of the center $Z_2$. The Polyakov line in this case is[4]

$$\begin{aligned}L(x)&=\tfrac{1}{2}[e^{i\pi q(x)}+e^{-i\pi q(x)}]\\&=\tfrac{1}{2}[e^{i\pi q_0}e^{i\pi\tilde{q}(x)}+e^{-i\pi q_0}e^{-i\pi\tilde{q}(x)}]\end{aligned} \tag{4.20}$$

and it is evident that $\langle L(x)\rangle=0$ because averaging over $q_0$ is factorized and

$$\langle e^{\pm i\pi q_0}\rangle_{q_0}=0\ . \tag{4.21}$$

However the zero-mode $q_0$ factors are canceled in the two-point correlation function:

$$\begin{aligned}\langle L(x)L(0)\rangle&=\tfrac{1}{2}\langle\cos[\tilde{q}(x)-\tilde{q}(0)]\rangle\\&=\tfrac{1}{2}e^{\langle\tilde{q}(x)\tilde{q}(0)\rangle}\end{aligned} \tag{4.22}$$

which is nonzero and has a finite limit at $x\to\infty$. Thus we have $\langle L(x)L(0)\rangle\to\frac{1}{2}$ when $x\to\infty$ and at the same time $\langle L(x)\rangle=0$. The breakdown of the clusterization property is connected with the factorization of the zero mode $q_0$. This mode is absolutely delocalized and one cannot use the naive rule $\langle A(x)B(0)\rangle\to\langle A(x)\rangle\langle B(0)\rangle$ at $x\to\infty$.

Because the $Z_N$ symmetry is unbroken at high temperatures one can conclude that the analysis of the instability of the confining phase at high temperatures which was done in [12] is incomplete. However, we must remember that the factorization of the fermion determinant (4.15) was obtained only for massless case. For massive fermions the effective action is

$$\mathrm{Tr}\ln[\gamma_\mu D^\mu-m]=\mathrm{Tr}\ln\gamma_\mu D^\mu-\mathrm{Tr}\sum_{n=1}^{\infty}\frac{m^n}{(\gamma_\mu D^\mu)^n} \tag{4.23}$$

and only in first term there is a factorization; other terms may mix constant and variable modes. However their contribution is suppressed by powers $\beta m$ and one can think about their importance at temperatures $T<m$. It is still unclear what is the order parameter corresponding to the Hagedorn transition in the theory with light fermions with mass $m\le\sqrt{g^2N/\pi}$. In any case the usual picture of the $Z_N$ symmetry breaking is wrong in this case and one has the strange picture of unbroken $Z_N$ with $\langle L(x)\rangle=0$, but with nonzero $\langle L(\infty)L(0)\rangle$, which means deconfining.

## V. DISCUSSION AND CONCLUSION

In this paper we tried to address some at first sight different but, from our point of view, related problems connected with the $Z_N$ symmetry in hot gauge theories. We demonstrated that by allowing the nontrivial $Z_N$ vacua to exist one may have a real problem when studying the metastable states which have impossible thermo-

[4]Here we are considering the simplest case with the winding number $k=1$. The generalization for arbitrary $k$ is trivial.

dynamic properties. We tried to present arguments that such bubbles cannot exist as metastable states at all. However we did not explore in this paper another possibility, which seems unrealistic, but cannot be excluded completely. This is the idea that if after all our attempts the metastable states will survive one must consider them as some kind of states with *inverse population*, i.e., the states with *negative* temperature. Let us remember that both negative specific heat and negative entropy proportional to $T^3$ and for negative $T$ they will change the sign; i.e., in the case when they were negative at positive $T$ they will be positive for negative $T$. This idea seems rather strange because we still have a paradox—what we have started from was the gas of quarks and gluons at high positive temperatures. How one gets a negative temperature is a big question. One may speculate that we can reach this region passing through infinite $T$. If such a thing would be possible one could imagine something like a quark-gluon laser in hot gauge theories. Despite all its strangeness this idea deserves further analysis.

We also considered the connection between the existence of $Z_N$ bubbles and the instabilities of the confining phase at high temperatures. Using two-dimensional QCD with adjoint fermions as an example we demonstrated that the situation with the breaking of the $Z_N$ symmetry is not so simple as one could imagine when dealing with "simple" two-dimensional models. In fact, for the massless case we have proved that the $Z_N$ symmetry cannot be broken at all; however, the correlation function of the two Polyakov lines $\langle L(x)L(0)\rangle$ is not going to zero when $x\to\infty$ and we are in the deconfining phase at high temperatures. It is unclear how the mass of the fermion affects this situation. We conjectured that the same physics must be correct for light enough fermions with a mass $m$ less than $\sqrt{g^2N}$. For heavy fermions with $m^2>g^2N$ one may have another phase and may be even the breaking of global $Z_N$. It is amusing that in this theory one has hidden supersymmetry precisely at $m^2=g^2N/\pi$ as was shown by Kutasov in [12]. Does it mean that the supersymmetry point is a phase transition point and one has two phases in a QCD with adjoint fermions? This is a very intriguing possibility.

It is also unclear how reliable the effective action is in the four-dimensional gauge theories at high temperatures. What is known is the effective action for the constant field and one assumes that it is possible to substitute the constant field $A_0$ with a general $x$-dependent field $A_0(x)$. We just saw, however, that this recipe is not universal; the two-dimensional example is a good demonstration of the fact that it is absolutely wrong in this case. There is no such strong statement as a factorization of a constant mode contribution to the effective action in a four-dimensional case, but maybe we still missed something else in a four-dimensional case. In any case it seems that the problem of $Z_N$ phases and related with it the problem of the instabilities of the confining phase[5] definitely deserve further investigations.

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[5]Which may give us unique information about structure of QCD strings, if string description of QCD exists.

[1] A. M. Polyakov, Phys. Lett. **72B**, 477 (1978); L. Susskind, Phys. Rev. D **20**, 2610 (1979); B. Svetitsky and L. G. Yaffe, Nucl. Phys. **B210**, 423 (1982).
[2] D. J. Gross, R. D. Pisarski, and L. G. Yaffe, Rev. Mod. Phys. **53**, 43 (1981).
[3] N. Weiss, Phys. Rev. D **24**, 475 (1981); Can. J. Phys. **59**, 1686 (1981); Phys. Rev. D **25**, 2667 (1982).
[4] L. McLerran and B. Svetitsky, Phys. Rev. D **24**, 450 (1981); J. Kuti, J. Polonyi, and K. Szlachanyi, Phys. Lett. **98B**, 199 (1981).
[5] T. Bhattacharya, A. Gocksch, C. Korthals Altes, and R. D. Pisarski, Phys. Rev. Lett. **66**, 998 (1991).
[6] V. Dixit and M. C. Ogilvie, Phys. Lett. B **269**, 353 (1991); J. Ignatius, K. Kajantie, and K. Rummukainen, Phys. Rev. Lett. **68**, 737 (1992).
[7] V. M. Belyaev, I. I. Kogan, G. W. Semenoff, and N. Weiss, Phys. Lett. B **277**, 331 (1992).
[8] A. Gocksch and R. D. Pisarski, Nucl. Phys. **B402**, 657 (1993).
[9] A. V. Smilga, Bern University Report No. BUTP-93/3 (unpublished).
[10] S. Dalley and I. R. Klebanov, Phys. Rev. D **47**, 2517 (1993).
[11] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[12] D. Kutasov, Chicago Report Nos. EFI-93-30, hep-th/9306013 (unpublished).
[13] G. Bhanot, K. Demeterfi, and I. R. Klebanov, Phys. Rev. D **48**, 4980 (1993).
[14] K. Demeterfi, I. R. Klebanov, and G. Bhanot, Princeton Report Nos. PUPT-1427/IASSNS-HEP-93/59, hep-th/9311015.
[15] I. I. Kogan, Pis'ma Zh. Eksp. Teor. Fiz. **45**, 556 (1987) [JETP Lett. **45**, 709 (1987)]; B. Sathiapalan, Phys. Rev. D **35**, 3277 (1987).
[16] J. Atick and E. Witten, Nucl. Phys. **B310**, 291 (1988).
[17] A. A. Abrikosov, Jr. and I. I. Kogan, Zh. Eksp. Teor. Fiz. **96**, 418 (1989) [Sov. Phys. JETP **69**, 235 (1989)]; Int. J. Mod. Phys. A **6**, 1501 (1991).

[18] J. Polchinski, Phys. Rev. Lett. **68**, 1267 (1992).
[19] S. K. Blau, M. Visser, and A. Wipf, Int. J. Mod. Phys. A **4**, 1467 (1989).
[20] A. Roberge and N. Weiss, Nucl. Phys. **B275**, 734 (1986).
[21] R. Anishetty, J. Phys. G **10**, 423 (1984); K. J. Dahlem, Z. Phys. C **29**, 553 (1985); V. M. Belyaev and V. L. Eletsky, *ibid.* **45**, 355 (1990); V. M. Belyaev, Phys. Lett. B **254**, 153 (1991).
[22] Y. Hosotani, Phys. Lett. **126B**, 309 (1983); D. J. Toms, *ibid.* **126B**, 445 (1983).
[23] A. T. Davies and A. McLachlan, Nucl. Phys. **B317**, 237 (1989); M. Burgess and D. J. Toms, Ann. Phys. (N.Y.) **210**, 438 (1991).
[24] C.-L. Ho and Y. Hosotani, Nucl. Phys. **B345**, 445 (1990); C.-L. Ho, Mod. Phys. Lett. A **5**, 2021 (1990).
[25] J. Illiopoulos, C. Itzykson, and A. Martin, Rev. Mod. Phys. **47**, 165 (1975); L. O'Raifeartaigh, A. Wipf, and H. Yoneyama, Nucl. Phys. **B271**, 653 (1986).